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Corrected proof
Available online 30.10.2025

research paper
оригинални научни рад
DOI: 10.57016/MV-ICMJ2670

$\mathcal{K}(2)$-SUPERSYMMETRIES OF MODULES OF DIFFERENTIAL OPERATORS

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Abstract. Let $\mathfrak{F}_\lambda^2$ be the space of tensor densities of degree $\lambda \in \mathbb{C}$ on the supercircle $S^{1|2}$. We consider the space $\mathfrak{D}_{\lambda,\mu}^{2,k}$ of k-th order linear differential operators from $\mathfrak{F}_\lambda^2$ to $\mathfrak{F}_\mu^2$ as a module over the superalgebra $\mathcal{K}(2)$ of contact vector fields on $S^{1|2}$ and we compute the superalgebra $\mathcal{K}_{\lambda,\mu}^{2,k}$ of endomorphisms on $\mathfrak{D}_{\lambda,\mu}^{2,k}$ commuting with the $\mathcal{K}(2)$-action. We prove that this algebra is trivial except for $\lambda = 0$.

1. Introduction

Let M be an n-dimensional manifold and $\mathrm{Vect}(M)$ the Lie algebra of vector fields on M. For every $\lambda \in \mathbb{C}$, we consider the space $\mathcal{F}_\lambda(M)$ of tensor densities of degree λ on M (i.e., the space of sections of the line bundle $\Delta_\lambda(M) = |\Lambda^n T^* M|^{\otimes\lambda}$ over M). Clearly, $\mathcal{F}_0(M) \cong C^\infty(M)$ as a $\mathrm{Vect}(M)$-module.

Denote $\mathcal{D}_{\lambda,\mu}(M) := \mathrm{Homdiff}\big(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M)\big)$ as the space of linear differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}\mu(M)$. This space is an associative (and therefore a Lie) algebra with a filtration by the order of differentiation:

$$\mathcal{D}_{\lambda,\mu}^0(M) \subset \mathcal{D}_{\lambda,\mu}^1(M) \cdots \subset \mathcal{D}_{\lambda,\mu}^k(M) \subset \cdots .$$

The study of these two-parameter $\mathrm{Vect}(M)$-module families, namely the classification of these modules, has been the subject of several works. Let us cite, for example, [1–3, 5, 8, 10].

Obviously, the classification of $\mathrm{Vect}(M)$-modules $\mathcal{D}_{\lambda,\mu}(M)$ is obtained through the study of the existence of isomorphisms between modules $\mathcal{D}_{\lambda,\mu}^k(M)$, i.e., linear bijective maps that are invariant under the $\mathrm{Vect}(M)$-action on these spaces. More generally, one can consider linear operators acting on differential operators (not necessarily bijective) that commute with the $\mathrm{Vect}(M)$-action (or specifically with the action of a given subalgebra of $\mathrm{Vect}(M)$), i.e., linear maps $T : \mathcal{D}_{\lambda,\mu}^k(M) \to \mathcal{D}_{\lambda,\mu}^k(M)$ satisfying, for all X in $\mathrm{Vect}(M)$,

$$[\mathcal{L}_X^{\lambda,\mu}, T] := \mathcal{L}_X^{\lambda,\mu} \circ T - T \circ \mathcal{L}_X^{\lambda,\mu} = 0,$$

2020 Mathematics Subject Classification: 53D55
Keywords and phrases: Contact structure; differential operators; densities.

where $\mathcal{L}_X^{\lambda,\mu}$ stands for the action of the vector field X on the space $\mathcal{D}^k_{\lambda,\mu}(M)$. Such an operator T is called a *symmetry* of the module $\mathcal{D}^k_{\lambda,\mu}(M)$.

The most important example of symmetries in the case where $M = \mathbb{R}$ (or S^1) is the conjugation of differential operators. This map associates to an operator A its adjoint operator $A^\star$. If $A \in \mathcal{D}^k_{\lambda,\mu}(\mathbb{R})$, then $A^\star \in \mathcal{D}^k_{1-\mu,1-\lambda}(\mathbb{R})$, so this map defines a symmetry if and only if $\lambda + \mu = 1$.

In [7], the algebra of symmetries of the module $\mathcal{D}^k_{\lambda,\mu}(S^1)$ was investigated; a complete description and classification for all integer k were provided in this paper.

In [4,11,12], we were interested in the study of analogous superstructures. Namely, we considered the superspace $\mathfrak{D}\lambda,\mu$ of linear differential operators $A : \mathfrak{F}_\lambda \to \mathfrak{F}_\mu$, where $\mathfrak{F}_\lambda$ and $\mathfrak{F}_\mu$ are the spaces of tensor densities on the supercircle $S^{1|1}$ of degree λ and μ, respectively.

Naturally, the Lie superalgebra $\mathrm{Vect}\mathbb{C}(S^{1|1})$ acts on $\mathfrak{D}_{\lambda,\mu}$. However, in [11], we restricted ourselves to the orthosymplectic superalgebra $\mathfrak{osp}(1|2)$, which can be realized as a subalgebra of $\mathrm{Vect}\mathbb{C}(S^{1|1})$, and we studied what we called the algebra of *orthosymplectic supersymmetries* of the module $\mathfrak{D}^k_{\lambda,\mu}$ – that is, the algebra of endomorphisms of $\mathfrak{D}^k_{\lambda,\mu}$ that commute with the $\mathfrak{osp}(1|2)$-action.

In [12], we studied the more interesting setting: the algebra $\mathfrak{C}^k_{\lambda,\mu}$ of *contact supersymmetries*. We considered the space $\mathfrak{D}_{\lambda,\mu}$ as a module over the superalgebra $\mathcal{K}(1)$ of contact vector fields on $S^{1|1}$. In this context, we computed the space $\mathfrak{C}^k_{\lambda,\mu}$ of linear maps on $\mathfrak{D}^k_{\lambda,\mu}$ commuting with the $\mathcal{K}(1)$-action. We established several results similar to the S^1 case.

A slightly more interesting result, unlike the case of orthosymplectic supersymmetries, is the stability of the dimension of $\mathfrak{C}^k_{\lambda,\mu}$ for $k \geq 3$. This result is due to the fact that any contact supersymmetry is completely determined by its restriction to the subspace of second-order operators. The particular values $k = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ are investigated. For all (λ, μ), a complete description of the algebra $\mathfrak{C}^k_{\lambda,\mu}$ for these values of k is provided.

In [4], we considered the general case of the supercircle $S^{1|n}$, $n \in \mathbb{N}^*$. Naturally, the Lie superalgebra $\mathrm{Vect}\mathbb{C}(S^{1|n})$ of vector fields on $S^{1|n}$ – and in particular, the Lie subalgebra $\mathcal{K}(n)$ of contact vector fields on $S^{1|n}$ – acts on the superspace $\mathfrak{D}^{k,n}\lambda,\mu$ of linear differential operators of order at most k from $\mathfrak{F}^n_\lambda$ to $\mathfrak{F}^n_\mu$, where $\mathfrak{F}^n_\lambda$ and $\mathfrak{F}^n_\mu$ are the spaces of tensor densities on the supercircle $S^{1|n}$ of degree λ and μ, respectively.

Evidently, computations in this general case are quite involved and complex. Our main result in this paper was the characterization of $\mathfrak{aff}(n|1)$-supersymmetries – that is, the endomorphisms of $\mathfrak{D}^{n,k}_{\lambda,\mu}$ that commute with the $\mathfrak{aff}(n|1)$-action, where $\mathfrak{aff}(n|1)$ is the affine subalgebra of $\mathrm{Vect}\mathbb{C}(S^{1|n})$, which can be realized as a subalgebra of $\mathcal{K}(n)$.

In the present paper, we focus our study on the case of the supercircle $S^{1|2}$. We consider the superspace $\mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$ of linear differential operators of order at most k from $\mathfrak{F}^2_\lambda$ to $\mathfrak{F}^2_\mu$, where $\mathfrak{F}^2_\lambda$ and $\mathfrak{F}^2_\mu$ are the spaces of tensor densities on the supercircle $S^{1|2}$ of degree λ and μ, respectively.

Naturally, the Lie superalgebra $\mathrm{Vect}_{\mathbb{C}}(S^{1|2})$ of vector fields on $S^{1|2}$, and its Lie subalgebra $\mathcal{K}(2)$ of contact vector fields on $S^{1|2}$, act on $\mathfrak{D}^{2,k}_{\lambda,\mu}$. Using the results of [4], we are able in this work to compute the algebras $\mathcal{K}^{2,k}_{\lambda,\mu}$, $k \in \frac{1}{2}\mathbb{N}^*$, of $\mathcal{K}(2)$-supersymmetries – i.e., the algebra of linear operators $T : \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2}) \to \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$ commuting with the $\mathcal{K}(2)$-action on $\mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$. We prove that the algebra $\mathcal{K}^{2,k}_{\lambda,\mu}$ is trivial except when $\lambda = 0$.

2. Basic definitions and tools

In this section, we recall the main definitions and facts related to the geometry of the supercircle $S^{1|2}$. For more details see [5, 6, 9].

2.1 $\mathcal{K}(2)$-supersymmetries

We consider the supercircle $S^{1|2}$ with local coordinates (x, θ_1, θ_2), where x is the even variable and θ_1, θ_2 are the odd variables, i.e., $\theta_1\theta_2 = -\theta_2\theta_1$. The superalgebra $C^\infty_{\mathbb{C}}(S^{1|2})$ of smooth functions on $S^{1|2}$ consists of elements of the form

$$F = f_0 + \sum_{s=1}^{2} \sum_{1 \le i_1 < i_2 \le 2} f_{i_1 i_2}(x)\theta_{i_1}\theta_{i_2},$$

where $f_0, f_{i_1 i_2} \in C^\infty_{\mathbb{C}}(S^1)$.

Let us introduce the standard contact structure given by the following 1-form:

$$\alpha_2 = dx + \sum_{i=1}^{2} \theta_i d\theta_i.$$

On the space $C^\infty_{\mathbb{C}}(S^{1|2})$, we consider the contact bracket

$$\{F, G\} = FG' - F'G - \frac{1}{2}(-1)^{|F|} \sum_{i=1}^{2} \overline{D}_i(F) \cdot \overline{D}_i(G),$$

where $\overline{D}_i = \frac{\partial}{\partial\theta_i} - \theta_i \frac{\partial}{\partial x}$ and $|F|$ is the parity of F.

Note that the derivations $\overline{D}_i$ are the generators of 2-extended supersymmetry and generate the kernel of the form α_2 as a module over the ring of functions.

Let $\mathrm{Vect}(S^{1|2})$ be the superspace of vector fields on $S^{1|2}$:

$$\mathrm{Vect}(S^{1|2}) = \left\{ F_0\partial_x + \sum_{i=1}^{2} F_i\partial_i \mid F_i \in C^\infty(S^{1|2}) \right\},$$

where $\partial_i = \frac{\partial}{\partial\theta_i}$ and $\partial_x = \frac{\partial}{\partial x}$.

We consider the superspace $\mathcal{K}(2)$ of contact vector fields on $C^\infty(S^{1|2})$. That is, $\mathcal{K}(2)$ is the superspace of vector fields on $S^{1|2}$ preserving the distribution defined by the 1-form α_2:

$$\mathcal{K}(2) = \left\{ X \in \mathrm{Vect}^\infty_{\mathbb{C}}(S^{1|2}) \mid \text{there exists } F \in C^\infty(S^{1|2}) \text{ such that } \mathfrak{L}_X(\alpha_2) = F\alpha_2 \right\},$$

where $\mathfrak{L}_X$ denotes the Lie derivative along the vector field X.

The Lie superalgebra $\mathcal{K}(2)$ is spanned by the fields of the form:

$$X_F = F\partial_x - \frac{1}{2}\sum_{i=1}^{2}(-1)^{|F|}\overline{D}_i(F)\overline{D}_i, \quad \text{where } F \in C^\infty(S^{1|2}). \tag{1}$$

The function F is said to be the contact Hamiltonian of the field X_F. The bracket in $\mathcal{K}(2)$ can be written as: $[X_F, X_G] = X_{\{F,G\}}$.

Note that in the The Lie superalgebra $\mathcal{K}(2)$ contains two important subalgebra, the *orthosymlectic* Lie superalgebra $\mathfrak{osp}(2|2) \subset \mathcal{K}(2)$ generated by

$$\mathfrak{osp}(2|2) = \mathrm{Span}(X_1, X_x, X_{x^2}, X_{\theta_1}, X_{\theta_2}, X_{\theta_1\theta_2}, X_{x\theta_1}, X_{x\theta_2})$$

and the *affine* superalgebra $\mathfrak{aff}(2|1)$ which is subalgebra of $\mathfrak{osp}(2|2)$, and then of $\mathcal{K}(2)$ spanned by

$$\mathfrak{aff}(2|1) = \mathrm{Span}(X_1, X_x, X_{\theta_1}, X_{\theta_2}, X_{\theta_1\theta_2}).$$

2.2 Weighted densities on $S^{1|2}$

For any $\lambda \in \mathbb{C}$, we define the space of λ-densities on $S^{1|2}$ as $\mathfrak{F}^2_\lambda = \left\{F\alpha_2^\lambda \mid F \in C^\infty(S^{1|2})\right\}$. As a vector space, $\mathfrak{F}^2_\lambda$ is isomorphic to $C^\infty_{\mathbb{C}}(S^{1|2})$. The Lie derivative of the density $G\alpha_2^\lambda$ along the vector field $X_F \in \mathcal{K}(2)$ is given by the rule

$$\mathfrak{L}^\lambda_{X_F}(G\alpha_2^\lambda) = \mathfrak{L}^\lambda_{X_F}(G)\alpha_2^\lambda, \quad \text{with} \quad \mathfrak{L}^\lambda_{X_F} = X_F + \lambda F'.$$

The space $\mathfrak{F}^2_\lambda$ is thus a module over the contact Lie superalgebra $\mathcal{K}(2)$. Obviously, we can easily see that:

1) The adjoint $\mathcal{K}(2)$-module is isomorphic to $\mathfrak{F}^2_{-1}$.

2) As a $\mathcal{K}(1)$-module, $\mathfrak{F}^2_\lambda = \mathfrak{F}^1_\lambda \oplus \prod(\mathfrak{F}^1_{\lambda+\frac{1}{2}})$.

2.3 Differential operators on weighted densities

We denote by $\mathfrak{D}^2_{\lambda,\mu}$ the space of differential operators from $\mathfrak{F}^2_\lambda$ to $\mathfrak{F}\mu^2$ for any $\lambda, \mu \in \mathbb{C}$. We can express any element $A \in \mathfrak{D}^2_{\lambda,\mu}$ in terms of the vector fields $\overline{D}_i = \partial_i - \theta_i\partial_x$, $i = 1, 2$. Indeed, since $\overline{D}_i^2 = -\partial_x$ and $\partial_i = \overline{D}i - \theta_i\overline{D}i^2$ for all $i = 1, 2$, we can write the operator A as a finite sum

$$A = \sum \ell = (\ell_1, \ell_2) b_\ell \overline{D}_1^{\ell_1}\overline{D}2^{\ell_2}, \tag{2}$$

where the coefficients $b\ell$ are smooth functions on $S^{1|2}$ and $\ell \in \mathbb{N}^2$. That is, for all $F = f\alpha_2^\lambda \in \mathfrak{F}^2_\lambda$,

$$A(F) = \sum_{\ell=(\ell_1,\ell_2)} b_\ell(x,\theta)\overline{D}_1^{\ell_1}\overline{D}_2^{\ell_2}(f)\alpha_2^\mu.$$

For $k \in \frac{1}{2}\mathbb{N}$, we denote by $\mathfrak{D}^{2,k}_{\lambda,\mu}$ the subspace of $\mathfrak{D}^2_{\lambda,\mu}$ consisting of differential operators of the form

$$A = \sum_{\substack{\ell=(\ell_1,\ell_2)\in\mathbb{N}\times\mathbb{N} \\ |\ell|=\ell_1+\ell_2\le 2k}} b_\ell \overline{D}_1^{\ell_1}\overline{D}_2^{\ell_2}. \tag{3}$$

The superspace $\mathfrak{D}^{2,k}_{\lambda,\mu}$ is then a $\mathcal{K}(2)$-module for the natural action:

$$\mathfrak{L}^{\lambda,\mu}_{X_F}(A) = \mathfrak{L}^{\mu}_{X_F} \circ A - (-1)^{|A||F|} A \circ \mathfrak{L}^{\lambda}_{X_F}, \quad X_F \in \mathcal{K}(2).$$

Thus, clearly, we have the filtration:

$$\mathfrak{D}^{2,0}_{\lambda,\mu} \subset \mathfrak{D}^{2,\frac{1}{2}}_{\lambda,\mu} \subset \mathfrak{D}^{2,1}_{\lambda,\mu} \subset \mathfrak{D}^{2,\frac{3}{2}}_{\lambda,\mu} \subset \cdots \subset \mathfrak{D}^{2,\ell-\frac{1}{2}}_{\lambda,\mu} \subset \mathfrak{D}^{2,\ell}_{\lambda,\mu} \cdots$$

Note that we can write the operator A in (3) uniquely in the form

$$A = \sum_{m=0}^{2k} \sum_{\substack{(s,\epsilon_1,\epsilon_2)\in\mathbb{N}\times\{0,1\}^2 \\ 2s+\epsilon_1+\epsilon_2=m}} a^m_{s,\epsilon_1,\epsilon_2} \partial^s_x, \overline{D}_1^{\epsilon_1} \overline{D} 2^{\epsilon_2},$$

where the coefficients $a^m s, \epsilon$ are smooth functions on $S^{1|2}$. We will adopt this notation in the sequel.

2.4 $\mathcal{K}(2)$-supersymmetries

A linear operator

$$T : \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2}) \to \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2}) \tag{4}$$

is said to be local if it preserves the supports of its arguments: $\mathrm{Supp}\Big(T(A)\Big) \subset \mathrm{Supp}(A)$, for all $A \in \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$. That is, for every open subset $U \subset S^{1|2}$, we have

$$A_{|U} = 0 \Rightarrow T(A)_{|U} = 0; \forall A \in \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2}).$$

The map T is non-local if there exists some $A \in \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$ vanishing on an open subset $U \subset S^{1|2}$ such that $T(A)$ does not vanish on U.

In [7], it has been proven that, in the S^1-case, the large class of symmetries of the modules $\mathfrak{D}^{k}_{\lambda,\mu}(S^1)$ – that is, linear operators $T : \mathfrak{D}^{k}_{\lambda,\mu}(S^1) \to \mathfrak{D}^{k}_{\lambda,\mu}(S^1)$ commuting with the $\mathrm{Vect}(S^1)$ action on $\mathfrak{D}^{k}_{\lambda,\mu}(S^1)$ is given by local operators. Indeed, the only non-local linear operator T commuting with the $\mathrm{Vect}(S^1)$-action might exist when $(\lambda,\mu) = (0,1)$ and is given by

$$T\Big(\sum_{\ell=0}^{k} (\frac{d}{dx})^{\ell}\Big) = \Big(\int_{S^1} a_0(x)\Big) \circ d,$$

where d is the de Rham differential.

Thus, we focus our study in this work on the large class of linear local operators $T : \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2}) \to \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$ commuting with the $\mathcal{K}(2)$-action on $\mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$, which we call $\mathcal{K}(2)$-supersymmetries of the modules $\mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$. The superalgebra of such operators will be denoted by $\mathcal{K}^{2,k}_{\lambda,\mu}$.

Thanks to the renowned Peetre theorem in classical differential geometry, if T is a local operator as in (4), then for all $(\ell,\epsilon_1,\epsilon_2) \in \mathbb{N}\times\{0,1\}^2$ and for all $(s,\epsilon'_1,\epsilon'_2) \in \mathbb{N}\times\{0,1\}^2$, the operator that associates to the function $a \in C^\infty(S^{1|2})$ the component with respect to $\partial^s \overline{D}_1^{\epsilon'_1} \overline{D}_2^{\epsilon'_2}$ of $T\big(a\partial^l\, \overline{D}_1^{\epsilon_1} \overline{D}_2^{\epsilon_2}\big)$ is a local operator acting between superfunctions. Hence, the coefficients of the operator $T\big(a\partial^l\, \overline{D}_1^{\epsilon_1} \overline{D}_2^{\epsilon_2}\big)$ appear as derivations of the function a.

That is, for all $0 \leq m \leq 2k$ and for all $(\ell, \epsilon_1, \epsilon_2) \in \mathbb{N} \times \{0, 1, \}^2$ such that $2\ell + \epsilon_1 + \epsilon_2 = m$, there exist an integer $\mathcal{M}_m$ and some functions $T_{s_1,\epsilon_1',\epsilon_2'}^{s_2,\epsilon_1'',\epsilon_2''} \in \mathbb{C}^\infty(S^{1|2})$ such that

$$T\Big(\sum_{\substack{(s,\epsilon_1,\epsilon_2)\in\mathbb{N}\times\{0,1\}^2\\ 2s+\epsilon_1+\epsilon_2=m}} a^m_{s,\epsilon_1,\epsilon_2}\partial_x^s\,\overline{D}_1^{\epsilon_1}\overline{D}_2^{\epsilon_2}\Big) = \tag{5}$$

$$\sum_{\substack{(s,\epsilon_1,\epsilon_2)\in\mathbb{N}\times\{0,1\}^2\\ 2s+\epsilon_1+\epsilon_2=m}} \sum_{\substack{(s_1,\epsilon_1',\epsilon_2')\in\mathbb{N}\times\{0,1\}^2\\ 2s_1+\epsilon_1'+\epsilon_2'\leq m}} \sum_{\substack{(s_2,\epsilon_1'',\epsilon_2'')\in\mathbb{N}\times\{0,1\}^2\\ 2s_2+\epsilon_1''+\epsilon_2''\leq M_m}} T^{\epsilon_1,\epsilon_2,\epsilon_1',\epsilon_2',\epsilon_1'',\epsilon_2''}_{s,s_1,s_2}\partial_x^{s_2}\overline{D}_1^{\epsilon_1''}\overline{D}_2^{\epsilon_2''}(a^m_{s,\epsilon_1,\epsilon_2})\partial_x^{s_1}\overline{D}_1^{\epsilon_1'}\overline{D}_2^{\epsilon_2'}$$

3. The algebra $\mathcal{K}^{2,k}_{\lambda,\mu}$

In [4], we have computed, for all $k \in \frac{1}{2}\mathbb{N}$, the superalgebra of $\mathfrak{aff}(2|1)$-*supersymmetries*, i.e., the set of (local) endomorphisms $T : \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2}) \to \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$ commuting with the $\mathfrak{aff}(2|1)$-action on $\mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$, that is

$$[\mathfrak{L}^{\lambda,\mu}_{X_F}, T] := \mathfrak{L}^{\lambda,\mu}_{X_F}\circ T - (-1)^{|T||F|}T\circ\mathfrak{L}^{\lambda,\mu}_{X_F} = 0, \quad X_F \in \mathfrak{aff}(2|1).$$

Let us recall this result

THEOREM 3.1. *Let* $A = \sum_{m=0}^{2k}\sum_{\substack{(s,\epsilon_1,\epsilon_2))\in\mathbb{N}\times\{0,1\}^2\\ 2s+\epsilon_1+\epsilon_2=m}} a^m_{s,\epsilon_1,\epsilon_2}\partial_x^s\,\overline{D}_1^{\epsilon_1}\overline{D}_2^{\epsilon_2} \in \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$ *and* $T(A)$ *as in* (5). *Then the operator* T *commutes with the* $\mathfrak{aff}(2|1)$*-action on* $\mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$ *if and only if* $T(A)$ *reads as*

$$T(A) = \sum_{m=0}^{2k}\sum_{\substack{(s,\epsilon_1,\epsilon_2))\in\mathbb{N}\times\{0,1\}^2\\ 2s+\epsilon_1+\epsilon_2=m}} \sum_{\substack{(t,\epsilon_1',\epsilon_2',\epsilon_1'',\epsilon_2'')\in\mathbb{N}\times\{0,1\}^4\\ 2t+\epsilon_1'+\epsilon_2'+\epsilon_1''+\epsilon_2''\leq m\\ \epsilon_1+\epsilon_2-\epsilon_1'-\epsilon_2'-\epsilon_1''-\epsilon_2''\in 2\mathbb{Z}}}$$

$$T^{\epsilon_1,\epsilon_2,\epsilon_1',\epsilon_2',\epsilon_1'',\epsilon_2''}_{s,t}\partial_x^t\overline{D}_1^{\epsilon_1''}\overline{D}_2^{\epsilon_2''}(a_{s,\epsilon_1,\epsilon_2})\partial_x^{s+\frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_1'-\epsilon_2'-\epsilon_1''-\epsilon_2''-t)}\overline{D}_1^{\epsilon_1'}\overline{D}_2^{\epsilon_2'}, \tag{6}$$

where $T^{\epsilon_1,\epsilon_2,\epsilon_1,\epsilon_2,\epsilon_1,\epsilon_2}_{s,t}$ *are constant scalars satisfying the following conditions:*

1) $T^{\epsilon_1,\epsilon_2,\epsilon_1',\epsilon_2',\epsilon_1'',\epsilon_2''}_{s,t} = 0$ *if* $(\epsilon_1,\epsilon_2),(\epsilon_1',\epsilon_2') \in \{(1,1),(0,0)\}$, $(\epsilon_1'',\epsilon_2'') \in \{(1,0),(0,1)\}$ *(respectively* $(\epsilon_1,\epsilon_2),(\epsilon_1'',\epsilon_2'') \in \{(1,1),(0,0)\}$, $(\epsilon_1',\epsilon_2') \in \{(1,0),(0,1)\}$, *respectively* $(\epsilon_1',\epsilon_2'),(\epsilon_1'',\epsilon_2'') \in \{(1,1),(0,0)\}$, $(\epsilon_1,\epsilon_2) \in \{(1,0),(0,1)\}$*).*

2) $T^{\epsilon_1,\epsilon_2,1-\epsilon_1',1-\epsilon_2',\epsilon_1'',\epsilon_2''}_{s,t} = T^{\epsilon_1,\epsilon_2,\epsilon_1',\epsilon_2',1-\epsilon_1'',1-\epsilon_2''}_{s,t}$ *if* $(\epsilon_1,\epsilon_2) \in \{(1,1),(0,0)\}$, $(\epsilon_1',\epsilon_2') = (0,1),(\epsilon_1'',\epsilon_2'') = (1,0)$ *or* $(\epsilon_1',\epsilon_2') = (1,0),(\epsilon_1'',\epsilon_2'') = (0,1)$, *(respectively* $T^{\epsilon_1,\epsilon_2,\epsilon_1',\epsilon_2',1-\epsilon_1'',1-\epsilon_2''}_{s,t} = T^{\epsilon_1,\epsilon_2,\epsilon_1',\epsilon_2',1-\epsilon_1'',1-\epsilon_2''}_{s,t}$ *if* $(\epsilon_1',\epsilon_2') \in \{(1,1),(0,0)\}$, $(\epsilon_1,\epsilon_2) = (0,1),(\epsilon_1'',\epsilon_2'') = (1,0)$ *or* $(\epsilon_1,\epsilon_2) = (1,0),(\epsilon_1'',\epsilon_2'') = (0,1)$, *respectively* $T^{1-\epsilon_1,1-\epsilon_2,\epsilon_1',\epsilon_2',\epsilon_1'',\epsilon_2''}_{s,t} = T^{\epsilon_1,\epsilon_2,1-\epsilon_1',1-\epsilon_2',\epsilon_1'',\epsilon_2''}_{s,t}$ *if* $(\epsilon_2'',\epsilon_2'')\{(1,1),(0,0)\}$, *(*$(\epsilon_1,\epsilon_2) = (0,1)$, $(\epsilon_1',\epsilon_2') = (1,0)$ *or* $(\epsilon_1,\epsilon_2) = (1,0),(\epsilon_1',\epsilon_2') = (0,1)$*).*

3) $T_{s,t}^{\epsilon_1,\epsilon_2,1-\epsilon'_1,1-\epsilon'_2,\epsilon''_1,\epsilon''_2} = -T_{s,t}^{\epsilon_1,\epsilon_2,\epsilon'_1,\epsilon'_2,1-\epsilon''_1,1-\epsilon''_2}$ *if* $(\epsilon_1,\epsilon_2) \in \{(1,1),(0,0)\}$, $(\epsilon'_1,\epsilon'_2) = (\epsilon''_1,\epsilon''_2) = (1,0)$ *or* $(\epsilon'_1,\epsilon'_2) = (\epsilon''_1,\epsilon''_2) = (0,1)$
(respectively $T_{s,t}^{1-\epsilon_1,1-\epsilon_2,\epsilon'_1,\epsilon'_2,\epsilon''_1,\epsilon''_2} = -T_{s,t}^{\epsilon_1,\epsilon_2,\epsilon'_1,\epsilon'_2,1-\epsilon''_1,1-\epsilon''_2}$ *if* $(\epsilon'_1,\epsilon'_2) \in \{(1,1),(0,0)\}$, $(\epsilon_1,\epsilon_2) = (\epsilon''_1,\epsilon''_2) = (1,0)$ *or* $(\epsilon_1,\epsilon_2) = (\epsilon''_1,\epsilon''_2) = (0,1)$,
respectively $T_{s,t}^{1-\epsilon_1,1-\epsilon_2,\epsilon'_1,\epsilon'_2,\epsilon''_1,\epsilon''_2} = -T_{s,t}^{\epsilon_1,\epsilon_1,1-\epsilon'_1,1-\epsilon'_2,\epsilon''_1,\epsilon''_2}$ *if* $(\epsilon''_1,\epsilon''_2) \in \{(1,1),(0,0)\}$, $(\epsilon_1,\epsilon_2) = (\epsilon'_1,\epsilon'_2) = (1,0)$ *or* $(\epsilon_1,\epsilon_2) = (\epsilon'_1,\epsilon'_2) = (0,1)$*).*

4) $T_{s,t}^{\epsilon_1,\epsilon_2,\epsilon'_1,\epsilon'_2,1-\epsilon''_1,1-\epsilon''_2} = 0$ *if* $(\epsilon_1,\epsilon_2) = (\epsilon'_1,\epsilon'_2) = (\epsilon''_1,\epsilon''_2) = (1,0)$ *or* $(1,0)$.

5) $T_{s,t}^{\epsilon_1,\epsilon_2,\epsilon'_1,\epsilon'_2,1-\epsilon''_1,1-\epsilon''_2} = -2T_{s,t}^{1-\epsilon_1,1-\epsilon_2,1-\epsilon'_1,1-\epsilon'_2,\epsilon''_1,\epsilon''_2} = -2T_{s,t}^{1-\epsilon_1,1-\epsilon_2,\epsilon'_1,\epsilon'_2,1-\epsilon''_1,1-\epsilon''_2}$ *if* $(\epsilon_1,\epsilon_2) = (1,0), (\epsilon'_1,\epsilon'_2) = (\epsilon'_1,\epsilon'_2) = (0,1)$ *or* $(\epsilon_1,\epsilon_2) = (0,1), (\epsilon'_1,\epsilon'_2) = (\epsilon'_1,\epsilon'_2) = (1,0)$.

6) $T_{s,t}^{\epsilon_1,\epsilon_2,\epsilon'_1,\epsilon'_2,1-\epsilon''_1,1-\epsilon''_2} = -2T_{s,t}^{1-\epsilon_1,1-\epsilon_2,\epsilon',1-\epsilon''_1,1-\epsilon''_2} = -2T_{s,t}^{\epsilon_1,\epsilon_2,1-\epsilon'_1,1-\epsilon'_2,1-\epsilon''_1,1-\epsilon''_2}$ *if* $(\epsilon_1,\epsilon_2) = (\epsilon'_1,\epsilon'_2) = (1,0), (\epsilon''_1,\epsilon''_2) = (0,1)$ *or* $(\epsilon_1,\epsilon_2) = (\epsilon'_1,\epsilon'_2) = (0,1), (\epsilon''_1,\epsilon''_2) = (1,0)$.

7) $T_{s,t}^{\epsilon_1,\epsilon_2,\epsilon'_1,\epsilon'_2,1-\epsilon''_1,1-\epsilon''_2} = -2T_{s,t}^{1-\epsilon_1,1-\epsilon_2,1-\epsilon'_1,1-\epsilon'_2,\epsilon''_1,\epsilon''_2} = -2T_{s,t}^{\epsilon_1,\epsilon_2,1-\epsilon'_1,1-\epsilon'_2,1-\epsilon''_1,1-\epsilon''_2}$ *if* $(\epsilon_1,\epsilon_2) = (\epsilon''_1,\epsilon''_2) = (1,0), (\epsilon'_1,\epsilon'_2) = (0,1)$ *or* $(\epsilon_1,\epsilon_2) = (\epsilon''_1,\epsilon''_2) = (0,1), (\epsilon'_1,\epsilon'_2) = (1,0)$.

Now, since $[X_{\theta_i}, X_{\theta_i f}] = X_f, \forall f \in C^\infty(S^1)$, as a superalgebra, $\mathcal{K}(2)$ is generated by the set of odd vector fields $X_{\theta_i f}, f \in C^\infty(S^1)$, $i = 1,2$ and by a classical argument we can just consider "polynomial vector fields", i.e., of the form $X_{x^n\theta_i}, n \in \mathbb{N}$, $i = 1,2$. Starting with an $\mathfrak{aff}(2|1)$-invariant linear operator $T : \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2}) \to \mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$ and given that, for $i = 1,2$, $[X_{x\theta_i}, X_{x\theta_i}] = X_{x^2}$ and $[X_1, X_{x^2\theta_i}] = X_{x\theta_i}$, for the $\mathfrak{aff}(2|1)$-invariant linear operator T, the invariance of T with respect to $X_{x\theta_i}$ and X_{x^2} holds as soon as the invariance with respect to $X_{x^2\theta_i}$ is. Thus, in our approach, the next step is to impose the invariance with respect the contact vectors fields $X_{x^2\theta_i}, i = 1,2$. Moreover, it is well known that, if we identify S^1 with $\mathbb{RP}^1$ with homogeneous coordinates $(x_1 : x_2)$ and choose the affine coordinate $x = x_1/x_2$, the vector fields $\frac{d}{dx}$, $x\frac{d}{dx}$, $x^2\frac{d}{dx}$ are globally defined and correspond to the standard projective structure on $\mathbb{RP}^1$. In this adapted coordinate the action of the algebra $\mathfrak{sl}(2) = \mathrm{Span}\left(\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\right)$ is well defined. Thus, in the corresponding adapted coordinate (x,θ_1,θ_2) of $S^{1|2}$, thanks to (1), for $i,j \in \{1,2\}$ such that $i \neq j$:

$$X_{x^2\theta_i} = \frac{1}{2}x^2(\theta_i\frac{d}{dx} + \frac{d}{d\theta_i}) + \theta_i\theta_j x\frac{d}{d\theta_j},$$

the vector field $X_{x^2\theta_i}$ for $i = 1,2$ is further globally defined.

Let us first establish the two following lemmas. Surely, we may have similar results with the vector field $X_{x^2\theta_2}$.

Lemma 3.2. *Let* $a \in C^\infty(S^{1|2})$*, thus we have*

1) $$\mathcal{L}^{\lambda,\mu}_{X_{x^2\theta_1}}(a\partial^\ell) = \mathcal{L}^{\mu-\lambda-\ell}_{X_{x^2\theta_1}}(a)\partial^\ell - \ell(2\lambda+\ell-1)\theta_1 a\partial^{\ell-1} - (-1)^{|a|}\ell x a\partial^{\ell-1}\overline{D}_1$$

$$- (-1)^{|a|}\ell\theta_1\theta_2 a\partial^{\ell-1}\overline{D}_2 - \frac{(-1)^{|a|}}{2}a\ell(\ell-1)\partial^{\ell-2}\overline{D}_1$$

2) $$\mathcal{L}^{\lambda,\mu}_{X_{x^2\theta_1}}(a\partial^\ell\overline{D}_1) = \mathcal{L}^{\mu-\lambda-\ell-\frac{1}{2}}_{X_{x^2\theta_1}}(a)\partial^\ell\overline{D}_1 + x\theta_2 a\partial^\ell\overline{D}_2 + (-1)^{|a|}ax(2\lambda+\ell)\partial^\ell - (2\lambda\ell+\ell^2)\theta_1 a\partial^{\ell-1}\overline{D}_1 + \theta_2 a\ell\partial^{\ell-1}\overline{D}_2 + (-1)^{|a|}\ell\theta_1\theta_2 a\partial^{\ell-1}\overline{D}_1\overline{D}_2 + \frac{(-1)^{|a|}}{2}(\ell(\ell-1)+4\lambda\ell)a\partial^{\ell-1}$$

3) $$\mathcal{L}^{\lambda,\mu}_{X_{x^2\theta_1}}(a\partial^\ell\overline{D}_2) = \mathcal{L}^{\mu-\lambda-\ell-\frac{1}{2}}_{X_{x^2\theta_1}}(a)\partial^\ell\overline{D}_2 - x\theta_2 a\partial^\ell\overline{D}_1 + (-1)^{|a|}\theta_1\theta_2 a(2\lambda+\ell)\partial^\ell - (2\lambda\ell+\ell^2)\theta_1 a\partial^{\ell-1}\overline{D}_2 - \ell\theta_2 a\partial^{\ell-1}\overline{D}_1 - (-1)^{|a|}\ell x a\partial^{\ell-1}\overline{D}_1\overline{D}_2 - \frac{(-1)^{|a|}}{2}\ell(\ell-1)a\partial^{\ell-2}\overline{D}_1\overline{D}_2$$

4) $$\mathcal{L}^{\lambda,\mu}_{X_{x^2\theta_1}}(a\partial^\ell\overline{D}_1\overline{D}_2) = \mathcal{L}^{\mu-\lambda-\ell-1}_{X_{x^2\theta_1}}(a)\partial^\ell\overline{D}_1\overline{D}_2 - (-1)^{|a|}\theta_1\theta_2 a(2\lambda+\ell+1)\partial^\ell\overline{D}_1 + (-1)^{|a|}(2\lambda+\ell+1)xa\partial^\ell\overline{D}_2 - 2\lambda\theta_2 a\partial^\ell - (2\lambda\ell+2\ell+\ell(\ell-1))\theta_1 a\partial^{\ell-1}\overline{D}_1\overline{D}_2 + (-1)^{|a|}a(2\lambda\ell+\ell+\frac{\ell(\ell-1)}{2})\partial^{\ell-1}\overline{D}_2.$$

Proof.

1) $$\begin{aligned}\mathcal{L}^{\lambda,\mu}_{X_{x^2\theta_1}}(a\partial^\ell_x) &= \mathcal{L}^{\mu}_{X_{x^2\theta_1}}\circ(a\partial^\ell_x) - (-1)^{|a|}(a\partial^\ell_x)\circ\mathcal{L}^{\lambda}_{X_{x^2\theta_1}}\\ &= x^2\theta_1 a'\partial^\ell_x + x^2\theta_1 a\partial^{\ell+1}_x + \frac{1}{2}\Big(x^2\overline{D}_1(a)\partial^\ell_x + (-1)^{|a|}x^2 a\partial^\ell_x\overline{D}_1\\ &+ 2\theta_1\theta_2 x\overline{D}_2(a)\partial^\ell_x + (-1)^{|a|}2\theta_1\theta_2 xa\partial^\ell_x\overline{D}_2 + 2\mu\theta_1 xa\partial^\ell_x\Big)\\ &- (-1)^{|a|}a\Big(\partial^\ell_x(x^2\theta_1\partial x + \frac{1}{2}x^2\overline{D}_1 + \theta_1\theta_2 x\overline{D}_2 + 2\lambda\theta_1 x)\Big)\\ &= \mathcal{L}^{\mu-\lambda-\ell}_{X_{x^2\theta_1}}(a)\partial^\ell_x - \ell(2\lambda+\ell-1)\theta_1 a\partial^{\ell-1} - (-1)^{|a|}lxa\partial^{\ell-1}\overline{D}_1\\ &- (-1)^{|a|}l\theta_1\theta_2 a\partial^{\ell-1}\overline{D}_2 - \frac{(-1)^{|a|}}{2}a\ell(\ell-1)\partial^{\ell-2}\overline{D}_1.\end{aligned}$$

2) $$\begin{aligned}\mathcal{L}^{\lambda,\mu}_{X_{x^2\theta_1}}(a\partial^\ell_x\overline{D}_1) &= \mathcal{L}^{\mu}_{X_{x^2\theta_1}}\circ(a\partial^\ell_x\overline{D}_1) - (-1)^{(|a|+1)}a\partial^\ell_x\overline{D}_1\circ\mathcal{L}^{\lambda}_{X_{x^2\theta_1}}\\ &= x^2\theta_1\partial_x(a\partial^\ell\overline{D}_1) + \frac{1}{2}\Big(x^2\overline{D}_1(a\partial^\ell_x\overline{D}_1) + 2\theta_1\theta_2 x\overline{D}_2(a\partial^\ell_x\overline{D}_1)\Big)\\ &+ 2\mu\theta_1 xa\partial^\ell_x\overline{D}_1 + (-1)^{|a|}a\partial^\ell_x\overline{D}_1\Big(x^2\theta_1 + \frac{1}{2}x^2\overline{D}_1 + x\theta_1\theta_2\overline{D}_2 + 2\lambda\theta_1 x\Big)\\ &= x^2\theta_1 a'\partial^\ell_x\overline{D}_1 + x^2\theta_1 a\partial^{\ell+1}_x\overline{D}_1 + \frac{1}{2}\Big(x^2\overline{D}_1(a)\partial^\ell\overline{D}_1 - (-1)^{|a|}x^2 a\partial^{\ell+1}_x\\ &+ 2\theta_1\theta_2 x\overline{D}_2(a)\partial^\ell\overline{D}_1 - 2(-1)^{|a|}\theta_1\theta_2 xa\partial^\ell_x\overline{D}_1\overline{D}_2\Big) + 2\mu\theta_1 xa\partial^\ell_x\overline{D}_1\\ &+ (-1)^{|a|}a\Big(- x^2\theta_1\partial^{\ell+1}_x\overline{D}_1 + \frac{1}{2}x^2\partial^{\ell+1} + x\theta_1\theta_2\partial^\ell_x\overline{D}_1\overline{D}_2 + x\theta_2\partial^\ell_x\overline{D}_2\end{aligned}$$

$$
\begin{aligned}
&-(2\lambda+2\ell+1)\theta_1 x\partial_x^\ell\overline{D}_1+(2\lambda+\ell)x\partial_x^\ell+\ell(2\lambda+\frac{\ell-1}{2})\partial_x^{\ell-1}\\
&-\ell(2\lambda+\ell)\theta_1\partial_x^{\ell-1}\overline{D}_1+\ell\theta_2\partial_x^{\ell-1}\overline{D}_2+\ell\theta_1\theta_2\partial^{l-1}\overline{D}_1\overline{D}_2\Big)\\
&=\mathcal{L}^{\mu-\lambda-\ell-\frac{1}{2}}_{X_{x^2\theta_1}}(a)\partial^\ell\overline{D}_1+x\theta_2 a\partial^\ell\overline{D}_2+(-1)^{|a|}ax(2\lambda+\ell)\partial^\ell\\
&-(2\lambda\ell+\ell^2)\theta_1 a\partial^{\ell-1}\overline{D}_1+\theta_2 a\ell\partial^{\ell-1}\overline{D}_2+(-1)^{|a|}\ell\theta_1\theta_2 a\partial^{\ell-1}\overline{D}_1\overline{D}_2\\
&+\frac{(-1)^{|a|}}{2}(\ell(\ell-1)+4\lambda\ell)a\partial^{\ell-1}.
\end{aligned}
$$

By an analogous calculation one can easily obtain 3) and 4). □

Lemma 3.3. *Let $a\in C^\infty(S^{1|2})$, thus we have*

1)
$$
\begin{aligned}
\partial_x^\ell\Big(\mathcal{L}^{\mu-\lambda}_{X_{x^2\theta_1}}(a)\Big)&=\mathcal{L}^{\mu-\lambda+\ell}_{X_{x^2\theta_1}}(\partial^l(a))+\ell x\partial_x^{\ell-1}\overline{D}_1(a)+\ell\theta_1\theta_2\partial_x^{\ell-1}\overline{D}_2(a)\\
&+\frac{\ell(\ell-1)}{2}\partial_x^{\ell-2}\overline{D}_1(a)+\ell(2(\mu-\lambda)+\ell-1)\theta_1\partial_x^{\ell-1}(a).
\end{aligned}
$$

2)
$$
\begin{aligned}
\partial_x^\ell\overline{D}_1\Big(\mathcal{L}^{\mu-\lambda}_{X_{x^2\theta_1}}(a)\Big)&=-\mathcal{L}^{\mu-\lambda+\ell+\frac{1}{2}}_{X_{x^2\theta_1}}(\partial_x^\ell\overline{D}_1(a))+2(\mu-\lambda)x\partial_x^\ell(a)+\theta_2 x\partial_x^\ell\overline{D}_2(a)\\
&+(2\ell(\mu-\lambda)+\frac{\ell(\ell-1)}{2})\partial_x^{\ell-1}(a)-\ell(2(\mu-\lambda)+\ell)\theta_1\partial_x^{\ell-1}\overline{D}_1(a)\\
&+\ell\theta_2\partial_x^{\ell-1}\overline{D}_2(a)+\ell\theta_1\theta_2\partial_x^{\ell-1}\overline{D}_1\overline{D}_2(a).
\end{aligned}
$$

3)
$$
\begin{aligned}
\partial_x^\ell\overline{D}_2\Big(\mathcal{L}^{\mu-\lambda}_{X_{x^2\theta_1}}(a)\Big)&=-\mathcal{L}^{\mu-\lambda+\ell+\frac{1}{2}}_{X_{x^2\theta_1}}(\partial_x^\ell\overline{D}_2(a))+(2(\mu-\lambda)+\ell)\theta_1\theta_2\partial_x^\ell(a)\\
&+\ell\theta_2\partial_x^{\ell-1}\overline{D}_1(a)-\theta_2 x\partial_x^\ell\overline{D}_1(a)-\ell(2(\mu-\lambda)+\ell)\theta_1\partial_x^{\ell-1}\overline{D}_2(a)\\
&-\ell x\partial_x^{\ell-1}\overline{D}_1\overline{D}_2(a)-\frac{\ell(\ell-1)}{2}\partial_x^{\ell-2}\overline{D}_1\overline{D}_2(a).
\end{aligned}
$$

4)
$$
\begin{aligned}
\partial_x^\ell\overline{D}_1\overline{D}_2\Big(\mathcal{L}^{\mu-\lambda}_{X_{x\theta_1}}(a)\Big)&=\mathcal{L}^{\mu-\lambda+\ell+1}_{X_{x^2\theta_1}}(\partial_x^\ell\overline{D}_1\overline{D}_2(a))+2(\mu-\lambda)\theta_2\partial_x^\ell(a)\\
&+(2(\mu-\lambda)+\ell+1)\theta_1\theta_2\partial_x^\ell\overline{D}_1(a)-(2(\mu-\lambda)+\ell+1)x\partial_x^\ell\overline{D}_2(a)\\
&+\ell(2(\mu-\lambda)+\ell+1)\theta_1\partial_x^{\ell-1}\overline{D}_1\overline{D}_2(a)\\
&-\ell(2(\mu-\lambda)+\frac{\ell+1}{2})\partial_x^{\ell-1}\overline{D}_2(a).
\end{aligned}
$$

Proof.

1)
$$
\begin{aligned}
\partial_x^\ell(\mathcal{L}^{\mu-\lambda}_{X_{x^2\theta_1}}(a))&=\partial_x^\ell\Big(x^2\theta_1 a'+\frac{1}{2}x^2\partial_x^\ell\overline{D}_1(a)+x\theta_1\theta_2\overline{D}_2(a)+2(\mu-\lambda)\theta_1 xa\Big)\\
&=x^2\theta_1\partial_x^{\ell+1}(a)+\frac{1}{2}x^2\partial_x^\ell\overline{D}_1(a)+x\theta_1\theta_2\partial_x^\ell\overline{D}_2(a)\\
&+2(\mu-\lambda+\ell)\theta_1 x\partial_x^\ell(a)+lx\partial_x^{\ell-1}\overline{D}_1(a)+\ell\theta_1\theta_2\partial_x^{\ell-1}\overline{D}_2(a)\\
&+\frac{\ell(\ell-1)}{2}\partial_x^{\ell-2}\overline{D}_1(a)+\ell(2(\mu-\lambda)+\ell-1)\theta_1\partial_x^{\ell-1}(a)\\
&=\mathcal{L}^{\mu-\lambda+\ell}_{X_{x^2\theta_1}}(\partial^l(a))+\ell x\partial_x^{\ell-1}\overline{D}_1(a)+\ell\theta_1\theta_2\partial_x^{\ell-1}\overline{D}_2(a)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\ell(\ell-1)}{2}\partial_x^{\ell-2}\overline{D}_1(a)+\ell(2(\mu-\lambda)+\ell-1)\theta_1\partial_x^{\ell-1}(a).\\
2)\quad \partial_x^\ell\overline{D}_1(\mathcal{L}^{\mu-\lambda}_{X_{x\theta_1}}(a)) &= \partial_x^\ell\overline{D}_1\Big(x^2\theta_1 a'+\frac{1}{2}x^2\overline{D}_1(a)+x\theta_1\theta_2\overline{D}_2(a)+2(\mu-\lambda)\theta_1 xa\Big)\\
&= -\mathcal{L}^{\mu-\lambda+\ell+\frac{1}{2}}_{X_{x^2\theta_1}}(\partial_x^l\overline{D}_1(a))+2(\mu-\lambda)x\partial_x^\ell(a)+\theta_2 x\partial_x^\ell\overline{D}_2(a)\\
&+(2\ell(\mu-\lambda)+\frac{\ell(\ell-1)}{2})\partial_x^{\ell-1}(a)-\ell(2(\mu-\lambda)+\ell)\theta_1\partial_x^{\ell-1}\overline{D}_1(a)\\
&+\ell\theta_2\partial_x^{\ell-1}\overline{D}_2(a)+\ell\theta_1\theta_2\partial_x^{\ell-1}\overline{D}_1\overline{D}_2(a).
\end{aligned}
$$

By an analogous calculation one can easily obtain 3) and 4). □

Now, thanks to Lemmas 3.2 and 3.3, we can impose the invariance under the action of the vector field $X_{x^2\theta_1}$ (respectively $X_{x^2\theta_2}$) to an $\mathfrak{aff}(2|1)$-invariant linear operator $T:\mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})\to\mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$.

THEOREM 3.4. *Let $T:\mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})\to\mathfrak{D}^{2,k}_{\lambda,\mu}(S^{1|2})$ an $\mathfrak{aff}(2|1)$-invariant linear (local) operator. Then, T commutes with the actions of the vector fields $X_{x^2\theta_i}; i=1,2$ if and only if, for all $k\in\frac{1}{2}\mathbb{N}^*$ there exist scalar constants $\Upsilon^1_s,\cdots,\Upsilon^6_s$ such that*

1) $\forall s$ such that $2s+1\leq 2k$ (resp $2s+1\leq 2k+1$) and $\forall a_s,b_s\in C^\infty_{\mathbb{C}}(S^{1|2})$

$$T(a_s\partial^s\overline{D}_1+b_s\partial^s\overline{D}_2)=\Upsilon^1_s\Big(a_s\partial^s\overline{D}_1+b_s\partial^s\overline{D}_2\Big)+\Upsilon^2_s(a_s\partial^s\overline{D}_2-b_s\partial^s\overline{D}_1)$$

2) $\forall s$ such that $2s\leq 2k$ (resp $2s\leq 2k+1$) and $\forall c_s,d_s\in C^\infty_{\mathbb{C}}(S^{1|2})$

$$T\Big(c_s\partial^s+d_s\partial^{s-1}\overline{D}_1\overline{D}_2\Big)=\Upsilon^3_s c_s\partial^{s-1}\overline{D}_1\overline{D}_2+\Upsilon^4_s c_s\partial^s+\Upsilon^5_s d_s\partial^{s-1}\overline{D}_1\overline{D}_2+\Upsilon^6_s d_s\partial^s,$$

and the scalars $\Upsilon^1_s,\cdots,\Upsilon^6_s$ satisfy the following system:

$$
\left\{
\begin{array}{ll}
(2\lambda+s)\Upsilon^1_s-(2\lambda+s)\Upsilon^4_s=0, & (2\lambda+s)\Upsilon^2_s-s\Upsilon^6_s=0,\\
s(4\lambda+s-1)\Upsilon^1_s-s(4\lambda+s-1)\Upsilon^4_{s-1}=0, & s(4\lambda+s-1)\Upsilon^2_s-s(s-1)\Upsilon^6_{s-1}=0,\\
s(2\lambda+s)\Upsilon^2_s-s(2\lambda+s)\Upsilon^2_{s-1}=0, & s\Upsilon^1_s-s\Upsilon^1_{s-1}=0,\\
s\Upsilon^2_s-s\Upsilon^2_{s-1}=0, & s(2\lambda+s)\Upsilon^2_s-s(2\lambda+s)\Upsilon^2_{s-1}=0,\\
s\Upsilon^1_s-s\Upsilon^5_s=0, & s(s-1)\Upsilon^1_s-s(s-1)\Upsilon^5_{s-1}=0,\\
s\Upsilon^2_{s-1}+(2\lambda+s)\Upsilon^3_s=0, & (s+1)(2\lambda+s)\Upsilon^4_s-(s+1)(2\lambda+s)\Upsilon^4_{s+1}=0,\\
2\lambda\Upsilon^5_{s+1}-2\lambda\Upsilon^4_s=0, & s(2\lambda+s+1)\Upsilon^6_s-(s+1)(2\lambda+s)\Upsilon^6_{s+1}=0,\\
s(4\lambda+s+1)\Upsilon^3_{s+1}+s(s+1)\Upsilon^2_{s-1}=0, & 2\lambda\Upsilon^3_s=0,\\
s(4\lambda+s+1)\Upsilon^1_{s-1}-s(4\lambda+s+1)\Upsilon^5_{s+1}=0, & s(4\lambda+s+1)\Upsilon^3_{s+1}+s(s+1)\Upsilon^2_{s-1}=0,\\
(s+1)(2\lambda+s)\Upsilon^3_s-s(2\lambda+s+1)\Upsilon^3_{s+1}=0, & s(2\lambda+s+1)\Upsilon^5_s-s(2\lambda+s+1)\Upsilon^5_{s+1}=0,\\
(2\lambda+s+1)\Upsilon^3_{s+1}+(s+1)\Upsilon^2_s=0, & s(4\lambda+s+1)\Upsilon^2_{s-1}-s(s-1)\Upsilon^6_{s+1}=0,\\
(s+1)\Upsilon^4_{s+1}-(s+1)\Upsilon^1_s=0, & s(s+1)\Upsilon^4_{s+1}-s(s+1)\Upsilon^1_{s-1}=0,\\
(2\lambda+s+1)\Upsilon^5_{s+1}-(2\lambda+s+1)\Upsilon^1_s=0, & (2\lambda+s+1)\Upsilon^2_s-(s+1)\Upsilon^6_{s+1}=0.
\end{array}
\right. \tag{7}
$$

Proof. Let $A = a_s\partial^s\overline{D}_1 + b_s\partial^s\overline{D}_2$. Upon using (5) and Theorem 3.1, $T(A)$ reads:

$$\begin{aligned}
T(A) = \sum_{t=0}^{s-1} &\Big(T_{s,t}^{(1,0),(1,0),(1,1)}\partial^t\overline{D}_1\overline{D}_2(a_s) + T_{s,t}^{(1,0),(1,0),(1,1)}\partial^t\overline{D}_1\overline{D}_2(b_s)\Big)\partial^{s-t-1}\overline{D}_1 \\
&+ \Big(T_{s,t}^{(1,0),(1,0),(1,1)}\partial^t\overline{D}_1\overline{D}_2(a_s) + T_{s,t}^{(1,0),(1,0),(1,1)}\partial^t\overline{D}_1\overline{D}_2(b_s)\Big)\partial^{s-t-1}\overline{D}_2 \\
&+ \Big(T_{s,t}^{(1,0),(1,1),(1,0)}\partial^t\overline{D}_1(a_s) + T_{s,t}^{(1,0),(1,1),(0,1)}\partial^t\overline{D}_2(a_s) \\
&+ T_{s,t}^{(0,1),(1,1),(1,0)}\partial^t\overline{D}_1(b_s) + T_{s,t}^{(0,1),(1,1),(0,1)}\partial^t\overline{D}_2(b_s)\Big)\partial^{s-t-1}\overline{D}_1\overline{D}_2 \\
&+ \sum_{t=0}^{s}\Big(T_{s,t}^{(1,0),(1,0),(0,0)}\partial^t(a_s) + T_{s,t}^{(0,1),(1,0),(0,0)}\partial^t(b_s)\Big)\partial^{s-t}\overline{D}_1 \\
&+ \Big(T_{s,t}^{(1,0),(1,0),(0,0)}\partial^t(a_s) + T_{s,t}^{(0,1),(1,0),(0,0)}\partial^t(b_s)\Big)\partial^{s-t}\overline{D}_2 \\
&+ \Big(T_{s,t}^{(1,0),(0,0),(1,0)}\partial^t\overline{D}_1(a_s) + T_{s,t}^{(1,0),(0,0),(0,1)}\partial^t\overline{D}_2(a_s) \\
&+ T_{s,t}^{(0,1),(0,0),(1,0)}\partial^t\overline{D}_1(b_s) + T_{s,t}^{(0,1),(0,0),(0,1)}\partial^t\overline{D}_2(b_s)\Big)\partial^{s-t}
\end{aligned}$$

with the additional conditions:

$$\begin{cases}
T_{s,t}^{(1,0),(1,0),(1,1)} = -T_{s,t}^{(0,1),(0,1),(1,1)}; & T_{s,t}^{(1,0),(0,1),(1,1)} = T_{s,t}^{(0,1),(1,0),(1,1)}; \\
T_{s,t}^{(1,0),(1,1),(1,0)} = -T_{s,t}^{(0,1),(1,1),(0,1)}; & T_{s,t}^{(1,0),(0,0),(1,0)} = -T_{s,t}^{(0,1),(0,0),(0,1)}; \\
T_{s,t}^{(1,0),(1,1),(0,1)} = T_{s,t}^{(0,1),(1,1),(1,0)}; & T_{s,t}^{(1,0),(0,0),(0,1)} = T_{s,t}^{(0,1),(0,0),(1,0)}; \\
T_{s,t}^{(1,0),(1,0),(0,0)} = -T_{s,t}^{(0,1),(0,1),(0,0)}; & T_{s,t}^{(1,0),(0,1),(0,0)} = T_{s,t}^{(0,1),(0,1),(1,1)};
\end{cases}$$

or with change of notations

$$\begin{aligned}
T(A) = \sum_{t=0}^{s-1} &T_{s,t}^{1}\Big(\partial_x^t\overline{D}_1\overline{D}_2(a_s)\partial^{s-t-1}\overline{D}_1 + \partial_x^t\overline{D}_1\overline{D}_2(b_s)\partial^{s-t-1}\overline{D}_2\Big) \\
&+ T_{s,t}^{2}\Big(\partial_x^t\overline{D}_1\overline{D}_2(a_s)\partial^{s-t-1}\overline{D}_2 - \partial_x^t\overline{D}_1\overline{D}_2(b_s)\partial^{s-t-1}\overline{D}_1\Big) \\
&+ \sum_{t=0}^{s-1} T_{s,t}^{3}\partial_x^t\Big(\overline{D}_1(a_s) + \overline{D}_2(b_s)\Big)\partial^{s-t-1}\overline{D}_1\overline{D}_2 + \sum_{t=0}^{s} T_{s,t}^{4}\partial_x^t\Big(\overline{D}_1(a_s) + \overline{D}_2(b_s)\Big)\partial^{s-t} \\
&+ \sum_{t=0}^{s-1} T_{s,t}^{5}\partial_x^t\Big(\overline{D}_2(a_s) - \overline{D}_1(b_s)\Big)\partial^{s-t-1}\overline{D}_1\overline{D}_2 + \sum_{t=0}^{s} T_{s,t}^{6}\partial_x^t\Big(\overline{D}_2(a_s) - \overline{D}_1(b_s)\Big)\partial^{s-t} \\
&+ T_{s,t}^{7}\Big(\partial_x^t(a_s)\partial^{s-t}\overline{D}_1 + \partial_x^t(b_s)\partial^{s-t}\overline{D}_2\Big) + T_{s,t}^{8}\Big(\partial_x^t(a_s)\partial^{s-t}\overline{D}_2 - \partial_x^t(b_s)\partial^{s-t}\overline{D}_1\Big).
\end{aligned}$$

Similarly, if $A = c_{s+1}\partial^{s+1} + d_{s+1}\overline{D}_1\overline{D}_2$ we can write $T(A)$ of the form

$$\begin{aligned}
T(A) = \sum_{t=0}^{s} &T_{s+1,t}^{9}\partial_x^{s+1,t}\overline{D}_1\overline{D}_2(c_{s+1})\partial^{s-t} + \sum_{t=0}^{s-1} T_{s+1,t}^{10}\partial_x^t\overline{D}_1\overline{D}_2(c_{s+1})\partial^{s-t-1}\overline{D}_1\overline{D}_2 \\
&+ \sum_{t=0}^{s} T_{s+1,t}^{11}\Big(\partial_x^t\overline{D}_1(c_{s+1})\partial^{s-t}\overline{D}_1 + \partial_x^t\overline{D}_2(c_{s+1})\partial^{s-t}\overline{D}_2\Big)
\end{aligned}$$

$$+ T^{12}_{s+1,t}\Big(\partial_x^t\overline{D}_2(c_{s+1})\partial^{s-t}\overline{D}_1 - \partial_x^t\overline{D}_1(c_{s+1})\partial^{s-t}\overline{D}_2\Big) + T^{13}_{s+1,t}\partial_x^t(c_{s+1})\partial^{s-t}\overline{D}_1\overline{D}_2$$
$$+ \sum_{t=0}^{s+1} T^{14}_{s+1,t}\partial_x^t(c_{s+1})\partial^{s-t+1} \sum_{t=0}^{s-1} T^{15}_{s+1,t}\partial_x^t\overline{D}_1\overline{D}_2(d_{s+1})\partial^{s-t-1}\overline{D}_1\overline{D}_2$$
$$+ \sum_{t=0}^{s} T^{16}_{s+1,t}\partial_x^t\overline{D}_1\overline{D}_2(d_{s+1})\partial^{s-t} + T^{17}_{s+1,t}\Big(\partial_x^t\overline{D}_1(d_{s+1})\partial^{s-t}\overline{D}_1 + \partial_x^t\overline{D}_2(d_{s+1})\partial^{s-t}\overline{D}_2\Big)$$
$$+ T^{18}_{s+1,t}\Big(\partial_x^t\overline{D}_2(d_{s+1})\partial^{s-t}\overline{D}_1 - \partial_x^t\overline{D}_1(d_{s+1})\partial^{s-t}\overline{D}_2\Big) + T^{19}_{s+1,t}\partial_x^t(d_{s+1})\partial^{s-t}\overline{D}_1\overline{D}_2$$
$$+ \sum_{t=0}^{s+1} T^{20}_{s+1,t}\partial_x^t(d_{s+1})\partial^{s-t+1}$$

Finally, thanks to Lemmas 3.2 and 3.3, we impose the conditions

$$[T, \mathcal{L}^{\lambda,\mu}_{X_{x^2\theta_1}}](a_s\partial^s\overline{D}_1 + b_s\partial^s\overline{D}_2) = 0 \quad \text{and} \quad [T, \mathcal{L}^{\lambda,\mu}_{X_{x^2\theta_1}}](c_{s+1}\partial^{s+1} + d_{s+1}\partial^s\overline{D}_1\overline{D}_2) = 0.$$

By a direct computation the theorem is thus proved. □

Now, we are able to compute the dimension of the algebra $\mathcal{K}^{2,k}_{\lambda,\mu}$ for all k in $\frac{1}{2}\mathbb{N}^*$.

Theorem 3.5. *Let $k \in \frac{1}{2}\mathbb{N}^*$. Then* $\dim\Big(\mathcal{K}^{2,k}_{\lambda,\mu}\Big) = \begin{cases} 2 & \text{if } \lambda = 0 \\ 1 & \text{otherwise.}\end{cases}$

Proof. By solving the system (7), we easily obtain that, if $\lambda \in \mathbb{R}^*$, $\Upsilon^2_s = \Upsilon^3_s = \Upsilon^6_s = 0$ for all s and $\Upsilon^1_s = \Upsilon^4_s = \Upsilon^5_s$ are constant. In this case, the algebra $\mathcal{K}^{2,k}_{\lambda,\mu}$ is trivial. If $\lambda = 0$, we get $\Upsilon^1_s = \Upsilon^4_s = \Upsilon^5_s$, $\Upsilon^2_s = -\Upsilon^3_s = \Upsilon^6_s$ and $\Upsilon^1_s, \Upsilon^2_s$ are constant and then $\mathcal{K}^{2,k}_{0,\mu} = \mathrm{Span}(Id, T_0)$ where T_0 is given by:

$$T_0\Big(\alpha\partial^{s+1} + \beta\partial^s\overline{D}_1\overline{D}_2\Big) = \alpha\partial^s\overline{D}_1\overline{D}_2 - \beta\partial^{s+1},$$
$$T_0\Big(\alpha\partial^s\overline{D}_1 + \beta\partial^s\overline{D}_2\Big) = \alpha\partial^s\overline{D}_2 - \beta\partial^s\overline{D}_1,$$

$\forall s \in \mathbb{N}; \forall\alpha, \beta \in C^\infty_{\mathbb{C}}(S^{1|2})$. We must prove that T_0 is still $\mathcal{K}(2)$-invariant. Indeed, Let $X_F, F = f\theta_1$ ($f \in C^\infty_{\mathbb{C}}(S^1)$) an odd vector field in $\mathcal{K}(2)$ and $A = \alpha\partial^{s+1} + \beta\partial^s\overline{D}_1\overline{D}_2$ ($s \in \mathbb{N}, \alpha, \beta \in C^\infty_{\mathbb{C}}(S^{1|2})$). Then $\mathfrak{L}^{0,\mu}_{X_F}(A) = \mathfrak{L}^\mu_{X_F} \circ A - (-1)^{|A||F|}A \circ \mathfrak{L}^0_{X_F}$, where

$$\begin{aligned}
\mathfrak{L}^\mu_{X_F} \circ A &= (X_F + \mu F') \circ A \\
&= \Big(F\partial_x - \frac{1}{2}\sum_{i=1}^{2}(-1)^{|F|}\overline{D}_i(F)\overline{D}_i + \mu F'\Big) \circ A \\
&= \Big(F\partial_x + \frac{1}{2}\sum_{i=1}^{2}\overline{D}_i(F)\overline{D}_i + \mu F'\Big) \circ A \\
&= \Big(f\theta_1\partial + \frac{1}{2}\big(f\overline{D}_1 + \theta_1\theta_2 f'\overline{D}_2\big) + \mu f'\theta_1\Big) \circ \Big(\alpha\partial^{s+1} + \beta\partial^s\overline{D}_1\overline{D}_2\Big) \\
&= \theta_1 f\alpha'\partial^{s+1} + \theta_1 f\alpha\partial^{s+2} + \theta_1 f\beta'\partial^s\overline{D}_1\overline{D}_2 + \theta_1 f\beta\partial^{s+1}\overline{D}_1\overline{D}_2 \\
&+ \frac{1}{2}\Big(f\overline{D}_1(\alpha)\partial^{s+1} + (-1)^{|\alpha|}f\alpha\partial^{s+1}\overline{D}_1 + f\overline{D}_1(\beta)\partial^s\overline{D}_1\overline{D}_2 - (-1)^{|\beta|}f\beta\partial^{s+1}\overline{D}_2
\end{aligned}$$

$$
\begin{aligned}
&+\theta_1\theta_2 f'\overline{D}_2(\alpha)\partial^{s+1}+(-1)^{|\alpha|}\theta_1\theta_2 f'\alpha\partial^{s+1}\overline{D}_2+\theta_1\theta_2 f'\overline{D}_2(\beta)\partial^{s}\overline{D}_1\overline{D}_2\\
&+(-1)^{|\beta|}\theta_1\theta_2 f'\beta\partial^{s+1}\overline{D}_1\Big)+\mu\theta_1 f'\alpha\partial^{s+1}+\mu\theta_1 f'\beta\partial^{s}\overline{D}_1\overline{D}_2,
\end{aligned}
$$

and

$$
\begin{aligned}
A\circ\mathfrak{L}^0_{X_F}&=A\circ\Big(F\partial_x-\frac{1}{2}\sum_{i=1}^{2}(-1)^{|F|}\overline{D}_i(F)\overline{D}_i\Big)\\
&=\Big(\alpha\partial^{s+1}+\beta\partial^{s}\overline{D}_1\overline{D}_2\Big)\circ\Big(f\theta_1\partial+\frac{1}{2}\big(f\overline{D}_1+\theta_1\theta_2 f'\overline{D}_2\big)\Big)\\
&=\alpha\Big[\sum_{i=0}^{s+1}C^i_{s+1}\Big(\theta_1 f^{(i)}\partial_x^{s+2-i}+\frac{1}{2}f^{(i)}\partial^{s+1-i}\overline{D}_1+\frac{1}{2}f^{(i+1)}\partial^{s+1-i}\overline{D}_2\Big)\Big]\\
&+\beta\Big[\sum_{i=0}^{s}C^i_{s}\Big(\frac{1}{2}\theta_1\theta_2 f^{(i+2)}\partial^{s-i}\overline{D}_1+\frac{1}{2}\theta_1\theta_2 f^{(i+1)}\partial^{s+1-i}\overline{D}_1-\frac{1}{2}f^{(i)}\partial^{s+1-i}\overline{D}_2\\
&-\frac{1}{2}f^{(i+1)}\partial^{s-i}\overline{D}_2+\theta_1 f^{(i)}\partial^{s+1-i}\overline{D}_1\overline{D}_2+\theta_1 f^{(i+1)}\partial^{s-i}\overline{D}_1\overline{D}_2\Big)\Big].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathfrak{L}^{0,\mu}_{X_F}(A)&=\mathfrak{L}^{\mu-s-1}_{X_F}(\alpha)\partial^{s+1}+\mathfrak{L}^{\mu-s-1}_{X_F}(\beta)\partial^{s}\overline{D}_1\overline{D}_2\\
&-\Big[\sum_{i=1}^{s}C^{i+1}_{s+1}\theta_1 f^{(i+1)}\Big((-1)^{|\alpha|}\alpha\partial^{s+1-i}+(-1)^{|\beta|}\beta\partial^{s-i}\overline{D}_1\overline{D}_2\Big)\Big]\\
&-\frac{1}{2}\Big[\sum_{i=0}^{s}C^{i+1}_{s+1}f^{(i+1)}\Big((-1)^{|\alpha|}\alpha\partial^{s-i}\overline{D}_1+(-1)^{|\beta|}\beta\partial^{s-i}\overline{D}_2\Big)\Big]\\
&-\frac{1}{2}\Big[\sum_{i=0}^{s}C^{i+1}_{s+1}\theta_1\theta_2 f^{(i+2)}\Big((-1)^{|\alpha|}\alpha\partial^{s-i}\overline{D}_2-(-1)^{|\beta|}\beta\partial^{s-i}\overline{D}_1\Big)\Big],
\end{aligned}
$$

and hence,

$$
\begin{aligned}
\big(T\circ\mathfrak{L}^{0,\mu}_{X_F}\big)(A)&=-\mathfrak{L}^{\mu-s-1}_{X_F}(\beta)\partial^{s+1}+\mathfrak{L}^{\mu-s-1}_{X_F}(\alpha)\partial^{s}\overline{D}_1\overline{D}_2\\
&-\Big[\sum_{i=1}^{s}C^{i+1}_{s+1}\theta_1 f^{(i+1)}\Big(-(-1)^{|\beta|}\beta\partial^{s+1-i}+(-1)^{|\alpha|}\alpha\partial^{s-i}\overline{D}_1\overline{D}_2\Big)\Big]\\
&-\frac{1}{2}\Big[\sum_{i=0}^{s}C^{i+1}_{s+1}f^{(i+1)}\Big(-(-1)^{|\beta|}\beta\partial^{s-i}\overline{D}_1+(-1)^{|\alpha|}\alpha\partial^{s-i}\overline{D}_2\Big)\Big]\\
&-\frac{1}{2}\Big[\sum_{i=0}^{s}C^{i+1}_{s+1}\theta_1\theta_2 f^{(i+2)}\Big(-(-1)^{|\beta|}\beta\partial^{s-i}\overline{D}_2-(-1)^{|\alpha|}\alpha\partial^{s-i}\overline{D}_1\Big)\Big].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\Big(\mathfrak{L}^{0,\mu}_{X_F}\circ T\Big)(A)&=-\mathfrak{L}^{\mu-s-1}_{X_F}(\beta)\partial^{s+1}+\mathfrak{L}^{\mu-s-1}_{X_F}(\alpha)\partial^{s}\overline{D}_1\overline{D}_2\\
&-\Big[\sum_{i=1}^{s}C^{i+1}_{s+1}\theta_1 f^{(i+1)}\Big(-(-1)^{|\beta|}\beta\partial^{s+1-i}+(-1)^{|\alpha|}\alpha\partial^{s-i}\overline{D}_1\overline{D}_2\Big)\Big]
\end{aligned}
$$

$$-\frac{1}{2}\Big[\sum_{i=0}^{s} C_{s+1}^{i+1} f^{(i+1)}\Big(-(-1)^{|\beta|}\beta\partial^{s-i}\overline{D}_1+(-1)^{|\alpha|}\alpha\partial^{s-i}\overline{D}_2\Big)\Big]$$
$$-\frac{1}{2}\Big[\sum_{i=0}^{s} C_{s+1}^{i+1}\theta_1\theta_2 f^{(i+2)}\Big(-(-1)^{|\beta|}\beta\partial^{s-i}\overline{D}_2-(-1)^{|\alpha|}\alpha\partial^{s-i}\overline{D}_1\Big)\Big].$$

Now, clearly, T_0 is an even linear operator, further more

$$[\mathfrak{L}_{X_F}^{\lambda,\mu}, T_0] := \mathfrak{L}_{X_F}^{0,\mu}\circ T_0-(-1)^{|T_0||F|}T_0\circ\mathfrak{L}_{X_F}^{0,\mu}=\mathfrak{L}_{X_F}^{0,\mu}\circ T_0-T_0\circ\mathfrak{L}_{X_F}^{0,\mu},$$

that gives $[\mathfrak{L}_{X_F}^{\lambda,\mu}, T_0](A)=0$. By a similar calculation, we obtain the same result if we take $A=\alpha\partial^s\overline{D}_1+\beta\partial^s\overline{D}_2$, and then consider the case X_F, where $F=f\theta_2$. □

References

[1] N. Belghith, M. Ben Ammar, N. Ben Fraj, *Differential Operators on the Weighted Densities on the Supercircle $S^{1|n}$*, arXiv:1306.0101v3 [math.DG].

[2] J. Boujelben, T. Bichr, K. Tounsi, *Bilinear differential operators: Projectively equivariant symbol and quantization maps*, Tohoku. Math. J., **67(4)** (2015), 481–493.

[3] J. Boujelben, T. Bichr, Z. Saoudi, K. *Tounsi, Modules of n-ary differential operators over the orthosymplectic superalgebra* $\mathfrak{osp}(1|2)$, Proc. Indian Acad. Sci. (Math. Sci.), **131(11)** (2021), 1–26.

[4] J. Boujelben , I. Safi, Z. Saoudi, K. Tounsi, *Symmetries of modules of differential operators on the supercircle $S^{1|n}$*, Indian. J. Pure. Appl. Math, (2021). https://doi.org/10.1007/s13226-021-00164-y

[5] C. Duval, V. Ovsienko, *Space of second order linear differential operators as a module over the Lie algebra of vector fields*, Adv. in Math., **132(2)** (1997), 316–333.

[6] D. Leites, *Introduction to the theory of supermanifolds*, Usp. Math. Nauk, **35(1)** (1980), 3–57.

[7] H. Gargoubi, P. Monthonet, V. Ovsienko, *Symetries of modules of differential operators*, J. Nonlinear. Math. Phys., **12** (2005), 348–380.

[8] H. Gargoubi, *Sur la géométrie de l'espace des opérateurs différentiels linéaires sur* $\mathbb{R}$, Bull. Soc. Roy. Sci. Liège, **69(1)** (2000), 21–47.

[9] H. Gargoubi, N. Mellouli, V. Ovsienko, *Differential operators on supercircle: Conformally equivariant quantization and symbol calculus*, Lett. Math. Phys., **79(1)** (2007), 51–65.

[10] H. Gargoubi, V. Ovsienko, *Modules of differential operators on the real line*, Funct. Anal. Appl., **35(1)** (2001), 13–18.

[11] I. Safi, K. Tounsi, *Orthosymplectic supersymmetries of modules of differential operators*, Bull. Soc. Roy. Sci. Liège, **83** (2014), 35–48.

[12] I. Safi, Z. Saoudi, K. Tounsi, *Supersymmetries of modules of differential operators*, Math. Rep., **21(71), 3,** (2019), 311–337.

(received 03.10.2023; in revised form 01.03.2025; available online 30.10.2025)

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