EXTREMAL $F$-INDEX OF A GRAPH WITH $k$ CUT EDGES

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Abstract. The so called forgotten index or $F$-index is defined as the sum of cubes of vertex degrees of a molecular graph. In this paper, we have obtained the upper and lower bounds of $F$-index for the graphs with $k$ cut edges and also we have characterized the extremal graphs.

1. Introduction

Let $G = (V, E)$ be a simple connected graph with the vertex set $V(G)$ and the edge set $E(G)$. The set of vertices adjacent to a vertex $v$ in $G$ is denoted by $N_G(v)$ and $d_G(v) = |N_G(v)|$ denotes the degree of the vertex $v$ in $G$. If an end vertex of an edge is of degree one, then the edge is called a pendant edge. A cut-edge is an edge of $G$ which when removed from $G$, leaves it disconnected. The path, cycle and star of $n$ vertices are denoted by $P_n$, $C_n$ and $K_{1,n-1}$ respectively. If two simple connected graphs $G_1$ and $G_2$ are concatenated at a common vertex $v$, we denote the new graph as $G_1vG_2$. The vertex set of $G_1vG_2$ is $V(G_1) \cup V(G_2)$, where $V(G_1) \cap V(G_2) = \{v\}$, and edge set of $G_1vG_2$ is $E(G_1) \cup E(G_2)$. Similarly, the concatenation of the graphs $G_1$ and $G_2$ by adding an edge $uv$ between the graphs $G_1$ and $G_2$ such that $u \in V(G_1)$ and $v \in V(G_2)$ is denoted by $G_1uvG_2$.

The first Zagreb index being one of the oldest degree based topological index, is defined as

$$M_1(G) = \sum_{v \in V(G)} [d_G(v)]^2.$$ 

In the paper where Zagreb indices were introduced for the first time by Gutman and Trinajstić [4], it was shown that total $\pi$-electron energy has correlation with both the sum of squares and sum of cubes of the vertex degrees of the underlying molecular graph. Eventually, the sum of squares of vertex degrees became known as the first Zagreb index, but the sum of cubes of vertex degrees remained unnoticed.
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by researchers until a recent work of Furtula and Gutman, where they have named it “forgotten” topological index, or $F$-index [3]. Formally, $F$-index is defined as

$$F(G) = \sum_{v \in V(G)} [d_G(v)]^3.$$ 

It is easy to follow that $M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$ and $F(G) = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2].$

Finding the extremal values or bounds for the topological indices of graphs, as well as related problems of characterizing the extremal graphs are among the most studied research topics in mathematical chemistry during last few decades. Extremal trees with respect to $F$-index have been found by Abdo et al. [1]. Ordering of unicyclic graphs with respect to $F$-index have also been done [6]. Extremal Zagreb indices of graphs with a given number of cut edges have been studied by Chen [2].

Let $G_{n,k}$ be the set of graphs with $n$ vertices and $k$ cut edges. Let $\{e_1, e_2, \ldots, e_k\}$ be the set of cut edges of $G$. The connected graphs with $k$ cut edges have been considered by many researchers while finding extremal topological indices [2,5,7,8]. If an edge $e$ of $G$ is on a cycle in $G$, $e$ cannot be a cut-edge. Every edge of $G$ is a cut-edge if and only if $G$ is a tree. Since the extremal trees with respect to $F$-index have already been studied, we assume that $G$ contains at least one cycle. Thus the number of its cut edges is at most $n - 3$. Therefore, in the following discussion we consider $1 \leq k \leq n - 3$.

In this paper, we investigate the graphs in $G_{n,k}$ and determine the graphs with the largest and smallest $F$-indices among them.

2. Some transformations which increase the $F$-index

Since addition of an edge $e$ to $G$ increases the degrees of each of the end vertices of $e$ by one, and deletion of an edge $e$ from $G$ reduces the degrees of each of the end vertices of $e$ by one, the following proposition is obvious. By $G + e$ we mean the graph obtained by adding an edge $e = uv \notin E(G), u, v \in V(G)$ to $G$. Similarly, by $G - e$ we mean the graph obtained by deleting the edge $e$ from $G$.

**Proposition 2.1.** Let $G = (V,E)$ is a simple connected graph.

(i) If $e = uv \notin E(G), u, v \in V(G)$, then $F(G) < F(G + e)$;

(ii) If $e \in E(G)$, then $F(G) > F(G - e)$.

![Figure 1: Transformation A.](image-url)
Lemma 2.2. Let $G_1uvG_2$ and $G_1uvG_2vK_2$ be two graphs as shown in Figure 1, with $|V(G_1)|, |V(G_2)| \geq 2$, where $G_1, G_2$ have no cut-edges and $uv$ is a non-pendant cut edge of $G_1uvG_2$. Then $F(G_1uvG_2) > F(G_1uvG_2vK_2)$.

Proof. Let $G^* = G_1uvG_2$ and $G^{**} = G_1uvG_2vK_2$. By the definition of $F$-index, we have
\[
F(G^{**}) - F(G^*) = d_{G^*}^{2} - d_{G^*}^{2} - d_{G^*}^{2} - d_{G^*}^{2} = [d_G(v)]^3 + [d_G(u)]^3 - 3d_G(v) - 3d_G(u) + [d_G(v) + d_G(u)] [d_G(v) - d_G(u)] + 1 > 0.
\]

Hence, $F(G_1uvG_2vK_2) > F(G_1uvG_2)$.

Lemma 2.3. Let $G_1uvK_{1,r}$ and $G_1uvK_{1,r+1}$ be two graphs as shown in Figure 2, with $|V(G_1)| \geq 2$, where $G_1$ is a 2-edge connected graph and $uv$ is a non-pendant cut edge of $G_1uvK_{1,r}$. Then $F(G_1uvK_{1,r+1}) > F(G_1uvK_{1,r})$.

Proof. Let $G^* = G_1uvK_{1,r}$ and $G^{**} = G_1uvK_{1,r+1}$. By the definition of $F$-index, we have
\[
F(G^{**}) - F(G^*) = d_{G^*}^{2} - d_{G^*}^{2} - d_{G^*}^{2} = [d_G(v)]^3 + [d_G(u)]^3 - 3d_G(v) - 3d_G(u) + [d_G(v) + d_G(u)] [d_G(v) - d_G(u)] + 1 > 0.
\]

Hence, $F(G_1uvK_{1,r+1}) > F(G_1uvK_{1,r})$.

Lemma 2.4. Let $G$ be a graph and $u,v$ be two vertices of $G$ such that $u_1,u_2,\ldots,u_s$ are the pendant edges adjacent to $u$ and $v_1,v_2,\ldots,v_t$ are pendant edges adjacent to $v$. $G' = G - \{uv_1, uv_2, \ldots, uv_t\} + \{vu_1, vu_2, \ldots, vu_s\}$, $G'' = G - \{vv_1, vv_2, \ldots, vv_t\} + \{uw_1, uw_2, \ldots, uw_t\}$ and $|V(G_0)| \geq 3$, as shown in Figure 3. Then either $F(G') > F(G)$ or $F(G') > F(G)$.

Proof. By the definition of $F$-index we have,
\[
F(G') - F(G) = d_{G'}^{2} - d_{G'}^{2} - d_{G'}^{2} = [d_G(v)]^3 + [d_G(u)]^3 - 3d_G(v) - 3d_G(u) + [d_G(v) + d_G(u)] [d_G(v) - d_G(u)] + 1 > 0.
\]
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$$F(G') - F(G) = d^3_{G'}(v) + d^3_{G'}(u) - d^3_G(v) - d^3_G(u)$$

$$= [(d_G(v) - t)^3 + (d_G(u) + t)^3 - d^3_G(v) - d^3_G(u)]$$

$$= 3t[d_G(u) + d_G(v)][d_G(u) - d_G(v) + t].$$

Since either $d_G(v) \geq d_G(u)$ or $d_G(u) \geq d_G(v)$, we have therefore either $F(G') > F(G)$ or $F(G') > F(G)$. $\square$

**Figure 3**: Transformation C.

**Lemma 2.5.** Let $G \in \mathcal{G}_{n,k}$. Any non-pendant cut edge of $G$ can be transformed to a pendant cut-edge by applying the transformation $A$ and Transformation $B$ repeatedly, so that we obtain a graph $G^* = S_1 u S_2 u \ldots u S_l$ as shown in Figure 4, where each $S_i, 1 \leq i \leq l$ has no non-pendant cut-edges. Clearly, the $F$-index increases at each application of the Transformation $A$ and Transformation $B$.

**Figure 4**: The graph $G^*$.

**Lemma 2.6.** By repeating Transformation $C$, we can attach all the pendant edges at the same vertex to get a graph of the form $H_1, H_2$ or $H_3$ as shown in Figure 5, so that $F$-index increases at each application of Transformation $C$.
3. Upper bound of $F$-index for the graphs in $G_{n,k}$

**Theorem 3.1.** For all connected graphs $G$ in $G_{n,k}$, $F(G) \leq (n-k-1)^4 + (n-k)^3 + k$, and the maximum $F$-index is achieved uniquely at $K^k_n$, where $K^k_n$ is the graph obtained by joining $k$ pendant vertices to one vertex of $K_{n-k}$.

**Proof.** From Lemma 2.6, it follows that we have three candidate graphs $H_1$, $H_2$ and $H_3$ in $G_{n,k}$, as shown in Figure 5, for the upper bound of $F$-index.

We consider three claims as follows.

**Claim 1.** For the graphs $H_1$ and $H_2$, we have $F(H_1) \leq F(H_2)$.

If we add an edge $e \notin E(G)$ to $G$, then by Proposition 2.1 we have $F(G + e) > F(G)$. The sub-graphs $K_{n_i+1}$ ($i = 1, 2, \ldots, l$) are obtained by adding some edges to the 2-edge connected graphs $S_i$ ($i = 1, 2, \ldots, l$). Thus the graph $H_2$ is obtained by adding some edges to the graph $H_1$. Hence $F(H_1) \leq F(H_2)$. Equality holds if and only if $H_1 \cong H_2$.

**Claim 2.** For the graphs $H_2$ and $H_3$, we have $F(H_2) \leq F(H_3)$.

If we add edges between every two vertices of the sub-graphs $K_{n_i+1}$ ($i = 1, 2, \ldots, l$) of $H_2$, $H_2$ will be changed into the graph $H_3$. Thus the graph $H_3$ is obtained by adding some edges to the graph $H_2$. Therefore by Proposition 2.1 we have $F(H_2) \leq F(H_3)$. Equality holds if and only if $H_2 \cong H_3$.

**Claim 3.** For any graph $G \in G_{n,k}$, $F(G) \leq F(H_3)$.

Suppose that $G \in G_{n,k}$ and $G$ is not isomorphic to $H_3$. Then by Lemma 2.6, Claim 1. and Claim 2., we have $F(G) \leq F(H_3)$. Clearly, $F(G) = F(H_3)$, only when $G$ isomorphic to $H_3$.

Combining the above three claims we have, $H_3 \cong K^k_n$ has maximum $F$-index $F(K^k_n) = (n-k-1)^4 + (n-k)^3 + k$, and the theorem follows. □
4. Two transformations which decrease the F-index

**Lemma 4.1.** Let $G$ be a graph and $u, v$ be two vertices of $G$ such that $u_1, u_2, \ldots, u_s$ are the pendant vertices adjacent to $u$ and $v_1, v_2, \ldots, v_t$ are pendant vertices adjacent to $v$. Let $G_3 = G - v_1 + v_2v_1$, as shown in Figure 6. Then $F(G_3) < F(G)$.

![Figure 6: Transformation D.](image)

**Proof.** By the definition of F-index we have,

$$F(G) - F(G_3) = d_G^3(v) + d_G^3(v_2) - d_{G_3}^3(v) - d_{G_3}^3(v_2)$$

$$= d_G^3(v) + 1 - (d_G(v) - 1)^3 - 2^3 = 3d_G^2(v) - 3d_G(v) - 6$$

$$= 3d_G(v)(d_G(v) - 1) - 6 > 0 \quad \text{(since } d_G(v) \geq 3\text{)}.$$  

Hence $F(G) > F(G_3)$. □

**Lemma 4.2.** Let $G$ be a graph and $u, v$ be two vertices of $G$ such that $u_1, u_2, \ldots, u_s$ are the pendant vertices adjacent to $u$ and $v_1, v_2, \ldots, v_t$ are pendant vertices adjacent to $v$. Let $G_4 = G - u_1 + v_1u_1$, as shown in Figure 7. Then $F(G_4) < F(G)$.

![Figure 7: Transformation E.](image)

**Proof.** By the definition of F-index we have,

$$F(G) - F(G_4) = d_G^3(u) + d_G^3(v_1) - d_{G_4}^3(u) - d_{G_4}^3(v_1) = d_G^3(u) + 1 - (d_G(u) - 1)^3 - 2^3$$

$$= 3d_G(u)(d_G(u) - 1) - 6 > 0 \quad \text{(since } d_G(u) \geq 3\text{)}.$$  

Hence $F(G) > F(G_4)$. □
5. Lower bound of $F$-index for the graphs in $G_{n,k}$

**Theorem 5.1.** For the connected graphs $G$ in $G_{n,k}$, $F(G) \geq 4(2n + 3)$, and the minimum $F$-index is obtained uniquely at $C_{n-k}^k$, where $C_{n-k}^k$ is a graph obtained by joining a path of length $k$ to one vertex of $C_{n-k}$.

Proof. From Lemma 4.1 and Lemma 4.2, we have three graphs in $G_{n,k}$, namely $G_5$, $G_6$ and $G_7$ as shown in Figure 8, for the lower bound of $F$-index.

We consider three claims as follows.

**Claim 1.** For the graphs $G_5$ and $G_6$, we have $F(G_5) \geq F(G_6)$.

The graphs $C_{n+1}^i$ $(i = 1, 2, \ldots, l)$ are obtained by deleting some edges from the 2-edge connected graphs $S_i$ $(i = 1, 2, \ldots, l)$. Thus the graph $G_6$ is obtained by deleting some edges from the graph $G_5$. Therefore by Proposition 2.1, we have $F(G_5) \geq F(G_6)$. Equality holds if and only if $G_5 \cong G_6$.

**Claim 2.** For the graphs $G_6$ and $G_7$, we have $F(G_6) \geq F(G_7)$.

By the definition of $F$-index we have,

$$F(G_6) - F(G_7) = \{(n-2)2^3 + (2l+1)^3 + 1^3\} - \{(n-2)2^3 + 3^3 + 1^3\} \geq 0 \text{ (since } l \geq 1)$$

Hence, $F(G_6) \geq F(G_7)$. Equality holds if and only if $G_6 \cong G_7$.

**Claim 3.** For a graph $G \in G_{n,k}$, $F(G) \geq F(G_7)$.

The proof follows directly from Lemma 4.1, Lemma 4.2, Claim 1. and Claim 2. Combining the above three claims, we have $G_7 \cong C_{n-k}^k$ has minimum $F$-index $F(C_{n-k}^k) = (n-2)2^3 + 3^3 + 1^3 = 4(2n + 3)$ and the theorem follows.

\[\square\]

**References**

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