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GEODESIC VECTORS ON 5-DIMENSIONAL HOMOGENEOUS NILMANIFOLDS

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Abstract. In this paper, firstly we study geodesic vectors for the m-th root homogeneous Finsler space admitting (α, β)-type. Then we obtain the necessary and sufficient condition for an arbitrary non-zero vector to be a geodesic vector for the m-th root homogeneous Finsler metric under mild conditions. Finally, we consider a quartic homogeneous Finsler metric on a simply connected nilmanifold of dimension five equipped with an invariant Riemannian metric and an invariant vector field. We study its geodesic vectors and classify the set of all the homogeneous geodesics on 5-dimensional nilmanifolds.

1. Introduction

Nilpotent Lie groups play an important role in geometric analysis, mathematical physics and harmonic analysis [10]. Among all nilpotent Lie groups, one of the most important and interesting Lie groups that have attracted special attention is the five-dimensional nilmanifold. In Finsler geometry, a connected Finsler manifold containing a transitive nilpotent Lie group of isometries is called a nilmanifold. Wilson [22] proved that a Lie group G contains a unique nilpotent Lie subgroup which acts transitively on a homogeneous nilmanifold and which is also normal in G. These results were then applied by Lauret [15] for the construction of non-abelian homogeneous nilmanifolds of dimensions three and four, up to isometry. As an example, he studied the structure of two-step nilmanifolds of dimension five with a 2-dimensional center. Later, Homolya and Kowalski [11] extended these results and studied the structure of all simply connected 5-dimensional two-step nilpotent Lie groups with left-invariant Riemannian metrics up to isometry. Figula and Nagy [9] have classified the isometry equivalence classes and nilmanifolds up to isometry of an arbitrary dimension and studied the 5-dimensional nilmanifolds of nilpotency greater than two. The geometry

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of two-step nilpotent Lie groups equipped with an invariant Finsler metric was studied by Tóth and Kovács [21]. In particular, they calculated the flag curvature and the Chen-Rund connection form and found the relative geodesic equations for some special groups. Moghaddam [16] has studied the existence of left-invariant Randers metrics of Berwald type on the 2-step nilpotent Lie groups of dimension five and computed the flag curvature of these metrics. Nasehi [17] has extended these results for the Douglas-type Randers metric. Bahmandoust and Latifi [3] constructed the set of all homogeneous geodesics of exponential metrics on 5-dimensional two-step nilpotent Lie groups. In [12], Kaur et. al. derived the formula for the flag curvature of a homogeneous Finsler space with a generalized Kropina metric.

Shimada developed the theory of the m-th root metric [20], which plays an important role in physics, general relativity, gravitation, and the theory of seismic ray [1, 2, 18, 19]. For a smooth manifold M with tangent bundle TM, a Finsler metric F defined by $F = \sqrt[m]{a_{i_1 i_2 \dots i_m} y^{i_1} y^{i_2} \dots y^{i_m}}$ is called the m-th root Finsler metric. The third root metric $F = \sqrt[3]{a_{i_1 i_2 i_3} y^{i_1} y^{i_2} y^{i_3}}$ and the fourth root metric $F = \sqrt[4]{a_{i_1 i_2 i_3 i_4} y^{i_1} y^{i_2} y^{i_3} y^{i_4}}$ are called cubic and quartic metrics, respectively. The general m-th root metric $F = \sqrt[m]{y^{i_1} y^{i_2} \dots y^{i_m}}$ is called the Berwald-Moór metric [4,5]. Kim and Park [13] introduced the notion of the m-th root Finsler metric, which admits the (α, β)-type. Ebrahimi et. al. [8] computed the flag curvature formula for a homogeneous cubic Finsler metric and investigated the necessary and sufficient conditions for a non-zero vector to be a geodesic vector.

This paper is organized as follows.

Section 2 contains some basic information about Finsler spaces, homogeneous Finlser spaces and nilpotent Lie groups. In Section 3, we study the necessary and sufficient condition for a nonzero vector of a homogeneous Finsler space with the m-th root metric to be a geodesic vector. In Section 4, we study the set of all geodesics on the two-step nilpotent Lie group of dimension five for the quartic Finsler metric. Finally, in Section 5, we study the set of all geodesic vectors for the above metrics on 5-dimensional nilpotent Lie groups with nilpotency greater than two.

2. Preliminaries

Definition 2.1 ([6]). Let M be an n-dimensional manifold and let TM be its tangent bundle. A Finsler metric on M is a non-negative function $F : TM \longrightarrow \mathbb{R}^+$ satisfying the following properties:

(i) F is smooth on slit tangent bundle $TM \setminus \{0\}$.

(ii) $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0, x \in M$ and $y \in T_x M$.

(iii) The Hessian metric

$$g_{ij} = \frac{1}{2} \left[\frac{\partial^2 F^2}{\partial y^i \partial y^j} \right]$$

is positive definite for every element of $TM \setminus \{0\}$.

Alternatively, the bilinear symmetric form g_y of $T_xM \times T_xM$ to the real line is defined by

$$g_y(u,v) = \frac{1}{2}\left[\frac{\partial^2}{\partial s \partial t} F^2(y+su+tv)\right]\Bigg|_{s=t=0}$$

is positive definite.

Definition 2.2 ([6]). The geodesic spray G on a smooth manifold M of dimension n is a vector field defined on the slit tangent bundle by

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where
$$G^i := \frac{1}{4} g^{ip}\left\{\frac{\partial^2 F^2}{\partial x^j \partial y^p} y^j - \frac{\partial F^2}{\partial x^p}\right\}, \quad i = 1, 2, \ldots, n.$$

Definition 2.3 ([6]). Let G and $c(t)$ be the geodesic spray and the non-constant curve on a smooth manifold M respectively. If the coordinate $c^i(t)$ of $c(t)$ satisfies the condition $\ddot{c}^i(t) + 2G^i\big(c(t), \dot{c}(t)\big) = 0$, then G^i and $c(t)$ be the geodesic coefficients and the geodesic of the spray on M.

Theorem 2.4 ([14]). *Let G be a connected Lie group with a Lie algebra $\mathfrak{m}$ and a left-invariant Finsler metric g_y, then $y \in \mathfrak{m}$ is a geodesic vector if and only if $g_y(y, [y, z]) = 0$, for all $z \in \mathfrak{m}$.*

Definition 2.5 ([21]). A Finsler metric F on a connected Lie group G is called a left-invariant Finsler metric if $F(x, y) = F(e, (L_{x^{-1}})_* y) \ \ \forall\, x \in G, y \in T_xG$, where L_x is a left translation on G.

Lemma 2.6 ([13]). *Let F be an m-th root Finsler metric on a smooth manifold M. If F is a homogeneous function of degree one of non-degenerate form $\alpha = \sqrt{a_{ij}y^i y^j}$ and a one-form $\beta = b_i(x)y^i$, then it can be written as*

$$F = \sqrt[m]{\sum_{s=0}^{t} c_{m-2s}\alpha^{2s}\beta^{m-2s}}, \quad t \le \frac{m}{2}, \tag{1}$$

where c_{m-2s} are real constants.

Definition 2.7 ([7]). A homogeneous Finsler space (M, F) is a Finsler space in which the group of isometries $I(M, F)$ acts transitively on the manifold M.

Definition 2.8 ([17]). A Finsler manifold N is called a homogeneous nilmanifold if the isometry group $I(N)$ of N contains a nilpotent Lie subgroup acting transitively on N.

Definition 2.9 ([16]). A two-step nilpotent Lie group equipped with a left-invariant Riemannian metric is called a 2-step homogeneous nilmanifold.

3. Geodesics of the m-th root homogeneous Finsler metric

In this section we consider the m-th root Finsler metrics admitting (α, β)-type on a homogeneous manifold $M = G/H$ equipped with an invariant Riemannian metric g and an invariant vector field $\tilde{x}$. Then we prove that the vector $x = \tilde{x}(H)$ is a geodesic vector of g if and only if it is a geodesic vector of F. Finally, we show that under certain conditions, a nonzero vector y is a geodesic vector of F if and only if y is a geodesic vector of g.

Let F be the m-th root homogeneous Finsler metric admitting (α, β)-types induced by the Riemannian metric g and an invariant vector field x defined by (1), which can be written in the following form

$$F = \left[\sum_{s=0}^{t} c_{m-2s} g^s(y,y) g^{m-2s}(x,y) \right]^{\frac{1}{m}}, \quad t \leq \frac{m}{2}.$$

For $p, q \in \mathbb{R}$ we know that $g_y(u,v) = \frac{1}{2} \frac{\partial^2}{\partial p \partial q} F^2(y + pu + qv)\Big|_{p=q=0}$. After some calculations we therefore get

$$\begin{aligned} g_y(u,v) = {} & \frac{(2-m)}{m^2} F^{2-2m} \left[\sum_{s=0}^{t} c_{m-2s} g^{s-1}(y,y) g^{m-2s-1}(x,y) \Big(2s g(y,v) g(x,y) \right. \\ & \left. + (m-2s) g(y,y) g(x,v) \Big) \right] \left[\sum_{s=0}^{t} c_{m-2s} g^{s-1}(y,y) g^{m-2s-1}(x,y) \Big(2s g(y,u) g(x,y) \right. \\ & \left. + (m-2s) g(y,y) g(x,u) \Big) \right] + \frac{1}{m} F^{2-m} \left[\sum_{s=0}^{t} c_{m-2s} g^{s-2}(y,y) g^{m-2s-2}(x,y) \right. \\ & \Big(4s(s-1) g(y,u) g(y,v) g^2(x,y) + 2s g(y,y) g(u,v) g^2(x,y) \\ & + 2s(m-2s) g(y,y) g(x,u) g(y,v) g(x,y) + 2s(m-2s) g(y,y) g(y,u) g(x,v) g(x,y) \\ & \left. + (m-2s)(m-2s-1) g(x,u) g(x,v) g^2(y,y) \Big) \right]. \end{aligned} \tag{2}$$

THEOREM 3.1. *Let $(G/H, F)$ be an m-th root homogeneous Finsler space with a Lie algebra $\mathfrak{m}$ equipped with an invariant Riemannian metric g and an invariant vector field $\tilde{x}$ such that $x = \tilde{x}(H)$. Then x is a geodesic vector of $(G/H, F)$ if and only if x is a geodesic vector of $(G/H, g)$.*

Proof. Using the formula (2), for all $z \in \mathfrak{m}$, we obtain

$$\begin{aligned} g_x(x, [x,z]_{\mathfrak{m}}) = {} & \frac{(2-m)}{m^2} F^{2-2m} \left[\sum_{s=0}^{t} c_{m-2s} g^{m-s-2}(x,x) \Big(m g(x,x) g(x, [x,z]_{\mathfrak{m}}) \Big) \right] \\ & \left[\sum_{s=0}^{t} c_{m-2s} g^{m-s-2}(x,x) \Big(m g^2(x,x) \Big) \right] + \frac{1}{m} F^{2-m} \left[\sum_{s=0}^{t} c_{m-2s} g^{m-s-4}(x,x) \right. \end{aligned}$$

$$\Big(4s(s{-}1)g^3(x,x)g(x,[x,z]_{\mathfrak{m}}){+}2sg^3(x,x)g(x,[x,z]_{\mathfrak{m}})$$
$$+2s(m{-}2s)g^3(x,x)g(x,[x,z]_{\mathfrak{m}}){+}2s(m{-}2s)g^3(x,x)g(x,[x,z]_{\mathfrak{m}})$$
$$+(m{-}2s)(m{-}2s{-}1)g^3(x,x)g(x,[x,z]_{\mathfrak{m}})\Big)\Bigg] = \frac{F^2}{g(x,x)}g(x,[x,z]_{\mathfrak{m}}).$$

For $g(x,x) \neq 0$ we therefore have $g_x(x,[x,z]_{\mathfrak{m}}) = 0$ if and only if $g(x,[x,z]_{\mathfrak{m}}) = 0$. This completes the proof. □

Theorem 3.2. *Let $(G/H, F)$ be a homogeneous Finsler space with m-th root of the Finsler metric F arising from an invariant Riemannian metric g and an invariant vector field $\tilde{x}$ such that $x = \tilde{x}(H)$. Let y be a non-zero vector of Lie algebra $\mathfrak{m}$ such that $\sum_{s=0}^{t} 2sc_{m-2s}g^{s-1}(y,y)g^{m-2s}(x,y) \neq 0$ and for all $z \in \mathfrak{m}$, $g(x,[y,z]_{\mathfrak{m}}) = 0$. Then y is a geodesic vector of (M,g) if and only if it is a geodesic vector of (M,F).*

Proof. From (2) we can write

$$g_{y_{\mathfrak{m}}}(y_{\mathfrak{m}},[y,z]_{\mathfrak{m}}) =$$
$$\Big(2sg(x,y_{\mathfrak{m}})g(y_{\mathfrak{m}},[y,z]_{\mathfrak{m}})$$
$$+(m{-}2s)g(y_{\mathfrak{m}},y_{\mathfrak{m}})g(x,[y,z]_{\mathfrak{m}})\Big)\Bigg]\Bigg[\sum_{s=0}^{t} c_{m-2s}g^{s-1}(y_{\mathfrak{m}},y_{\mathfrak{m}})g^{m-2s-1}(x,y_{\mathfrak{m}})$$
$$\Big(mg(y_{\mathfrak{m}},y_{\mathfrak{m}})g(x,y_{\mathfrak{m}})\Big)\Bigg]{+}\frac{1}{m}F^{2-m}\Bigg[\sum_{s=0}^{t} c_{m-2s}g^{s-2}(y_{\mathfrak{m}},y_{\mathfrak{m}})g^{m-2s-2}(x,y_{\mathfrak{m}})$$
$$\Big(2s(m{-}1)g(y_{\mathfrak{m}},y_{\mathfrak{m}})g^2(x,y_{\mathfrak{m}})g(y_{\mathfrak{m}},[y,z]_{\mathfrak{m}}){+}(m{-}2s)(m{-}1)g(x,y_{\mathfrak{m}})$$
$$g^2(y_{\mathfrak{m}},y_{\mathfrak{m}})g(x,[y,z]_{\mathfrak{m}})\Big)\Bigg]. \tag{3}$$

Since $g(x,[y,z]_{\mathfrak{m}}) = 0$ for every $z \in \mathfrak{m}$, (3) can be written in the form

$$g_{y_{\mathfrak{m}}}(y_{\mathfrak{m}},[y,z]_{\mathfrak{m}}) = \frac{(2{-}m)}{m^2}F^{2-2m}\Bigg[\sum_{s=0}^{t} 2sc_{m-2s}g^{s-1}(y_{\mathfrak{m}},y_{\mathfrak{m}})g^{m-2s}(x,y_{\mathfrak{m}})g(y_{\mathfrak{m}},[y,z]_{\mathfrak{m}})\Bigg]$$
$$\Bigg[\sum_{s=0}^{t} mc_{m-2s}g^{s}(y_{\mathfrak{m}},y_{\mathfrak{m}})g^{m-2s}(x,y_{\mathfrak{m}})\Bigg]$$
$$+\frac{1}{m}F^{2-m}\Bigg[\sum_{s=0}^{t} 2s(m{-}1)c_{m-2s}g^{s-1}(y_{\mathfrak{m}},y_{\mathfrak{m}})g^{m-2s}(x,y_{\mathfrak{m}})g(y_{\mathfrak{m}},[y,z]_{\mathfrak{m}})\Bigg].$$

After simplification we get

$$g_{y_{\mathfrak{m}}}(y_{\mathfrak{m}},[y,z]_{\mathfrak{m}}) = \frac{F^{2-m}}{m}\Bigg[\sum_{s=0}^{t} 2sc_{m-2s}g^{s-1}(y_{\mathfrak{m}},y_{\mathfrak{m}})g^{m-2s}(x,y_{\mathfrak{m}})\Bigg]g(y_{\mathfrak{m}},[y,z]_{\mathfrak{m}}).$$

Hence, y is a geodesic vector of (M,g) if and only if it is a geodesic vector of (M,F). □

4. Two-step nilpotent Lie groups of dimension five

In this section we consider a quartic Finsler metric admitting (α, β)-type induced by an invariant Riemannian metric and an invariant vector on a two-step nilmanifold of dimension five. Then we study geodesic vectors and examine the set of all geodesics on the two-step nilpotent Lie group of dimension five for the metric mentioned above.

4.1 Lie algebra with 1-dimensional center

Let $\mathfrak{p}$ be the Lie algebra of a two-step nilmanifold of dimension five with 1-dimensional center $\mathfrak{z}$ equipped with a left-invariant quartic homogeneous Finsler metric. Suppose that $\mathfrak{p}$ is equipped with an inner product $\langle , \rangle$ and N is a Lie group corresponding to the Lie algebra $\mathfrak{p}$. Let E_5 be a unit vector in the center $\mathfrak{z}$ and let $\mathfrak{b}$ be the orthogonal complement of the center $\mathfrak{z}$ in $\mathfrak{p}$. Homolya and Kowalski [11] have proved that there exists an orthonormal basis $\{E_1, E_2, E_3, E_4, E_5\}$ of $\mathfrak{p}$ with

$$[E_1, E_2] = \gamma E_5, \quad [E_3, E_4] = \delta E_5, \tag{4}$$

where $\gamma \geq \delta > 0$.
Let F be a left-invariant homogeneous quartic Finsler metric admitting (α, β)-type on the simply connected 2-step nilmanifold N of dimension five induced by the Riemannian metric g and the vector field $x = \sum_{i=1}^{5} x_i E_i$, defined by

$$F(y) = \sqrt[4]{c_1 g^2(y, y) + c_2 g(y, y) g^2(x, y) + c_3 g^4(x, y)},$$

then we have

$$\begin{aligned}
&g_y(u, v) = \\
&\frac{-1}{2F^6}\Bigg[\Big(2c_1 g(y, y)g(y, v)+c_2 g^2(x, y)g(y, v)+c_2 g(x, y)g(y, y)g(x, v)+2c_3 g^3(x, y)g(x, v)\Big) \\
&\Big(2c_1 g(y, y)g(y, u)+c_2 g^2(x, y)g(y, u)+c_2 g(x, y)g(y, y)g(x, u)+2c_3 g^3(x, y)g(x, u)\Big)\Bigg] \\
&+\frac{1}{2F^2}\Bigg[4c_1 g(y, u)g(y, v)+2c_1 g(y, y)g(u, v)+c_2 g^2(x, y)g(u, v)+2c_2 g(x, y)g(y, v)g(x, u) \\
&+2c_2 g(x, y)g(y, u)g(x, v)+c_2 g(y, y)g(x, u)g(x, v)+6c_3 g^2(x, y)g(x, u)g(x, v)\Bigg].
\end{aligned} \tag{5}$$

For any $z \in \mathfrak{p}$, using (5), we get

$$\begin{aligned}
g_y(y, [y, z]) &= \frac{1}{2F^2}\big[2c_1 g(y, y)+c_2 g^2(x, y)\big] g(y, [y, z]) \\
&+\frac{1}{2F^2}\big[c_2 g(x, y)g(y, y)+2c_3 g^3(x, y)\big] g(x, [y, z]) \\
&= \frac{c_2 g(x, y)g(y, y)+2c_3 g^3(x, y)}{2F^2}\left[g\left(x+\Big(\frac{2c_1 g(y, y)+c_2 g^2(x, y)}{c_2 g(x, y)g(y, y)+2c_3 g^3(x, y)}\Big)y, [y, z]\right)\right].
\end{aligned} \tag{6}$$

From Theorem 2.4, equation (6) and $z = E_j$, for $j = 1, 2, 3, 4, 5$ a vector $y = \sum_{i=1}^{5} y_i E_i$ of Lie algebra $\mathfrak{p}$ is a geodesic vector if and only if

$$g\Big(\sum_{i=1}^{5} x_i E_i + A \sum_{i=1}^{5} y_i E_i, \Big[\sum_{i=1}^{5} y_i E_i, E_j\Big]\Big) = 0, \tag{7}$$

where $A = \left(\dfrac{2c_1 \sum_{i=1}^{5} y_i^2 + c_2(\sum_{i=1}^{5} x_i y_i)^2}{\sum_{i=1}^{5} x_i y_i \big(c_1 \sum_{i=1}^{5} y_i^2 + 2c_3(\sum_{i=1}^{5} x_i y_i)^2\big)} \right)$.

With (4) and (7) we obtain that y is a geodesic vector if and only if

$$\gamma y_1\big(x_5+Ay_5\big) = 0,\ \gamma y_2\big(x_5+Ay_5\big) = 0,\ \delta y_3\big(x_5+Ay_5\big) = 0,\ \delta y_4\big(x_5+Ay_5\big) = 0. \tag{8}$$

Theorem 4.1. *Let F be a homogeneous Finsler metric on a two-step nilpotent Lie group of dimension five with a one-dimensional center, induced by an invariant Riemannian metric g and a left-invariant vector field $x = \sum_{i=1}^{5} x_i E_i$. Then $y \in \mathfrak{p}$ is a geodesic vector if and only if it satisfies* (8).

Corollary 4.2. *Let F be a homogeneous quartic Finsler metric induced by an invariant Riemannian metric g and a left invariant vector field $x = \sum_{i=1}^{5} x_i E_i$ on a two-step nilpotent Lie group of dimension five with a one-dimensional center. Then its geodesic vectors depend only on x_5.*

Corollary 4.3. *Let F be a quartic Finsler metric defined by a Riemannian metric g and a left-invariant vector field $x = \sum_{i=1}^{4} x_i E_i$ on a simply connected two-step nilmanifold of dimension five with a one-dimensional center. Then $y \in \mathfrak{p}$ is a geodesic vector if and only if $y \in span\{E_1, E_2, E_3, E_4\}$ or $y = \lambda E_5,\ \lambda \neq 0$.*

Theorem 4.4. *Let N be a Lie group with its Lie algebra $\mathfrak{p}$ and a quartic Finsler metric F. Let $x = \sum_{i=1}^{4} x_i E_i$ be a left-invariant vector field on the simply connected two-step nilmanifold of dimension five with a one-dimensional center. Then y is a geodesic vector of (N, g) if and only if it is a geodesic vector of (N, F).*

Proof. Let y be a geodesic vector of (N, g). Then

$$g(y, [y, E_i]) = 0 \quad \forall i = 1, 2, 3, 4, 5. \tag{9}$$

Using (4) we get

$$g(x, [y, E_i]) = 0. \tag{10}$$

So if we use (9) and (10) in (6), we get $g_y(y, [y, z]) = 0$.

Conversely, let y be a geodesic vector of (N, F). Then $g_y(y, [y, E_i]) = 0$ and from (4) we get $g(x, [y, E_i]) = 0$. Therefore, from (7), y is a geodesic vector of (N, g). □

4.2 Lie algebra with 2-dimensional center

Let N be a Lie group corresponding to the Lie algebra $\mathfrak{p}$ of the two-step nilmanifold of dimension five with 2-dimensional center $\mathfrak{z}$, equipped with the left-invariant quartic Finsler metric and a Riemannian metric g. In [11] Homolya and Kowalski proved that there exists an orthonormal basis $\{E_1, E_2, E_3, E_4, E_5\}$ of $\mathfrak{p}$ with

$$[E_1, E_2] = \gamma E_4, \qquad [E_1, E_3] = \delta E_5, \tag{11}$$

where $\{E_4, E_5\}$ is a basis for the center of $\mathfrak{p}$ and $\gamma \geq \delta > 0$.

Using (6), Theorem 2.4, a vector $y = \sum_{i=1}^{5} y_i E_i$ of $\mathfrak{p}$ is a geodesic vector if and only if

$$g\Big(\sum_{i=1}^{5} x_i E_i + A \sum_{i=1}^{5} y_i E_i, \Big[\sum_{i=1}^{5} y_i E_i, E_j\Big]\Big) = 0, \tag{12}$$

with $j = 1, 2, 3, 4, 5$.

If we substitute (11) into (12), we obtain that y is a geodesic vector if and only if

$$\gamma y_1\big(x_4 + Ay_4\big) = 0,\ \delta y_1\big(x_5 + Ay_5\big) = 0,\ \gamma y_2\big(x_4 + Ay_4\big) + \delta y_3\big(x_5 + Ay_5\big) = 0. \tag{13}$$

Theorem 4.5. *Let F be a homogeneous Finsler metric on a two-step nilpotent Lie group of dimension five with 2-dimensional center, induced by an invariant Riemannian metric g and a left-invariant vector field $x = \sum_{i=1}^{5} x_i E_i$. Then $y \in \mathfrak{p}$ is a geodesic vector if and only if it satisfies* (13).

Corollary 4.6. *Let F be a quartic Finsler metric with a Riemannian metric g and let $x = \sum_{i=1}^{5} x_i E_i$ be a left-invariant vector field on a simply connected two-step nilpotent Lie group of dimension five with 2-dimensional center. Then its geodesic vectors depend only on γ, δ, x_4 and x_5.*

Corollary 4.7. *Let $\mathfrak{p}$ be a Lie algebra and let F be a quartic metric defined by a left-invariant vector field x and an invariant Riemannian metric g on the simply connected two-step nilmanifold of dimension five with a center of dimension two. Then x is a geodesic vector of (N, F) if and only if x is a geodesic vector of (N, g).*

Theorem 4.8. *Suppose F is a homogeneous quartic Finsler metric induced by the left-invariant vector $x = \sum_{i=1}^{3} x_i E_i$ and by the invariant Riemannian metric g on the two-step nilmanifold of dimension five with a 2-dimensional center. Then $y \in \mathfrak{p}$ is a geodesic vector of (N, g) if and only if y is a geodesic vector of (N, F).*

Proof. From (11), we have $g(x, [y, E_i]) = 0$. Therefore, (6) can be rewritten as

$$g_y(y, [y, z]) = \frac{2c_1 g(y, y) + c_2 g^2(x, y)}{2F^2} g(y, [y, z]) \quad \forall\ i = 1, 2, 3, 4, 5.$$

Then $g(y, [y, z]) = 0$ if and only if $g_y(y, [y, z]) = 0$. This completes the proof. □

4.3 Lie algebra with 3-dimensional center

Let $\mathfrak{p}$ be a Lie algebra on a simply connected two-step nilmanifold of dimension five with three-dimensional center $\mathfrak{z}$. Suppose that N is a Lie group corresponding to the Lie algebra $\mathfrak{p}$ with a left-invariant quartic homogeneous Finsler metric. Homolya and Kowalski [11] showed that there exists an orthonormal basis $\{E_1, E_2, E_3, E_4, E_5\}$ of $\mathfrak{p}$ with

$$[E_1, E_2] = \gamma E_3, \tag{14}$$

where $\{E_3, E_4, E_5\}$ is a basis for the center of $\mathfrak{p}$ and $\gamma > 0$.

Then using (6) and Theorem 2.4, we get that a vector $y = \sum_{i=1}^{5} y_i E_i$ of $\mathfrak{p}$ is a geodesic vector if and only if

$$g\big(\sum_{i=1}^{5} x_i E_i + A \sum_{i=1}^{5} y_i E_i, \Big[\sum_{i=1}^{5} y_i E_i, E_j\Big]\big) = 0, \tag{15}$$

where $j = 1, 2, 3, 4, 5$.

From (14) and (15), we get the following conditions for y to be a geodesic vector:

$$\gamma y_1\big(x_3 + Ay_3\big) = 0, \quad \gamma y_2\big(x_3 + Ay_3\big) = 0. \tag{16}$$

Theorem 4.9. *Let F be a homogeneous Finsler metric on a two-step nilpotent Lie group of dimension five with a three-dimensional center induced by an invariant Riemannian metric g and a left-invariant vector field $x = \sum_{i=1}^{5} x_i E_i$. Then $y \in \mathfrak{p}$ is a geodesic vector if and only if it satisfies* (16).

Corollary 4.10. *Let (N, F) be a Finsler space with a quartic metric F induced by an invariant Riemannian metric g and a left-invariant vector field x on a simply connected two-step nilpotent Lie group N of dimension five with a three-dimensional center. Then its geodesic vectors depend only on x_3.*

Corollary 4.11. *Let F be a quartic Finsler metric with an invariant Riemannian metric g and a left-invariant vector field $x = x_1E_1 + x_2E_2 + x_4E_4 + x_5E_5$ on a simply connected two-step nilpotent Lie group of dimension five with the three-dimensional center. Then a vector $y \in \mathfrak{p}$ is a geodesic vector if and only if $y \in span\{E_1, E_2, E_4, E_5\}$ or $y \in span\{E_3, E_4, E_5\}$.*

Theorem 4.12. *Suppose F is a quartic Finlser metric on a simply connected two-step nilmanifold of dimension five with the three-dimensional center defined by a Riemannian metric g and the left-invariant vector field $x = x_1E_1 + x_2E_2 + x_4E_4 + x_5E_5$. Then a vector $y \in \mathfrak{p}$ is a geodesic vector of (N, g) if and only if it is a geodesic vector of (N, F).*

Proof. Using (14), we can write (6) as follows

$$g_y(y, [y, z]) = \frac{2c_1 g(y, y) + c_2 g^2(x, y)}{2F^2} g(y, [y, z]).$$

Therefore, y is a geodesic vector of (N, g) if and only if y is a geodesic vector of (N, F). □

5. 5-dimensional nilpotent Lie groups of nilpotency class greater than two

In this section we consider the quartic Finsler metrics on three-step and four-step nilmanifolds of dimension five. Then we study the set of all geodesic vectors for the above metrics on these nilpotent Lie groups. In [9] Figula and Nagy give a classification of 5-dimensional nilpotent Lie groups of nilpotency classes greater than two, up to isometry.

5.1 The three-step nilpotent Lie algebra $\mathfrak{l}_{5,5}$

Let N be a three-step nilpotent Lie group of dimension five with Lie algebra $\mathfrak{l}_{5,5}$ equipped with a left-invariant quartic metric F defined by a Riemannian metric g and a vector field $x = \sum_{i=1}^{n} x_i E_i$. Suppose that the set $\{E_1, E_2, E_3, E_4, E_5\}$ is an orthonormal basis of $\mathfrak{l}_{5,5}$ such that

$$[E_1, E_2] = \gamma E_4 + \delta E_5,\ [E_1, E_3] = \lambda E_5,\ [E_1, E_4] = \zeta E_5,\ [E_2, E_3] = \eta E_5, \tag{17}$$

where $\gamma, \zeta, \eta > 0$ and $\delta, \lambda \geq 0$.

Therefore, using Theorem 2.4 and (6), a non-zero vector $y = \sum_{i=1}^{5} y_i E_i$ of $\mathfrak{l}_{5,5}$ is a geodesic vector if and only if

$$g\Big(\sum_{i=1}^{5} x_i E_i + A\sum_{i=1}^{5} y_i E_i, \Big[\sum_{i=1}^{5} y_i E_i, E_j\Big]\Big) = 0 \quad \forall\, j = 1, 2, 3, 4, 5. \tag{18}$$

Using (17) and (18), we get that the geodesic vector y satisfies the following equations:

$$\begin{gathered}
\gamma y_2\big(x_4 + Ay_4\big) + (\delta y_2 + \lambda y_3 + \zeta y_4)\big(x_5 + Ay_5\big) = 0,\\
\gamma y_1\big(x_4 + Ay_4\big) + (\delta y_1 + \eta y_3)\big(x_5 + Ay_5\big) = 0,\\
(\lambda y_1 + \eta y_2)\big(x_5 + Ay_5\big) = 0,\\
\zeta y_1\big(x_5 + Ay_5\big) = 0.
\end{gathered} \tag{19}$$

THEOREM 5.1. *Let F be a homogeneous Finsler metric on a three-step nilpotent Lie group of dimension five with its Lie algebra $\mathfrak{l}_{5,5}$ induced by an invariant Riemannian metric g and a left-invariant vector field $x = \sum_{i=1}^{5} x_i E_i$. Then $y \in \mathfrak{p}$ is a geodesic vector if and only if it satisfies* (19)*.*

COROLLARY 5.2. *Let (N, F) be a Finsler space with a quartic metric F induced by an invariant Riemannian metric g and a left-invariant vector field x on a simply connected three-step nilpotent Lie group N of dimension five with Lie algebra $\mathfrak{l}_{5,5}$. Then its geodesic vectors depend only on $\gamma, \delta, \lambda, \zeta, \eta, x_4$ and x_5.*

COROLLARY 5.3. *If (N, F) is a Finsler space with a quartic metric F on a simply connected three-step nilpotent Lie group N of dimension five with Lie algebra $\mathfrak{l}_{5,5}$ defined by an invariant Riemannian metric g and an invariant vector field x, then x is a geodesic vector of (N, g) if and only if it is a geodesic vector of (N, F).*

THEOREM 5.4. *Let F be a quartic Finsler metric induced by an invariant Riemannian metric g and a left-invariant vector field $x = \sum_{i=1}^{3} x_i E_i$ on a three-step nilpotent Lie group of dimension five with the Lie algebra $\mathfrak{l}_{5,5}$. Then y is a geodesic vector of (N, F) if and only if it is a geodesic vector of (N, g).*

Proof. Using (17) and (6), we have

$$g_y(y, [y, z]) = \frac{2c_1 g(y, y) + c_2 g^2(x, y)}{2F^2} g(y, [y, z]).$$

Therefore, $g(y, [y, z]) = 0$ iff $g_y(y, [y, z]) = 0$. This completes the proof. $\square$

5.2 The three-step nilpotent Lie algebra $\mathfrak{l}_{5,9}$

Let $\mathfrak{l}_{5,9}$ be a Lie algebra on a three-step simply connected nilmanifold of dimension five equipped with the Lie group N, a left-invariant vector field x and a left-invariant quartic Finsler metric F such that the orthonormal basis $\{E_1, E_2, E_3, E_4, E_5\}$ satisfies

$$[E_1, E_2] = \gamma E_3 + \delta E_4 + \lambda E_5, \quad [E_1, E_3] = \zeta E_4, \quad [E_2, E_3] = \eta E_5, \tag{20}$$

where either $\gamma > 0, \eta > \zeta > 0$ and $\delta, \lambda \geq 0$ or $\gamma, \zeta > 0, \delta \geq 0, \lambda = 0$ and $\eta = \zeta$.

Now, using (6), a non-zero vector $y = \sum_{i=1}^{5} y_i E_i$ of $\mathfrak{l}_{5,9}$ is geodesic vector if and only if

$$g\Big(\sum_{i=1}^{5} x_i E_i + A \sum_{i=1}^{5} y_i E_i, \Big[\sum_{i=1}^{5} y_i E_i, E_j\Big]\Big) = 0 \quad \forall\, j = 1, 2, 3, 4, 5. \tag{21}$$

From (20) and (21), we get that y is a geodesic vector if and only if

$$\begin{gathered} \gamma y_2\big(x_3 + Ay_3\big) + (\delta y_2 + \zeta y_3)\big(x_4 + Ay_4\big) + \lambda y_2\big(x_5 + Ay_5\big) = 0, \\ \gamma y_1\big(x_3 + Ay_3\big) + \delta y_1\big(x_4 + Ay_4\big) + (\lambda y_1 + \eta y_3)\big(x_5 + Ay_5\big) = 0, \\ (\zeta y_1 + \eta y_2)\big(x_4 + Ay_4\big) + \eta y_2\big(x_5 + Ay_5\big) = 0. \end{gathered} \tag{22}$$

Theorem 5.5. *Let F be a homogeneous Finsler metric on a three-step nilpotent Lie group of dimension five with its Lie algebra $\mathfrak{l}_{5,9}$ induced by an invariant Riemannian metric g and a left-invariant vector field $x = \sum_{i=1}^{5} x_i E_i$. Then $y \in \mathfrak{p}$ is a geodesic vector if and only if it satisfies* (22).

Corollary 5.6. *Let N be a Lie group equipped with a quartic Finsler metric F and its Lie algebra $\mathfrak{l}_{5,9}$ on a simply connected three-step nilmanifold of dimension five. Then its geodesic vectors depends on $\gamma, \delta, \zeta, \lambda, \eta, x_3, x_4$ and x_5.*

Theorem 5.7. *Suppose N is a three-step nilpotent Lie group of dimension five with its Lie algebra $\mathfrak{l}_{5,9}$ and let F be a left-invariant quartic Finsler metric induced by a Riemannian metric g and a left-invariant vector field $x = \sum_{i=1}^{2} x_i E_i$. Then $y \in \mathfrak{l}_{5,9}$ is a geodesic vector of (N, g) if and only if y is a geodesic vector of (N, F).*

Proof. From (20), we have $g(x, [y, E_i]) = 0$. Therefore, (6) can be rewritten as

$$g_y(y, [y, z]) = \frac{2c_1 g(y, y) + c_2 g^2(x, y)}{2F^2} g(y, [y, z]).$$

This completes the proof. $\square$

5.3 The four-step nilpotent Lie algebra $\mathfrak{l}_{5,7}$

Let N be a four-step nilmanifold of dimension five with Lie algebra $\mathfrak{l}_{5,7}$ equipped with a left-invariant quartic Finsler metric F and a left-invariant vector field x. Suppose that the set $\{E_1, E_2, E_3, E_4, E_5\}$ is an orthonormal basis of $\mathfrak{l}_{5,7}$ with

$$[E_1, E_2] = \gamma E_3 + \delta E_4 + \lambda E_5, \quad [E_1, E_3] = \zeta E_4 + \eta E_5, \quad [E_1, E_4] = \vartheta E_5, \tag{23}$$

where $\gamma, \zeta, \vartheta > 0$ and either $\delta > 0$ or $\delta = 0$, $\eta \geq 0$.

Therefore, a non-zero vector $y=\sum_{i=1}^{5} y_iE_i$ of $\mathfrak{l}_{5,7}$ is a geodesic vector if and only if

$$g\Big(\sum_{i=1}^{5} x_iE_i + A\sum_{i=1}^{5} y_iE_i, \Big[\sum_{i=1}^{5} y_iE_i, E_j\Big]\Big) = 0 \quad \forall\ j=1,2,3,4,5. \tag{24}$$

From (23) and (24), we get that the geodesic vector y satisfies the following equations:

$$\begin{aligned}
\gamma y_2\big(x_3+Ay_3\big)+(\delta y_2+\zeta y_3)\big(x_4+Ay_4\big)+(\lambda y_2+\eta y_3+\vartheta y_4)\big(x_5+Ay_5\big)&=0,\\
\gamma y_1\big(x_3+Ay_3\big)+\delta y_1\big(x_4+Ay_4\big)+\lambda y_1\big(x_5+Ay_5\big)&=0,\\
\zeta y_1\big(x_4+Ay_4\big)+\eta y_1\big(x_5+Ay_5\big)&=0,\\
\vartheta y_1\big(x_5+Ay_5\big)&=0.
\end{aligned}\tag{25}$$

Theorem 5.8. *Let F be a homogeneous Finsler metric on a four-step nilpotent Lie group of dimension five with its Lie algebra $\mathfrak{l}_{5,7}$ induced by an invariant Riemannian metric g and a left-invariant vector field $x=\sum_{i=1}^{5} x_iE_i$. Then $y\in\mathfrak{p}$ is a geodesic vector if and only if it satisfies* (25).

Corollary 5.9. *Let N be a four-step nilpotent Lie group of dimension five with its Lie algebra $\mathfrak{l}_{5,7}$ and a left-invariant quartic Finsler metric F induced by an invariant Riemannian metric g and a left-invariant vector field x. Then its geodesic vectors depend only on $\gamma,\delta,\lambda,\zeta,\eta,\vartheta,x_3,x_4$ and x_5.*

Corollary 5.10. *If F is a quartic metric on a four-step nilmanifold of dimension five with its Lie algebra $\mathfrak{l}_{5,7}$ induced by an invariant Riemannian metric g and an invariant vector field x, then x is a geodesic vector of (N,F) if and only if x is a geodesic vector of (N,g).*

Theorem 5.11. *Suppose F is a quartic Finsler metric defined by an invariant Riemannian metric g and $x=\sum_{i=1}^{2} x_iE_i$ is a left-invariant vector field on the four-step nilpotent Lie group of dimension five with the Lie algebra $\mathfrak{l}_{5,7}$. Then $y\in\mathfrak{l}_{5,7}$ is a geodesic vector of (N,g) if and only if it is a geodesic vector of (N,F).*

Proof. Using (23), we can write (6) as follows

$$g_y(y,[y,z])=\frac{2c_1g(y,y)+c_2g^2(x,y)}{2F^2}g(y,[y,z]).$$

Therefore, y is a geodesic vector of (N,g) iff y is a geodesic vector of (N,F). □

5.4 The four-step nilpotent Lie algebra $\mathfrak{l}_{5,6}$

Let N be a four-step nilmanifold of dimension five equipped with Lie algebra $\mathfrak{l}_{5,6}$ and a left-invariant quartic homogeneous Finsler metric F such that the orthonormal basis $\{E_1,E_2,E_3,E_4,E_5\}$ satisfies the following condition

$$[E_1,E_2]=\gamma E_3+\delta E_4+\lambda E_5,\quad [E_1,E_3]=\zeta E_4+\eta E_5,\quad [E_1,E_4]=\vartheta E_5,\quad [E_2,E_3]=\mu E_5, \tag{26}$$

where $\gamma,\zeta,\vartheta,\mu>0$ and either $\delta>0$ or $\delta=0,\lambda\geq 0$.

From (6), a non-zero vector $y=\sum_{i=1}^{5} y_iE_i$ of $\mathfrak{l}_{5,6}$ is a geodesic vector if and only if

$$g\Big(\sum_{i=1}^{5} x_iE_i + A\sum_{i=1}^{5} y_iE_i, \Big[\sum_{i=1}^{5} y_iE_i, E_j\Big]\Big) = 0 \quad \forall\ j = 1,2,3,4,5. \tag{27}$$

From (26) and (27), we get that the geodesic vector y satisfies the following equations:

$$\begin{gathered}
\gamma y_2\big(x_3+Ay_3\big)+(\delta y_2+\zeta y_3)\big(x_4+Ay_4\big)+(\lambda y_2+\eta y_3+\vartheta y_4)\big(x_5+Ay_5\big)=0,\\
\gamma y_1\big(x_3+Ay_3\big)+\delta y_1\big(x_4+Ay_4\big)+(\lambda y_1+\mu y_3)\big(x_5+Ay_5\big)=0,\\
\zeta y_1\big(x_4+Ay_4\big)+(\eta y_1+\mu y_2)\big(x_5+Ay_5\big)=0,\\
\vartheta y_1\big(x_5+Ay_5\big)=0.
\end{gathered}\tag{28}$$

Theorem 5.12. *Let F be a homogeneous Finsler metric on a four-step nilpotent Lie group of dimension five with its Lie algebra $\mathfrak{l}_{5,6}$ induced by an invariant Riemannian metric g and a left-invariant vector field $x=\sum_{i=1}^{5} x_iE_i$. Then $y\in\mathfrak{p}$ is a geodesic vector if and only if it satisfies* (28).

Corollary 5.13. *Let (N,F) be a four-step nilpotent Lie group of dimension five with its Lie algebra $\mathfrak{l}_{5,6}$ and a left-invariant quartic metric F induced by a left-invariant Riemannian metric g and a left-invariant vector field x. Then its geodesic vectors only depends on $\gamma,\delta,\vartheta,\lambda,\zeta,\eta,\mu,x_3,x_4$ and x_5.*

Theorem 5.14. *Let N be a four-step nilpotent Lie group of dimension five equipped with a quartic Finsler metric F and a Lie algebra $\mathfrak{l}_{5,6}$ induced by an invariant Riemannian metric g and a left-invariant vector field $x=\sum_{i=1}^{2} x_iE_i$. Then y is a geodesic vector of (N,g) if and only if it is a geodesic vector of (N,F).*

Proof. Using (26) and (6), we get

$$g_y(y,[y,z]) = \frac{2c_1g(y,y)+c_2g(x,y)^2}{2F^2}g(y,[y,z]).$$

Therefore, y is a geodesic vector of (N,g) iff y is a geodesic vector of (N,F). □

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