

ALMOST YAMABE SOLITON AND ALMOST RICCI-BOURGUIGNON SOLITON WITH GEODESIC VECTOR FIELDS

Shahroud Azami

Abstract. The aim of this paper is to prove some results about almost Yamabe soliton and almost Ricci-Bourguignon soliton with special soliton vector field. In fact, we prove that every compact non-trivial almost Ricci-Bourguignon soliton with constant scalar curvature is isometric to a Euclidean sphere. Then we show that every compact almost Ricci-Bourguignon soliton whose soliton vector field is divergence-free is Einstein and its soliton vector field is Killing. Finally, we prove that a complete almost Ricci-Bourguignon soliton (M, g, V, λ, ρ) has V as the contact vector field of a contact manifold M with metric g and its Reeb vector field is geodesic, then it becomes a Ricci-Bourguignon soliton and g has constant scalar curvature. In particular, if V is strict, then g is a compact Sasakian Einstein.

1. Introduction

On an n -dimensional smooth manifold M , a Riemannian metric g and a non-vanishing vector field V define an almost Yamabe soliton [8] if there exists a smooth function λ on M such that

$$\mathcal{L}_V g = (\lambda - R)g, \quad (1)$$

respectively, an almost Ricci-Bourguignon soliton if there exists a smooth function λ on M such that

$$2Ric + \mathcal{L}_V g = 2(\lambda + \rho R)g, \quad (2)$$

where \mathcal{L}_V denotes the Lie derivative operator in the direction of the vector field V , Ric denotes the Ricci curvature tensor field of g , R is the scalar curvature of g , and ρ is a real constant. For λ constant, they reduce to a Yamabe soliton and a Ricci-Bourguignon soliton respectively. As in the case of almost Ricci solitons, almost

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Ricci-Bourguignon solitons give rise to a self-similar solution of the Ricci-Bourguignon flow

$$\frac{\partial}{\partial t}g = -2(Ric - \rho Rg), \quad g(0) = g_0.$$

This flow was first considered by Bourguignon [3] and then the short-time existence and uniqueness of the solution of the Ricci-Bourguignon flow on $[0, T)$ was shown by Catino et al. [5] for $\rho < \frac{1}{2(n-1)}$.

An almost Yamabe soliton (M, g, V, λ) (or an almost Ricci-Bourguignon soliton (M, g, V, λ, ρ)) is said to be shrinking, steady or expanding if λ is positive, zero or negative, respectively. If the potential vector field V is of gradient type, $V = \nabla f$, for a smooth function $f : M \rightarrow \mathbb{R}$, then an almost Yamabe soliton (resp. an almost Ricci-Bourguignon soliton) is called a gradient almost Yamabe soliton (or a gradient almost Ricci-Bourguignon soliton). An almost Ricci-Bourguignon soliton (M, g, V, λ, ρ) is called trivial if V is a Killing vector field, i.e. $\mathcal{L}_V g = 0$.

If $\rho = 0$, then the almost Ricci-Bourguignon soliton reduces to the almost Ricci soliton, which was first introduced by Pigola et al. [12], and after that many authors have obtained some properties of the almost Ricci soliton. Some characterization results for compact almost Ricci solitons were obtained in [1, 13, 14]. Gradient Ricci-Bourguignon solitons were studied in detail, for example in [6, 7], where the authors called them gradient ρ -Einstein solitons. Then Dwivedi [9] obtained some results on the almost Ricci-Bourguignon soliton.

We recall an operator \square acting on a smooth vector field V such that $\square V$ is a vector field and in a local coordinate system $\{x^i\}$ it has components $-(g^{jk}\nabla_j\nabla_k V^i + R_j^i V^j)$ where $R_j^i = g^{ik}R_{jk}$ and R_{jk} are the components of the Ricci tensor. From [16] we have the following definition.

DEFINITION 1.1. A vector field V on a Riemannian manifold (M, g) is called a geodesic vector field if $\square V = 0$.

Note that the condition for a geodesic vector field is also equivalent to the condition $g^{jk}\mathcal{L}_V \Gamma_{jk}^i = 0$ and this shows that a Killing vector field and an affine Killing vector field are special examples of a geodesic vector field. Moreover, this definition of a geodesic vector field is different from a vector field whose integral curves are geodesics (see [15, 16]). In this paper, we first give the following rigidity result for almost Yamabe soliton.

PROPOSITION 1.2. *If an almost Yamabe soliton (M, g, V, λ) has a divergence-free soliton vector field V , then V is a Killing vector field.*

Motivated by the almost Ricci soliton case, we will prove the following theorem and generalize results of [1].

THEOREM 1.3. *Let $(M^n, g, V, \lambda, \rho)$ be a compact oriented almost Ricci-Bourguignon soliton. If $d\mu$ denotes the volume form with respect to g , then*

$$\int_M \left| Ric - \frac{R}{n}g \right|^2 d\mu = \frac{n-2}{2n} \int_M g(\nabla R, V) d\mu. \quad (3)$$

Furthermore, if $n > 2$, the almost-Ricci-Bourguignon soliton is non-trivial and the scalar curvature is constant, then (M, g) is isometric to a Euclidean sphere and almost-Ricci-Bourguignon soliton is gradient.

Consequently, we have the following rigidity result for the almost Ricci-Bourguignon soliton.

COROLLARY 1.4. *If a compact Ricci-Bourguignon soliton (M, g, V, λ, ρ) has a divergence-free soliton vector field V , then V is a Killing vector field and g is an Einstein metric.*

REMARK 1.5. Every Killing vector field is divergence-free, but the converse need not be true in general. Therefore, Theorems 1.2 and 1.4 provide a condition under which the converse holds.

We now note that the above result holds for a Ricci-Bourguignon soliton without the compactness condition, as follows.

THEOREM 1.6. *If a Ricci-Bourguignon soliton (M, g, V, λ, ρ) has a divergence-free soliton vector field V , then for $\rho \neq \frac{1}{n}$ the vector field V is Killing and g is Einstein metric.*

We study almost Yamabe solitons and almost Ricci-Bourguignon solitons when the vector field of the soliton is a geodesic vector field. In fact, we show the following.

THEOREM 1.7. *In an almost Yamabe soliton (M, g, V, λ) , V is Killing if and only if V is a geodesic vector field in the sense of the Definition 1.1.*

We also prove that following is true.

THEOREM 1.8. *In an almost Ricci-Bourguignon soliton (M, g, V, λ, ρ) , $\lambda + \rho R$ is constant if and only if V is a geodesic vector field in the sense of the Definition 1.1.*

In the following we give some well-known definitions and basic formulas for the contact geometry of [2]. A $(2m + 1)$ -dimensional smooth manifold M is said to be a contact manifold if it carries a 1-form η on M such that $\eta \wedge (d\eta)^m \neq 0$ on M . For a contact 1-form η there exists a unique vector field ξ (it is called a Reeb vector field) such that $d\eta(\xi, \cdot) = 0$ and $\eta(\xi) = 1$. If we polarize $d\eta$ on the contact subbundle $\eta = 0$, we obtain a Riemannian metric g and a $(1, 1)$ tensor field ϕ such that $d\eta(X, Y) = g(X, \phi Y)$, $\eta(X) = g(\xi, X)$, $\phi^2 = -I + \eta \otimes \xi$, $\phi(\xi) = 0$, for any vector fields X, Y on M . If the vector field ξ is Killing, then the contact manifold is called K -contact manifold and in this case we have $Ric(\xi, X) = 2mg(\xi, X)$ for any vector field X on M . A vector field X on a contact manifold is called a contact vector field if $\mathcal{L}_X \eta = f\eta$ for a smooth function f on M , and it is called strict if $f = 0$. A contact metric g on M is called Sasakian if the almost-Kähler structure induced on the cone $(M \times \mathbb{R}^+, r^2g + dr^2)$ is Kähler.

Finally, we prove the following results for an almost Ricci-Bourguignon soliton on contact manifolds.

THEOREM 1.9. *If a complete $(2m+1)$ -dimensional almost Ricci-Bourguignon soliton (M, g, V, λ, ρ) has V as the contact vector field of a K -contact manifold M with metric g , then it becomes a Ricci-Bourguignon soliton and g has constant scalar curvature. In particular, if V is strict, then g is a compact Sasakian Einstein.*

REMARK 1.10. All Sasakian manifolds are K -contact but the converse need not be true except in dimension three. Therefore, Theorem 1.9 gives a condition under which the converse holds.

COROLLARY 1.11. *If a complete $(2m+1)$ -dimensional almost Ricci-Bourguignon soliton (M, g, V, λ, ρ) has V as contact vector field of a contact manifold M with metric g and its Reeb vector field is geodesic in the sense of the Definition 1.1, then it becomes a Ricci-Bourguignon soliton and g has constant scalar curvature. In particular, if V is strict, then g is a compact Sasakian Einstein.*

2. Proofs of the results

In this section, we prove our results.

Proof (Proposition 1.2). Taking trace of the almost Yamabe soliton equation (1) and the condition $\operatorname{div}V = 0$, we get $R = \lambda$. Using this in (1) shows that V is Killing. \square

Below we give a lemma that will be used throughout the paper. For a proof of the lemma we need the following formula from [15, pp. 23],

$$\mathcal{L}_V \Gamma_{ij}^k = \frac{1}{2} g^{kl} \{ \nabla_i (\mathcal{L}_V g_{jl}) + \nabla_j (\mathcal{L}_V g_{il}) - \nabla_l (\mathcal{L}_V g_{ij}) \}, \quad (4)$$

and
$$\nabla_l (\mathcal{L}_V \Gamma_{ij}^k) - \nabla_j (\mathcal{L}_V \Gamma_{il}^k) = \mathcal{L}_V R_{lji}^k, \quad (5)$$

where Γ_{ij}^k and R_{lji}^k are the Christoffel symbols and the components of the curvature tensor of the metric g in local coordinates, respectively.

LEMMA 2.1. *Let $(M^n, g, V, \lambda, \rho)$ be an almost Ricci-Bourguignon soliton. Then we have*

$$\mathcal{L}_V R = (1 - 2(n-1)\rho)\Delta R - 2\rho R^2 - 2\lambda R + 2|\operatorname{Ric}|^2 - 2(n-1)\Delta\lambda. \quad (6)$$

Proof. If we take the Lie derivative of the relation $g_{lk}g^{kj} = \delta_l^j$ along the vector field V , use the equation (2) and then multiply the equation obtained by g^{il} , we obtain

$$\mathcal{L}_V g^{ij} = 2R^{ij} - 2(\lambda + \rho R)g^{ij}. \quad (7)$$

If you substitute (7) into (4), you obtain

$$\mathcal{L}_V \Gamma_{ij}^k = \nabla^k R_{ij} - \nabla_j R_i^k - \nabla_i R_j^k - \nabla^k (\lambda + \rho R)g_{ij} + \nabla_j (\lambda + \rho R)\delta_i^k + \nabla_i (\lambda + \rho R)\delta_j^k. \quad (8)$$

By replacing (8) with (5) and using the Ricci identity, you get

$$\begin{aligned} \mathcal{L}_V R_{lji}^k &= \nabla_j \nabla_l R_i^k - \nabla_l \nabla_j R_i^k + \nabla_j \nabla_i R_l^k - \nabla_l \nabla_i R_j^k + \nabla_l \nabla^k R_{ij} - \nabla_j \nabla^k R_{il} \\ &+ (\nabla_l \nabla_i)(\lambda + \rho R)\delta_j^k - (\nabla_l \nabla^k)(\lambda + \rho R)g_{ij} - (\nabla_i \nabla_j)(\lambda + \rho R)\delta_l^k + (\nabla_j \nabla^k)(\lambda + \rho R)g_{il}. \end{aligned}$$

Contracting this equation with g^{lk} , you get

$$\begin{aligned} \mathcal{L}_V R_{ij} &= \nabla_j \nabla_i R - \nabla_k \nabla_j R_i^k - \nabla_k \nabla_i R_j^k \\ &+ \Delta R_{ij} - (\Delta(\lambda + \rho R))g_{ij} - (n-2)\nabla_i \nabla_j (\lambda + \rho R). \end{aligned} \quad (9)$$

Taking the Lie derivative of $R = g^{ij} R_{ij}$ along the vector field V and using the equations (7) and (9), we obtain (6). \square

Proof (Theorem 1.3). We can write (6) as

$$g(\nabla R, V) = (1 - 2(n-1)\rho)\Delta R - 2\rho R^2 - 2\lambda R + 2|Ric|^2 - 2(n-1)\Delta\lambda.$$

By integrating both sides of the last equation and applying the divergence theorem, we obtain that

$$\begin{aligned} \frac{1}{2} \int_M g(\nabla R, V) d\mu &= \int_M \left[|Ric|^2 - \rho R^2 - \lambda R \right] d\mu \\ &= \int_M \left[\left| Ric - \frac{R}{n}g \right|^2 - \frac{(n\rho - 1)R^2 + n\lambda R}{n} \right] d\mu. \end{aligned} \quad (10)$$

The contraction of (2) leads to $\text{div}V = n\lambda + (n\rho - 1)R$, then

$$\int_M R \text{div}V d\mu = \int_M (n\lambda R + (n\rho - 1)R^2) d\mu.$$

Since

$$\text{div}(RV) = g(\nabla R, V) + R \text{div}V, \quad (11)$$

we conclude

$$\int_M g(\nabla R, V) d\mu = - \int_M R \text{div}V d\mu. \quad (12)$$

Inserting (12) into (11) results in

$$\int_M g(\nabla R, V) d\mu = - \int_M (n\lambda R + (n\rho - 1)R^2) d\mu. \quad (13)$$

If you insert (13) into (10), you get (3). If the scalar curvature R is constant, then (3) implies that g is Einstein. Consequently, the equation (2) reduces to $\mathcal{L}_V g = 2(\lambda + \rho R - \frac{R}{n})g$. Assuming that λ is not constant, V is a non-homothetic conformal vector field on M . We set $h := \lambda + \rho R - \frac{R}{n}$. We have $\mathcal{L}_V g = 2hg$ and from [15, pp. 26], $\mathcal{L}_V R_{ij} = (n-2)\nabla_i \nabla_j h - (\Delta h)g_{ij}$. If we take the Lie derivative of the relation $R_{ij} = \frac{R}{n}g_{ij}$ along V , we get

$$\left(\Delta h + \frac{2R}{n}h \right) g_{ij} = (2-n)\nabla_i \nabla_j h. \quad (14)$$

If you take the trace of the above equation, you get $\Delta h = -\frac{R}{n-1}h$ and this shows that

$$\Delta h^2 = 2|\nabla h|^2 + 2h\Delta h = 2|\nabla h|^2 - \frac{2R}{n-1}h^2.$$

If you integrate the above equation over the compact M and using the divergence

theorem, you get

$$\int_M |\nabla h|^2 d\mu = \frac{R}{n-1} \int_M h^2 d\mu.$$

This implies that R is positive. From (14) and $\Delta h = -\frac{R}{n-1}h$ we conclude

$$\nabla_i \nabla_j h = -\frac{R}{n(n-1)} h g_{ij}. \quad (15)$$

Obata's Theorem [11] now implies that (M, g) is isomorphic to a Euclidean sphere of radius $\sqrt{\frac{n(n-1)}{R}}$. We can write (15) as $\mathcal{L}_{\nabla h} g = -\frac{2R}{n(1-n)} h g$ or equivalently $\mathcal{L}_{-\frac{R}{n(1-n)} \nabla h} g = 2h g$. Since V is also conformal and satisfies $\mathcal{L}_V g = 2h g$, we can use the Hoge-de-Rham decomposition to conclude that $V = -\frac{R}{n(1-n)} \nabla h + Z$, where Z is a Killing vector field, so V is the gradient of a smooth function and thus the proof of the theorem is complete. \square

Proof (Corollary 1.4). If we replace $\operatorname{div} V = 0$ in equation (3), then $\int_M |Ric - \frac{R}{n}g|^2 d\mu = 0$, which implies that $Ric = \frac{R}{n}g$, i.e. g is Einstein metric. Substituting $Ric = \frac{R}{n}g$ and $(1-n\rho)R = n\lambda$ into (2) gives $\mathcal{L}_V g = 0$, which shows that the vector field V is Killing. \square

Proof (Theorem 1.6). In this case, the equation $(1-n\rho)R = n\lambda$ applies. Since λ is constant, we conclude that R is also constant. Thus, the formulas $(1-n\rho)R = n\lambda$ and (6) imply that $|Ric|^2 = \frac{R^2}{n}$. Substituting this into $|Ric|^2 - \frac{R^2}{n} = |Ric - \frac{R}{n}g|^2$ gives $Ric = \frac{R}{n}g$, i.e. g is Einstein metric. Finally, (2) implies that the vector field V is Killing. \square

Proof (Theorem 1.7). If you take the trace of the equation (1) and then take the covariant derivative of it in an orthonormal frame, you get

$$\nabla_j \nabla_i V^i = n \nabla_j (\lambda - R). \quad (16)$$

In addition, the differentiation of (1) results in

$$\nabla_i \nabla_j V^i + \nabla_i \nabla^i V_j = 2 \nabla_j (\lambda - R). \quad (17)$$

If you subtract equation (16) from (17), you get $R_{kj} V^k + \nabla_i \nabla^i V_j = (2-n) \nabla_j (\lambda - R)$. This shows that $\square V = (2-n) \nabla_j (\lambda - R)$. Therefore, equation (1) implies that V is a geodesic vector field if and only if V is Killing. \square

Proof (Theorem 1.8). First we take the trace of the equation (1) and then we take the covariant derivative of it in an orthonormal frame and get

$$\nabla_j \nabla_i V^i + \nabla_j R = n \nabla_j (\lambda + \rho R). \quad (18)$$

By differentiating (1) and using the twice contracted second Bianchi identity $2 \operatorname{div} Ric = 2 \nabla R$, we also arrive at the conclusion

$$\nabla_i \nabla_j V^i + \nabla_i \nabla^i V_j + \nabla_j R = 2 \nabla_j (\lambda + \rho R). \quad (19)$$

If you subtract equation (18) from (19), you obtain $R_{kj} V^k + \nabla_i \nabla^i V_j = (2-n) \nabla_j (\lambda + \rho R)$. This results in $\square V = (2-n) \nabla_j (\lambda + \rho R)$. Therefore, V is a geodesic vector field if and only if $\lambda + \rho R$ is constant. \square

Proof (Theorem 1.9). According to the definition of the contact manifold, we have $\omega = \eta \wedge (d\eta)^m \neq 0$, then ω is a volume element and

$$\mathcal{L}_V \omega = (\operatorname{div} V) \omega. \quad (20)$$

Equation $\mathcal{L}_V \eta = f\eta$ implies that $\mathcal{L}_V d\eta = d\mathcal{L}_V \eta = df \wedge \eta + f d\eta$. It follows from the equation (20) that $\operatorname{div} V = (m+1)f$.

On the other hand, from (2) we have $\operatorname{div} V = (2m+1)\lambda + ((2m+1)\rho - 1)R$. Then

$$[1 - (2m+1)\rho]R = (2m+1)\lambda - (m+1)f. \quad (21)$$

With the help of the formula $\eta(X) = g(\xi, X)$ we derive

$$(\mathcal{L}_V \eta)(X) = (\mathcal{L}_V g)(\xi, X) + g(\mathcal{L}_V \xi, X), \quad (22)$$

for any vector field X on M . From (2) and $\operatorname{Ric}(\xi, X) = 2mg(\xi, X)$ we get

$$(\mathcal{L}_V g)(\xi, X) = 2(\lambda + \rho R - 2m)g(\xi, X). \quad (23)$$

Inserting (23) and $\mathcal{L}_X \eta = f\eta$ into (22) results in $g(\mathcal{L}_V \xi, X) = f\eta(X) - 2(\lambda + \rho R - 2m)g(\xi, X)$ for any vector field X on M and this shows that

$$\mathcal{L}_V \xi = (f - 2\lambda - 2\rho R + 4m)\xi. \quad (24)$$

The inner product of (24) with ξ yields $g(\mathcal{L}_V \xi, \xi) = (f - 2\lambda - 2\rho R + 4m)$. With the Lie derivative of $g(\xi, \xi) = 1$ along V and using equation (2) and $\operatorname{Ric}(\xi, \xi) = 2m$ we also obtain $g(\mathcal{L}_V \xi, \xi) = 2m - \lambda - \rho R$.

If we compare the two values of $g(\mathcal{L}_V \xi, \xi)$, the result is $f = \lambda + \rho R - 2m$, then

$$\mathcal{L}_V \eta = (\lambda + \rho R - 2m)\eta, \quad \mathcal{L}_V \xi = (2m - \lambda - \rho R)\xi. \quad (25)$$

Let Q denote the Ricci operator defined by $g(QX, Y) = \operatorname{Ric}(X, Y)$ for arbitrary vector fields X, Y on M . If we take the Lie derivative of $d\eta(X, Y) = g(X, \phi Y)$ along the vector field V and use equation (2) and $\mathcal{L}_V \eta = f\eta$, we obtain

$$\eta(Y)\nabla f - (Yf)\xi + 2(f - 2\lambda - 2\rho R)\phi Y = -4Q(\phi Y) + 2(\mathcal{L}_X \phi)Y. \quad (26)$$

Substituting ξ for Y in (26) results in

$$\nabla f - (\xi f)\xi = 2(\mathcal{L}_X \phi)\xi. \quad (27)$$

If we now take the Lie derivative of $\phi(\xi) = 0$ along the vector field V again and use the second equation of (25), we obtain $(\mathcal{L}_X \phi)\xi = 0$ and insert it into (27), we obtain $\nabla f = (\xi f)\xi$, i.e. $df = (\xi f)\eta$. If you take the exterior derivative of this and then take the wedge product with η , you get $(\xi f)\eta \wedge d\eta = 0$. Since $\eta \wedge d\eta$ is nonzero everywhere then $\xi f = 0$, i.e. $df = 0$ and this shows that f is constant on M . Therefore, (26) reduces to $\mathcal{L}_X \phi = 2Q\phi - (2m + \lambda + \rho R)\phi$. Since $f = \lambda + \rho R - 2m$ and f is constant, we conclude that $\lambda + \rho R$ is constant and (21) shows that R is also constant. Thus λ is constant and the almost Ricci-Bourguignon soliton is simply a Ricci-Bourguignon soliton. This completes the proof of the first part of the theorem.

To prove the second part, we choose $f = 0$, then $\lambda + \rho R = 2m$ and thus we have from (21) that $R = 2m(2m+1)$. Since λ, R are constant, (6) implies that $|\operatorname{Ric} - 2mg|^2 = 0$, i.e. $\operatorname{Ric} = 2mg$. This shows that g is an Einstein metric. Since (M, g) is complete, (M, g) is compact according to Myers' theorem. From [4, 10] every compact K -contact manifold with an Einstein constant greater than -2 is Sasakian. This completes the proof of the theorem. \square

Proof (Corollary 1.11). Since the Reeb vector field is geodesic, we have from [13, Theorem 3] that the metric contact manifold is a K -contact manifold. Therefore, the Theorem 1.9 completes the proof. \square

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Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran

E-mail: azami@sci.ikiu.ac.ir