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ON THE PÓLYA FIELDS OF SOME REAL BIQUADRATIC FIELDS

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Abstract. Let $\mathcal{O}_K$ be the ring of integers of a number field K, and let $\mathrm{Int}(\mathcal{O}_K) = \{R \in K[X] \mid R(\mathcal{O}_K) \subset \mathcal{O}_K\}$. The Pólya group is the group generated by the classes of the products of the prime ideals with the same norm. The Pólya group $\mathcal{P}_O(K)$ is trivial if and only if the $\mathcal{O}_K$-module $\mathrm{Int}(\mathcal{O}_K)$ has a regular basis if and only if K is a Pólya field. In this paper, we give the structure of the first cohomology group of units of the real biquadratic number fields $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where $d_1 > 1$ and $d_2 > 1$ are two square-free integers with $(d_1, d_2) = 1$ and the prime 2 is not totally ramified in $K/\mathbb{Q}$. We then determine the Pólya groups and the Pólya fields of K.

1. Introduction

Let $\mathrm{Int}(\mathcal{O}_K) = \{R \in K[X] \mid R(\mathcal{O}_K) \subset \mathcal{O}_K\}$ be the ring of integer-valued polynomials on $\mathcal{O}_K$. In 1919 Pólya [11] and Ostrowski [10] were interested in whether the $\mathcal{O}_K$-module $\mathrm{Int}(\mathcal{O}_K)$ has a regular basis. According to Pólya, a basis $(g_n)_{n\in\mathbb{N}}$ of $\mathrm{Int}(\mathcal{O}_K)$ is said to be a regular basis if the $\deg(g_n) = n$ for each polynomial g_n. In 1982 Zantema [16] called field K for which the $\mathcal{O}_K$-module $\mathrm{Int}(\mathcal{O}_K)$ has a regular basis, Pólya-field.

Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ be the real biquadratic number field such that d_1 and d_2 are square-free integers with $(d_1, d_2) = 1$. Let $H^1(G_K, E_K)$ be the first cohomology group of the units of K. The studies on the Pólya groups and the Pólya fields of K started in 1982 by Zantema [16], who gave a result on Pólya real biquadratic fields using $H^1(G_K, E_K)$ (see Theorem 3.2 and Proposition 3.6 below). In 2011 Leriche [7] studied the real biquadratic fields K. She gave some Pólya fields of K by using the capitulation. In [4] characterized some Pólya groups and Pólya fields of K based on the structure of $H^1(G_K, E_K)$. In 2020, Tougma [14] also used $H^1(G_K, E_K)$ to determine some non-Pólya real biquadratic fields. In 2021 Maarefparvar [8] characterized the Pólya groups of some real biquadratic fields also with $H^1(G_K, E_K)$.

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It is clear that the structure of the first cohomology group of the units of K plays an important role in providing results on Pólya groups and Pólya fields of K. In this paper, we characterize $H^1(G_K, E_K)$, the first cohomology group of the units of $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where $(d_1, d_2) = 1$ and the prime 2 is not totally ramified in $K/\mathbb{Q}$. We therefore specify the Pólya groups of K in order to determine the Pólya fields of K.

2. Notations

In this work, we accept the following notations:

1. $d_1 > 1$ and $d_2 > 1$ are two square-free integers where $(d_1, d_2) = 1$.
2. $d_3 = d_1 d_2$.
3. $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$: a real biquadratic number field.
4. $\mathcal{O}_K$: the ring of integers of K.
5. $k_i = \mathbb{Q}(\sqrt{d_i})$: the quadratic subfields of K for $i = 1, 2, 3$.
6. $\epsilon_i = x_i + y_i\sqrt{d_i}$: the fundamental unit of $\mathbb{Q}(\sqrt{d_i})$, for $i = 1, 2, 3$.
7. $N(\gamma_i) = N_i(\gamma_i) = Norm_{k_i/\mathbb{Q}}(\gamma_i)$ where $\gamma_i \in k_i$, for $i = 1, 2, 3$.
8. E_K: the unit group of K over $\mathbb{Q}$.
9. G_K: the Galois group of K over $\mathbb{Q}$.
10. $\alpha_i = N(\epsilon_i + 1) = 2(x_i + 1) \in \mathbb{Q}$ if $N\epsilon_i = 1$ otherwise $\alpha_i = 1$, for $i = 1, 2, 3$.
11. $[\alpha_i]$: the class of α_i in $\mathbb{Q}^*/\mathbb{Q}^{*2}$, for $i = 1, 2, 3$.
12. $[d_i]$: the class of d_i in $\mathbb{Q}^*/\mathbb{Q}^{*2}$, for $i = 1, 2, 3$.
13. $\widetilde{H}$: the subgroup of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ generated by the images of d_1, d_2, d_3, α_1, α_2 and α_3.
14. $H^1(G_K, E_K)$: the first cohomology group of units of K.
15. e_p: the ramification index of a prime number p in $K/\mathbb{Q}$.
16. d_K: the discriminant of K over $\mathbb{Q}$.
17. t: the number of the prime divisors of d_K.
18. $E_n = (\mathbb{Z}/2\mathbb{Z})^n$, $n \in \mathbb{N}$.

3. Preliminaries

Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, d_1 and d_2 be two square-free integers such that $(d_1, d_2) = 1$. Let $k_1 = \mathbb{Q}(\sqrt{d_1})$, $k_2 = \mathbb{Q}(\sqrt{d_2})$ and $k_3 = \mathbb{Q}(\sqrt{d_3})$ with $d_3 = d_1 d_2$.

By [3, 15] we have when $(d_1, d_2) \equiv (1, 1) \pmod 4$, then $d_K = (d_1 d_2)^2$ (note that in this case we have the prime 2 is not dividing neither the discriminant of k_1, k_2 nor k_3). When $(d_1, d_2) \equiv (1, 2), (2, 1), (1, 3), (3, 1), (3, 3) \pmod 4$, so $d_K = (4d_1 d_2)^2$ (note that here the prime 2 is dividing the discriminant of two subfield of K). When $(d_1, d_2) \equiv (2, 3), (3, 2) \pmod 4$ then $d_K = (8d_1 d_2)^2$ (note that the prime 2 is dividing both the discriminant of k_1 and k_2 and k_3). Let e_2 be the ramification index of the prime number 2 in $K/\mathbb{Q}$. The prime 2 is the only prime can be totally ramified in $K/\mathbb{Q}$.

Remark 3.1. When we say that $e_2 = 4 = [K : \mathbb{Q}]$, the prime 2 is totally ramified in $K/\mathbb{Q}$, so we have $(d_1, d_2) \equiv (2, 3), (3, 2) \pmod 4$ and thus $N\epsilon_1 \neq N\epsilon_2 = N\epsilon_3 = 1$, $N\epsilon_2 \neq N\epsilon_1 = N\epsilon_3 = 1$ or $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$.

When we say that $e_2 \neq 4$, the prime 2 is not totally ramified in $K/\mathbb{Q}$, so we have either $e_2 = 1$ or 2 and hence $(d_1, d_2) \equiv (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (3, 3) \pmod 4$, therefore we have the following possibilities : $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$, $N\epsilon_1 = N\epsilon_2 = -1 \neq N\epsilon_3 = 1$, $N\epsilon_1 = N\epsilon_2 = 1 \neq N\epsilon_3 = -1$, $N\epsilon_j = 1 \neq N\epsilon_k = N\epsilon_3 = -1$, $N\epsilon_j = -1 \neq N\epsilon_k = N\epsilon_3 = 1$, $j \neq k \in \{1, 2\}$ or $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$.

Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, $k_1 = \mathbb{Q}(\sqrt{d_1})$, $k_2 = \mathbb{Q}(\sqrt{d_2})$ and $k_3 = \mathbb{Q}(\sqrt{d_3})$ where $d_3 = d_1 d_2$. Let $\epsilon_1 = x_1 + y_1\sqrt{d_1}$, $\epsilon_2 = x_2 + y_2\sqrt{d_2}$ and $\epsilon_3 = x_3 + y_3\sqrt{d_3}$ be the fundamental unit of k_1, k_2 and k_3 respectively. Let $H^1(G_K, E_K)$ be the first cohomology group of units of K. Recall $\alpha_i \in \mathbb{Q}$ such that $\alpha_i = N(\epsilon_i + 1) = 2(x_i + 1)$ when $N\epsilon_i = 1$ else $\alpha_i = 1$, for $i = 1, 2, 3$. $\widetilde{H}$ is the subgroup of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ generated by the images of d_1, d_2, d_3, α_1, α_2 and α_3 with $d_3 = d_1 d_2$. Setzer [13] gave the following theorem which gives the structure of the first cohomology group of units of K. Keep in mind that Zantema stated the following theorem (see [16, Section 4]).

Theorem 3.2 ([13, Theorems 4 and 5]). *$\widetilde{H} \simeq H^1(G_K, E_K)$, except for the next two cases in which $\widetilde{H}$ is canonically isomorphic to a subgroup of index 2 of $H^1(G_K, E_K)$:*
1. the prime 2 is totally ramified in $K/\mathbb{Q}$, and there exists integral $z_i \in k_i$, $i \in \{1, 2, 3\}$ such that $N_1(z_1) = N_2(z_2) = N_3(z_3) = \pm 2$,

2. all the quadratic subfields k_i contain units of norm -1 and $E_K = E_{k_1} E_{k_2} E_{k_3}$.

The theorem above follows directly from the proofs of [13, Theorems 4 and 5]. The theorem above is also stated in [4, Theorem 1.6].

Proposition 3.3 ([6, Satz 1]). *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, d_1 and d_2 be two square-free integers, so we have the following eight possibilities for a system of fundamental units of E_K :*
1. $\epsilon_u, \epsilon_v, \epsilon_w$;

2. $\sqrt{\epsilon_u}, \epsilon_v, \epsilon_w$ with $N\epsilon_u = 1$;

3. $\sqrt{\epsilon_u}, \sqrt{\epsilon_v}, \epsilon_w$ such that $N\epsilon_u = N\epsilon_v = 1$;

4. $\sqrt{\epsilon_u \epsilon_v}, \epsilon_v, \epsilon_w$ such that $N\epsilon_u = N\epsilon_v = 1$;

5. $\sqrt{\epsilon_u \epsilon_v}, \sqrt{\epsilon_w}, \epsilon_v$ where $N\epsilon_u = N\epsilon_v = N\epsilon_w = 1$;

6. $\sqrt{\epsilon_u\epsilon_v}, \sqrt{\epsilon_v\epsilon_w}, \sqrt{\epsilon_w\epsilon_u}$ *where* $N\epsilon_u = N\epsilon_v = N\epsilon_w = 1$;

7. $\sqrt{\epsilon_u\epsilon_v\epsilon_w}, \epsilon_v, \epsilon_w$ *where* $N\epsilon_u = N\epsilon_v = N\epsilon_w = 1$;

8. $\sqrt{\epsilon_u\epsilon_v\epsilon_w}, \epsilon_v, \epsilon_w$ *with* $N\epsilon_u = N\epsilon_v = N\epsilon_w = -1$ *where* $\{\epsilon_u, \epsilon_v, \epsilon_w\} = \{\epsilon_3, \epsilon_1, \epsilon_2\}$.

It is worth mentioning that the proposition mentioned above was stated by Benjamin et al. [1].

The following definition is a well known definition in the Pólya group theory. We refer the reader to [2, Definition II.3.8 and Proposition II.3.9]. The reader can also consult [9, Definition 1.2], as well as [7, Definition 2.2].

Definition 3.4 ([2, Definition II.3.8]). Let $\prod_q(K)$ be the product of all prime ideals of $\mathcal{O}_K$ with the norm $q \geq 2$. The Pólya group $\mathcal{P}_O(K)$ of a number field K is the subgroup of the class group generated by the classes of the ideals $\prod_q(K)$.

The proposition below is a famous result about the notion of Pólya field and group of a number field K which is mentioned in [7, Proposition 2.3] (also consult [16, Theorem 2.3]).

Proposition 3.5. *The group* $\mathcal{P}_O(K)$ *is trivial if and only if one of the following assertions is satisfied:*
1. the field K *is a Pólya field;*

2. all the ideals $\prod_q(K)$ *are principal;*

3. the $\mathcal{O}_K$*-module* $\mathrm{Int}(\mathcal{O}_K)$ *admits a regular basis.*

The following proposition mentioned by many authors in the field, we refer the reader to [9, Proposition 1.4] and [4, Proposition 1.3].

Proposition 3.6 ([16, Section 3]). *Let* $K/\mathbb{Q}$ *be a Galois extension and* d_K *be its discriminant. Denote by* e_p *the ramification index of a prime number* p *in* K*. Then, the following sequence is exact* $1 \to H^1(G_K, E_K) \to \oplus_{p/d_K}\mathbb{Z}/e_p\mathbb{Z} \to \mathcal{P}_O(K) \to 1$*. In particular,* $\mid H^1(G_K, E_K) \mid\mid \mathcal{P}_O(K) \mid = \prod_{p|d_K} e_p$.

And thus we have the following corollary.

Corollary 3.7. K *is a Pólya field if and only if* $\mid H^1(G_K, E_K) \mid = \prod_{p|d_K} e_p$.

The proposition below is stated in [9, Proposition 1.3].

Proposition 3.8 ([5, Theorem 106]). *Let* $k = \mathbb{Q}(\sqrt{d})$ *be a quadratic number field, where* d *is a square-free integer, and let* ϵ *be the fundamental unit of* k*. Let* z *be the number of ramified prime in the extension* $k/\mathbb{Q}$*. Then,*

$$\mathcal{P}_O(k) \simeq \begin{cases} E_{z-2} & \text{if } k \text{ is real and } N(\epsilon) = 1, \\ E_{z-1} & \text{otherwise.} \end{cases}$$

Note that we can identify all quadratic Pólya fields based on the characterization provided in the proposition above.

Proposition 3.9 ([16, Example 3.3]). *Let $k = \mathbb{Q}(\sqrt{d})$ be a quadratic number field, where d is a square-free integer, and let ϵ be the fundamental unit of k. Let p and q be two distinct odd prime numbers. Then, k is a Pólya field if and only if one of the following assertions is satisfied:*
1. $d = -2$, or -1, or 2, or $-p$ with $p \equiv 3 \pmod 4$, or p.

2. $d = 2p$ and either $p \equiv 3 \pmod 4$ or $p \equiv 1 \pmod 4$ and $N(\epsilon) = 1$.

3. $d = pq$ and either $p, q \equiv 3 \pmod 4$ or $p, q \equiv 1 \pmod 4$ and $N(\epsilon) = 1$.

The proposition above stated in [4, Proposition 1.4]. The reader may also find helpful [2, Proposition 3.1].

In this paper we use a result of Setzer [13] (Theorem 3.2) and a result of Kubota [6] (Proposition 3.3), to obtain the first cohomology group of units of real biquadratic number fields $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where d_1 and d_2 are two square-free integers with $(d_1, d_2) = 1$ and $e_2 \neq 4$. Then we use the result of Zantema [16] (Proposition 3.6) to give the Pólya groups of K. Finally, we derive the Pólya fields of K.

4. The first cohomology group of units of $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where $(d_1, d_2) = 1$ and $e_2 \neq 4$

We start this section by giving the following proposition.

Proposition 4.1 ([6]). *Let $k = \mathbb{Q}(\sqrt{d})$ such that $N\epsilon = 1$ and let m denote the square-free part of the positive integer $N(\epsilon + 1)$. Then $m > 1$, m divides the discriminant of k, $m \neq d$, and $\sqrt{m\epsilon} \in k$.*

Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $(d_1, d_2) = 1$ and $d_3 = d_1 d_2$. Let ϵ_i be the fundamental unit of $\mathbb{Q}(\sqrt{d_i})$ for $i = 1, 2, 3$. Recall that $[d_i]$ be the class of d_i in $\mathbb{Q}^*/\mathbb{Q}^{*2}$, for $i = 1, 2, 3$. Note that $[m_i]$ is the class of the squarefree part m_i of $N(\epsilon_i + 1)$ in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ such that $N\epsilon_i = 1$ for $i = 1, 2, 3$.

Proposition 4.2. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, d_1 and d_2 be two square-free integers such that $(d_1, d_2) = 1$. Let ϵ_i be the fundamental unit of $\mathbb{Q}(\sqrt{d_i})$ where $N\epsilon_i = 1$ for $i = 1, 2, 3$ and we let m_i, $i = 1, 2, 3$ as we have in the Proposition 4.1. Then, we have the following results.*
1. $\sqrt{\epsilon_3} \in K$ if and only if either $m_3 = d_1$ or d_2.

2. $\sqrt{\epsilon_1 \epsilon_2} \in K$ if and only if $m_1 = m_2 = 2$.

3. $\sqrt{\epsilon_j \epsilon_3} \in K$ for $j = 1$ or 2 if and only if either ($[m_j m_3] = [d_1], [d_2]$ or $[d_3]$ with $j = 1$ or 2) or ($m_j = m_3$ for $j = 1$ or 2).

4. $\sqrt{\epsilon_1 \epsilon_2 \epsilon_3} \in K$ if and only if either ($[m_1 m_2 m_3] = [d_1], [d_2]$ or $[d_3]$) or ($[m_1 m_2] = [m_3]$).

Proof. Let $k_i = \mathbb{Q}(\sqrt{d_i})$ such that $N\epsilon_i = 1$ for $i = 1, 2, 3$ and let m_i be the squarefree part of the positive integer $N(\epsilon_i + 1)$ for $i \in \{1, 2, 3\}$.

1. ($\Longrightarrow$) we use the contrapositive. We suppose that $m_3 \neq d_1$ and d_2. Since we have $\sqrt{m_3\epsilon_3} \in k_3$, then we get that $\sqrt{\epsilon_3} \notin K$. Let $m_3 = d_1$ or d_2, and since we have $\sqrt{m_3\epsilon_3} \in k_3$ so $\sqrt{\epsilon_3} \in K$.

2. Suppose that $m_1 \neq 2$ or $m_2 \neq 2$, we have $\sqrt{m_1\epsilon_1} \in k_1$ and $\sqrt{m_2\epsilon_2} \in k_2$ and thus we get that $\sqrt{m_1\epsilon_1 m_2\epsilon_2} \in K$. Therefore, we get that $\sqrt{\epsilon_1\epsilon_2} \notin K$ since $m_1 \neq 2$ or $m_2 \neq 2$, (we recall that $m_j > 1$, m_j divides the discriminant of k_j, $m_j \neq d_j$ for $j = 1, 2$ and $(d_1, d_2) = 1$). Now let $m_1 = m_2 = 2$, since $\sqrt{m_1\epsilon_1} \in k_1$ and $\sqrt{m_2\epsilon_2} \in k_2$ then $\sqrt{\epsilon_1\epsilon_2} \in K$.

3. We suppose that $[m_j m_3] \neq [d_1], [d_2]$ and $[d_3]$, and $m_j \neq m_3$ with $j \in \{1, 2\}$. We know that $\sqrt{m_j\epsilon_j} \in k_j$ with $j = 1$ or 2 and $\sqrt{m_3\epsilon_3} \in k_3$ (see the Proposition 4.1), so $\sqrt{m_j m_3 \epsilon_j \epsilon_3} \in K$ and since $[m_j m_3] \neq [d_1], [d_2]$ and $[d_3]$, and $m_j \neq m_3$. Then, $\sqrt{\epsilon_j\epsilon_3} \notin K$ for $j = 1, 2$. Reciprocally, we suppose either $[m_j m_3] = [d_1], [d_2]$ or $[d_3]$, or $m_j = m_3$, and since we have $\sqrt{m_j\epsilon_j} \in k_j$ with $j = 1$ or 2 and $\sqrt{m_3\epsilon_3} \in k_3$. So, $\sqrt{m_j\epsilon_j}\sqrt{m_3\epsilon_3} \in K$ and thus we get that $\sqrt{\epsilon_j\epsilon_3} \in K$ for $j = 1, 2$.

4. Assuming that $[m_1 m_2 m_3] \neq [d_1], [d_2], [d_3]$, and $[m_1 m_2] \neq [m_3]$. Since $\sqrt{m_1\epsilon_1} \in k_1$, $\sqrt{m_2\epsilon_2} \in k_2$, and $\sqrt{m_3\epsilon_3} \in k_3$, so $\sqrt{m_1\epsilon_1 m_2\epsilon_2 m_3\epsilon_3} \in K$ and therefore $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$. Now, suppose either $[m_1 m_2 m_3] = [d_1], [d_2]$ or $[d_3]$, or $[m_1 m_2] = [m_3]$. As $\sqrt{m_1\epsilon_1} \in k_1$ and $\sqrt{m_2\epsilon_2} \in k_2$ and then $\sqrt{m_3\epsilon_3} \in k_3$, thus we get that $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$. □

Example 4.3. Let $K = \mathbb{Q}(\sqrt{7}, \sqrt{55})$ where $d_1 = 7$, $d_2 = 5{\cdot}11 = 55$ and $d_3 = 7{\cdot}5{\cdot}11 = 385$. The fundamental units are $\epsilon_1 = 8 + 3\sqrt{7}$, $\epsilon_2 = 89 + 12\sqrt{55}$ and $\epsilon_3 = 95831 + 4884\sqrt{385}$ such that $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$. So, $\alpha_1 = 2(x_1 + 1) = 2(8 + 1) = 2 \cdot 3^2$, $\alpha_2 = 2(x_2+1) = 2(89+1) = 2^2 \cdot 3^2 \cdot 5$ and $\alpha_3 = 2(x_3+1) = 2(95831+1) = 2^4 \cdot 3^2 \cdot 11^3$. We have $m_1 = 2$, $m_2 = 5$ and $m_3 = 11$, so $m_2 m_3 = 5 \cdot 11 = d_2$ which means that $\sqrt{\epsilon_2\epsilon_3} \in K$.

In the following lemma we give in all cases the first cohomology group of units of $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where $(d_1, d_2) = 1$ and $e_2 \neq 4$.

Lemma 4.4. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, d_1 and d_2 be two square-free integers such that $(d_1, d_2) = 1$. Then*
1. $H^1(G_K, E_K) \simeq E_2$. If

(i) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ or

(ii) $N\epsilon_1 = N\epsilon_2 = -1$, $N\epsilon_3 = 1$ and $\sqrt{\epsilon_3} \in K$.

2. $H^1(G_K, E_K) \simeq E_3$. When

(i) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$,

(ii) $N\epsilon_1 = N\epsilon_2 = -1$, $N\epsilon_3 = 1$ and $\sqrt{\epsilon_3} \notin K$,

(iii) $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = -1$ with $j \neq k = 1, 2$,

(iv) $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = 1$ and either $\sqrt{\epsilon_3} \in K$ or $\sqrt{\epsilon_k\epsilon_3} \in K$ where $e_2 \neq 4$ and $j \neq k = 1, 2$ or

(v) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_1\epsilon_2} \in K$ and $\sqrt{\epsilon_1\epsilon_3} \in K$ and $\sqrt{\epsilon_2\epsilon_3} \in K$ where $e_2 \neq 4$.

3. $H^1(G_K, E_K) \simeq E_4$. If

(i) $N\epsilon_1 = N\epsilon_2 = 1$, $N\epsilon_3 = -1$,

(ii) $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = 1$, $\sqrt{\epsilon_3} \notin K$ and $\sqrt{\epsilon_k\epsilon_3} \notin K$ where $e_2 \neq 4$ and $j \neq k = 1,2$ or

(iii) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and either $\sqrt{\epsilon_3} \in K$ or $\sqrt{\epsilon_1\epsilon_2} \in K$ or $\sqrt{\epsilon_1\epsilon_3} \in K$ or $\sqrt{\epsilon_2\epsilon_3} \in K$ or $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ where $e_2 \neq 4$.

4. $H^1(G_K, E_K) \simeq E_5$. When

(i) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_3} \notin K$ and $\sqrt{\epsilon_1\epsilon_2} \notin K$ and $\sqrt{\epsilon_1\epsilon_3} \notin K$ and $\sqrt{\epsilon_2\epsilon_3} \notin K$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$ where $e_2 \neq 4$.

Proof. Recall that $\alpha_i = N(\epsilon_i + 1) = 2(x_i + 1) \in \mathbb{Q}$ if $N\epsilon_i = 1$ otherwise $\alpha_i = 1$, for $i = 1,2,3$, also $[\alpha_i]$ is the class of α_i in $\mathbb{Q}^*/\mathbb{Q}^{*2}$, for $i = 1,2,3$. According to Proposition 3.3, we have m_i is the squarefree part of $N(\epsilon_i + 1)$ where $N\epsilon_i = 1$ for $i = 1,2,3$ with $m_i > 1$, $m_i \mid d_{k_i}$ and $m_i \neq d_i$ for $i = 1,2,3$. Therefore, $\alpha_i = m_i w^2 = N(\epsilon_i + 1) = 2(x_i + 1)$ where $N\epsilon_i = 1$ for $i = 1,2,3$. As a result, we obtain that $[\alpha_i] = [m_i w^2] = [m_i][w^2] = [m_i]$ taking into account that $N\epsilon_i = 1$ for $i = 1,2,3$.

We know that $\widetilde{H}$ is the subgroup of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ generated by the images of d_1, d_2, d_3, α_1, α_2 and α_3 with $d_3 = d_1d_2$. In the following we study in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ whether $[d_1]$, $[d_2]$, $[d_3]$, $[\alpha_1]$, $[\alpha_2]$, and $[\alpha_3]$ are linearly independents. Note that $[d_3] = [d_1d_2]$ belongs to the subgroup generated by $[d_1]$ and $[d_2]$ in $\mathbb{Q}^*/\mathbb{Q}^{*2}$, in other words $[d_3] \in \langle [d_1], [d_2] \rangle$.

1. When $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$, then $[\alpha_1] = [\alpha_2] = [\alpha_3] = 1$. So, $\widetilde{H} = \langle [d_1], [d_2] \rangle$ i.e. $\widetilde{H} \simeq E_2$. As $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$, then we have to distinguish the two following cases

(i) when $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$, i.e. $E_K = \langle -1, \epsilon_1, \epsilon_2, \sqrt{\epsilon_1\epsilon_2\epsilon_3} \rangle$ so by Theorem 3.2, we get that $\widetilde{H} \simeq H^1(G_K, E_K) \simeq E_2$.

(ii) otherwise, i.e. $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$ and thus $E_K = \langle -1, \epsilon_1, \epsilon_2, \epsilon_3 \rangle = E_{k_1}E_{k_2}E_{k_3}$ where $E_{k_1} = \langle -1, \epsilon_1 \rangle$, $E_{k_2} = \langle -1, \epsilon_2 \rangle$, and $E_{k_3} = \langle -1, \epsilon_3 \rangle$. Thus, by using the Theorem 3.2, we get that $H^1(G_K, E_K) \simeq E_3$.

2. If $N\epsilon_1 = N\epsilon_2 = -1$ and $N\epsilon_3 = 1$, then $[\alpha_1] = [\alpha_2] = 1$. Since $N\epsilon_3 = 1$, then we have the two following cases:

(i) $\sqrt{\epsilon_3} \in K$ (in other words $E_K = \langle -1, \epsilon_1, \epsilon_2, \sqrt{\epsilon_3} \rangle$), so according to Proposition 4.2, $[\alpha_3] = [m_3] = [d_1]$ or $[d_2]$ so $[\alpha_3] \in \langle [d_1], [d_2] \rangle$ i.e. $\widetilde{H} = \langle [d_1], [d_2] \rangle$. Thus, we get that $\widetilde{H} \simeq H^1(G_K, E_K) \simeq E_2$.

(ii) Otherwise, $\sqrt{\epsilon_3} \notin K$ and thus we have $m_3 \neq d_1$ and d_2 then $[\alpha_3] = [m_3] \notin \langle [d_1], [d_2] \rangle$, i.e. $[d_1], [d_2]$ and $[\alpha_3]$ are linearly independents. So, $\widetilde{H} = \langle [d_1], [d_2], [\alpha_3] \rangle$ and thus we get that $\widetilde{H} \simeq H^1(G_K, E_K) \simeq E_3$.

3. When $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = -1$ such that $j \neq k = 1,2$. Then, $[\alpha_k] = [\alpha_3] = 1$ and $[\alpha_j] = [m_j] \notin \langle [d_1], [d_2] \rangle$ (since $m_j > 1$ and m_j divides the discriminant of k_j, $m_j \neq d_j$ for $j = 1,2$) and thus $\widetilde{H} = \langle [d_1], [d_2], [\alpha_j] \rangle$. So, $\widetilde{H} \simeq H^1(G_K, E_K) \simeq E_3$.

4. If $N\epsilon_1 = N\epsilon_2 = 1$ and $N\epsilon_3 = -1$, then $[\alpha_3] = 1$. As $N\epsilon_3 = -1$ therefore $(d_1, d_2) \equiv (1,2)$ or $(2,1) \pmod 4$. To say that $\sqrt{\epsilon_1\epsilon_2} \in K$ we must have $2 \mid d_{k_1}$ and $2 \mid d_{k_2}$ which is not our case. So $\sqrt{\epsilon_1\epsilon_2} \notin K$, then $[\alpha_k] \notin \langle [d_1], [d_2], [\alpha_j] \rangle$ with $j \neq k = 1,2$. Hence, $H^1(G_K, E_K) \simeq \widetilde{H} = \langle [d_1], [d_2], [\alpha_1], [\alpha_2] \rangle \simeq E_4$.

5. We assume $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = 1$ where $e_2 \neq 4$ and $j \neq k = 1,2$. Then, $[\alpha_j] = 1$ and since $N\epsilon_k = N\epsilon_3 = 1$, then we have to distinguish the three following cases.

(i) If $\sqrt{\epsilon_3} \in K$. So, by Proposition 4.2, we have $[\alpha_3] = [m_3] = [d_1]$ or $[d_2]$ hence $[\alpha_3] \in \langle [d_1], [d_2] \rangle$. On the other hand, we have $[\alpha_k] = [m_k] \notin \langle [d_1], [d_2] \rangle$ (recall that $m_k > 1$ and m_k divides the discriminant of $\mathbb{Q}(\sqrt{d_k})$, $m_k \neq d_k$ for $k = 1,2$ see the Proposition 4.1). Thence, $\widetilde{H} = \langle [d_1], [d_2], [\alpha_k] \rangle$ and as a result we get that $\widetilde{H} \simeq H^1(G_K, E_K) \simeq E_3$.

(ii) When $\sqrt{\epsilon_k\epsilon_3} \in K$ for $k = 1$ or 2. As stated in Proposition 4.2, we get that $[m_k m_3] = [d_1], [d_2]$ or $[d_3]$, or $m_k = m_3$ and thus we get that $[\alpha_3] \in \langle [d_1], [d_2], [\alpha_k] \rangle$ so $\widetilde{H} = \langle [d_1], [d_2], [\alpha_k] \rangle$. Thus, we have $\widetilde{H} \simeq H^1(G_K, E_K) \simeq E_3$.

(iii) Otherwise, i.e. $\sqrt{\epsilon_k\epsilon_3} \notin K$, for $k = 1,2$ and $\sqrt{\epsilon_3} \notin K$ then we have $[m_k m_3] \neq [d_1], [d_2]$ and $[d_3]$, and $m_k \neq m_3$, and then $m_3 \neq d_1$ and d_2. Therefore, $[\alpha_3] \notin \langle [d_1], [d_2], [\alpha_k] \rangle$ and thus $\widetilde{H} = \langle [d_1], [d_2], [\alpha_k], [\alpha_3] \rangle$ such that $k = 1,2$. Consequently, we get that $\widetilde{H} \simeq H^1(G_K, E_K) \simeq E_4$.

6. If $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ where $e_2 \neq 4$, then we have the following cases.

(i) If $\sqrt{\epsilon_3} \in K$, then we have $[\alpha_3] \in \langle [d_1], [d_2] \rangle$. We mention here that $\sqrt{\epsilon_1\epsilon_2} \notin K$ (since in this case we have $E_K = \langle -1, \epsilon_1, \epsilon_2, \sqrt{\epsilon_3} \rangle$). Therefore, $[\alpha_k] \notin \langle [d_1], [d_2], [\alpha_j] \rangle$ with $j \neq k = 1,2$ i.e. $[d_1], [d_2], [\alpha_1]$ and $[\alpha_2]$ are linearly independents. According to Theorem 3.2, we have $\widetilde{H} \simeq H^1(G_K, E_K) \simeq E_4$.

(ii) If $\sqrt{\epsilon_j\epsilon_3} \in K$, $j = 1,2$ then we have $[\alpha_3] \in \langle [d_1], [d_2], [\alpha_j] \rangle$. On the other hand, we have $\sqrt{\epsilon_k\epsilon_j} \notin K$ with $j \neq k = 1,2$ (since in this case we have $E_K = \langle -1, \epsilon_1, \epsilon_2, \sqrt{\epsilon_j\epsilon_3} \rangle$), so $[\alpha_k] \notin \langle [d_1], [d_2], [\alpha_j] \rangle$, $j \neq k = 1,2$. So, $\widetilde{H} = \langle [d_1], [d_2], [\alpha_k], [\alpha_j] \rangle$, $j \neq k = 1,2$. Therefore, we have $\widetilde{H} \simeq H^1(G_K, E_K) \simeq E_4$.

(iii) When $\sqrt{\epsilon_1\epsilon_2} \in K$, then we get that $[\alpha_1] = [\alpha_2] = [2]$. Note that $\sqrt{\epsilon_k\epsilon_3} \notin K$ for $k = 1,2$, which means that $[\alpha_3] \notin \langle [d_1], [d_2], [\alpha_k] \rangle$. Therefore, $\widetilde{H} = \langle [d_1], [d_2], [\alpha_k], [\alpha_3] \rangle$ where $k = 1,2$. Thus, we get that $\widetilde{H} \simeq H^1(G_K, E_K) \simeq E_4$.

(iv) If $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$, so we have $([\alpha_1\alpha_2\alpha_3] = [d_1], [d_2]$ or $[d_3])$ or $([\alpha_1\alpha_2] = [\alpha_3])$. We know that, $[\alpha_1]$, $[\alpha_2]$ $[\alpha_3] \notin \langle [d_1], [d_2] \rangle$. Note that $\sqrt{\epsilon_3} \notin K$, $\sqrt{\epsilon_1\epsilon_2} \notin K$, $\sqrt{\epsilon_1\epsilon_3} \notin K$ and $\sqrt{\epsilon_2\epsilon_3} \notin K$ (since $E_K = \langle -1, \epsilon_1, \epsilon_2, \ \sqrt{\epsilon_1\epsilon_2\epsilon_3} \rangle$), hence $[\alpha_3] \notin \langle [d_1], [d_2], [\alpha_k] \rangle$ with $k \in \{1,2\}$, but $[\alpha_3] \in \langle [d_1], [d_2], [\alpha_1], [\alpha_2] \rangle$. So, $H^1(G_K, E_K) \simeq \widetilde{H} \simeq E_4$.

(v) Otherwise, i.e. $\sqrt{\epsilon_3} \notin K$, $\sqrt{\epsilon_1\epsilon_2} \notin K$, $\sqrt{\epsilon_1\epsilon_3} \notin K$, $\sqrt{\epsilon_2\epsilon_3} \notin K$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$. Then, we get that $\widetilde{H} \simeq H^1(G_K, E_K) \simeq E_5$.

7. If $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_1\epsilon_2} \in K$ and $\sqrt{\epsilon_1\epsilon_3} \in K$ and $\sqrt{\epsilon_2\epsilon_3} \in K$ where $e_2 \neq 4$, note that in this case we have $E_K = \langle -1, \sqrt{\epsilon_1\epsilon_2}, \sqrt{\epsilon_2\epsilon_3}, \sqrt{\epsilon_1\epsilon_3} \rangle$. When

$\sqrt{\epsilon_1\epsilon_2} \in K$, then $[\alpha_k] \in \langle [d_1], [d_2], [\alpha_j] \rangle$ with $j \neq k = 1, 2$ and when $\sqrt{\epsilon_j\epsilon_3} \in K$ so $[\alpha_3] \in \langle [d_1], [d_2], [\alpha_j] \rangle$, $j = 1, 2$. So, $H^1(G_K, E_K) \simeq \widetilde{H} \simeq E_3$. □

We end this section by giving examples of the first cohomology group of units of some fields $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where $(d_1, d_2) = 1$ and $e_2 \neq 4$.

Example 4.5. Let $K = \mathbb{Q}(\sqrt{29}, \sqrt{65})$ such that $d_1 = 29$, $d_2 = 5 \cdot 13 = 65$ and $d_3 = 29 \cdot 65 = 1885$. The fundamental units are $\epsilon_1 = \frac{1}{2}(5 + \sqrt{29})$, $\epsilon_2 = 8 + \sqrt{65}$ and $\epsilon_3 = 521 + 12\sqrt{1885}$ where $N\epsilon_1 = N\epsilon_2 = -1$ and $N\epsilon_3 = 1$. So, we have $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = 2(521 + 1) = 2 \cdot 522 = 2^2 \cdot 3^2 \cdot 29$. Since we have m_3, the squarefree part of the integer $N(\epsilon_3 + 1) = 2(x_3 + 1) = \alpha_3 = 29 = d_1$, then $\sqrt{\epsilon_3} \in K$. Therefore, we get that $H^1(G_K, E_K) \simeq \widetilde{H} = \langle [29], [65] \rangle$, i.e. $H^1(G_K, E_K) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Example 4.6. Put $K = \mathbb{Q}(\sqrt{65}, \sqrt{38})$ such that $d_1 = 5 \cdot 13 = 65$, $d_2 = 2 \cdot 19 = 38$ and $d_3 = 65 \cdot 38 = 2470$. We have $\epsilon_1 = 8 + \sqrt{65}$, $\epsilon_2 = 37 + 6\sqrt{38}$ and $\epsilon_3 = 2426111 + 48816\sqrt{2470}$ such that $N\epsilon_1 = -1$ and $N\epsilon_2 = N\epsilon_3 = 1$. So, $\alpha_1 = 1$, $\alpha_2 = 2(37 + 1) = 2^2 \cdot 19$ and $\alpha_3 = 2(2426111 + 1) = 2^8 \cdot 3^6 \cdot 2 \cdot 13$. Note that m_2, the squarefree part of α_2 is 19 and $m_3 = 2 \cdot 13$. Therefore, we get that both $\sqrt{\epsilon_3} \notin K$ and $\sqrt{\epsilon_2\epsilon_3} \notin K$ since $m_3 = 2 \cdot 13 \neq (65 = d_1$ and $38 = d_2)$, also $2 \cdot 13 \cdot 19 \neq (65, 38$ and $2470)$. Hence, $H^1(G_K, E_K) \simeq \widetilde{H} = \langle [5 \cdot 13], [2 \cdot 19], [19], [2 \cdot 13] \rangle \simeq E_4$.

Example 4.7. Let $K = \mathbb{Q}(\sqrt{35}, \sqrt{23})$ such that $d_1 = 5 \cdot 7 = 35$, $d_2 = 23$ and $d_3 = 35 \cdot 23 = 805$. So, we have $\epsilon_1 = 6 + \sqrt{35}$, $\epsilon_2 = 24 + 5\sqrt{23}$ and $\epsilon_3 = \frac{1}{2}(1447 + 51\sqrt{805})$ such that $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$. And thus we get that $\alpha_1 = 2(6 + 1) = 2 \cdot 7$, $\alpha_2 = 2(24 + 1) = 2 \cdot 5^2$, and $\alpha_3 = 2(\frac{1447}{2} + 1) = 1449 = 3^2 \cdot 7 \cdot 23$. Therefore, we get that $m_1 = 2 \cdot 7$, $m_2 = 2$, and $m_3 = 7 \cdot 23$, and thus $m_1m_2m_3 = (2 \cdot 7)^2 \cdot 23$, hence $[m_1m_2m_3] = [d_2] = [23]$, which means that we have $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$. Therefore, we get that $H^1(G_K, E_K) \simeq \widetilde{H} = \langle [5 \cdot 7], [23], [2 \cdot 7], [2] \rangle \simeq E_4$.

5. The Pólya groups of the real biquadratic fields $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where $(d_1, d_2) = 1$ and $e_2 \neq 4$

Theorem 5.1. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, d_1 and d_2 be two square-free integers such that $(d_1, d_2) = 1$. Let t be the number of the prime divisors of d_K. Then*

1. $\mathcal{P}_O(K) \simeq E_{t-2}$. When

(i) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ or

(ii) $N\epsilon_1 = N\epsilon_2 = -1$, $N\epsilon_3 = 1$ and $\sqrt{\epsilon_3} \in K$.

2. $\mathcal{P}_O(K) \simeq E_{t-3}$. When

(i) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$,

(ii) $N\epsilon_1 = N\epsilon_2 = -1$, $N\epsilon_3 = 1$ and $\sqrt{\epsilon_3} \notin K$,

(iii) $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = -1$ with $j \neq k = 1, 2$,

(iv) $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = 1$ *and either* $\sqrt{\epsilon_3} \in K$ *or* $\sqrt{\epsilon_k\epsilon_3} \in K$ *where* $e_2 \neq 4$ *and* $j \neq k = 1, 2$ *or*

(v) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_1\epsilon_2} \in K$ *and* $\sqrt{\epsilon_1\epsilon_3} \in K$ *and* $\sqrt{\epsilon_2\epsilon_3} \in K$ *where* $e_2 \neq 4$.

3. $\mathcal{P}_O(K) \simeq E_{t-4}$. *When*

(i) $N\epsilon_1 = N\epsilon_2 = 1$, $N\epsilon_3 = -1$,

(ii) $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = 1$, $\sqrt{\epsilon_3} \notin K$ *and* $\sqrt{\epsilon_k\epsilon_3} \notin K$ *where* $e_2 \neq 4$ *and* $j \neq k = 1, 2$ *or*

(iii) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ *and either* $\sqrt{\epsilon_3} \in K$, $\sqrt{\epsilon_1\epsilon_2} \in K$, $\sqrt{\epsilon_1\epsilon_3} \in K$, $\sqrt{\epsilon_2\epsilon_3} \in K$ *or* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ *where* $e_2 \neq 4$.

4. $\mathcal{P}_O(K) \simeq E_{t-5}$. *When*

(i) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$, $\sqrt{\epsilon_3} \notin K$, $\sqrt{\epsilon_1\epsilon_2} \notin K$, $\sqrt{\epsilon_1\epsilon_3} \notin K$, $\sqrt{\epsilon_2\epsilon_3} \notin K$ *and* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$ *where* $e_2 \neq 4$.

Proof. As the prime 2 is not totally ramified in $K/\mathbb{Q}$ and since $K/\mathbb{Q}$ is a Galois extension and d_K is its discriminant. So, according to Proposition 3.6, we have $\mid H^1(G_K, E_K) \mid\mid \mathcal{P}_O(K) \mid = \prod_{p|d_K} e_p$ where e_p is the ramification index of the prime number p in $K/\mathbb{Q}$. Thus, we get that $\mid \mathcal{P}_O(K) \mid = \frac{\prod_{p|d_K} e_p}{|H^1(G_K,E_K)|}$. Hence, $\mathcal{P}_O(K) \simeq E_{t-s}$ where s satisfies $E_s \simeq H^1(G_K, E_K)$ and $\prod_{p|d_K} e_p = 2^t$ with t is the number of prime numbers dividing d_K. By Lemma 4.4, we have when $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ or $N\epsilon_1 = N\epsilon_2 = -1$, $N\epsilon_3 = 1$ and $\sqrt{\epsilon_3} \in K$, then $H^1(G_K, E_K) \simeq E_2$. Therefore, $\mathcal{P}_O(K) \simeq E_{t-2}$. Similarly, we get the other results of the theorem. □

6. The real biquadratic Pólya fields

Recall that in [3, 15] the discriminant d_K of $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ over $\mathbb{Q}$ is explicitly determined by:
1. $d_K = (d_1d_2)^2$ when $(d_1, d_2) \equiv (1, 1) \pmod 4$.

2. $d_K = (4d_1d_2)^2$ when $(d_i, d_j) \equiv (1, 2),\ (1, 3)$ or $(3, 3) \pmod 4$ with $i \neq j = 1, 2$.
Let p, p_1, p_2, p_3, p_4 and p' be prime integers congruent to $1 \pmod 4$. Let q, q_1, q_2, q_3 and q_4 be prime integers congruent to $3 \pmod 4$.

Now, we determine the real biquadratic Pólya fields and we start by the case of $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$.

Theorem 6.1. *Let* $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, *where* d_1 *and* d_2 *are two square-free integers such that* $(d_1, d_2) = 1$. *We assume* $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ *and put* $i \neq j \in \{1, 2\}$. *Then,* K *is a Pólya field if and only if one of the following assertions is satisfied:*
1. $d_i = p_1\ d_j = p_2$,

2. $d_i = p_1\ d_j = 2$,

3. $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$ *and either* $d_i = p_1\ d_j = p_2p_3$*, or* $d_i = p_1p_2\ d_j = 2$*, or* $d_i = p_1\ d_j = 2p_2$*.*

Proof. We have $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$, so by the Theorem 5.1, we have the two following cases:
1. When $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$, then $\mathcal{P}_O(K) \simeq E_{t-2}$, where t is the number of prime divisors of d_K. So, K is a Pólya field if and only if $t = 2$. Thus, we get either $d_i = p_1 \quad d_j = p_2$ or $d_i = p_1 \quad d_j = 2$. As stated in [12], $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ is always verifying whenever we have one of the two first items of the theorem.

2. When $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$, then $\mathcal{P}_O(K) \simeq E_{t-3}$. So, K is a Pólya field if and only if $t = 3$.

(i) Assuming $(d_i, d_j) \equiv (1,1) \pmod 4$. Then, K is a Pólya field if and only if $d_i = p_1\ d_j = p_2p_3$.

(ii) Now we assume that $(d_i, d_j) \equiv (1,2) \pmod 4$. So, we have either $d_i = p_1p_2\ d_j = 2$ or $d_i = p_1\ d_j = 2p_2$. $\square$

THEOREM 6.2. *Let* $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ *where* d_1 *and* d_2 *are two square-free integers such that* $(d_1, d_2) = 1$*. Assuming* $N\epsilon_j \neq N\epsilon_i = N\epsilon_3 = -1$ *such that* $i \neq j = 1, 2$*. So,* K *is a Pólya field if and only if one of the following assertions is satisfied:*
(i) $d_i = p_1 \quad d_j = p_2p_3$*,* *(ii)* $d_i = p_1 \quad d_j = 2p_2$*,* *(iii)* $d_i = 2 \quad d_j = p_1p_2$*.*
We assume $N\epsilon_1 = N\epsilon_2 = 1$ *and* $N\epsilon_3 = -1$*. Then,* K *is a Pólya field if and only if one of the following assertions is satisfied:*
(i) $d_i = p_1p_2 \quad d_j = p_3p_4$*,* *(ii)* $d_i = 2p_1 \quad d_j = p_2p_3$*.*

Proof. We have $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that d_1 and d_2 are two square-free integers such that $(d_1, d_2) = 1$.
1. As $N\epsilon_j \neq N\epsilon_i = N\epsilon_3 = -1$ such that $i \neq j = 1, 2$. So, by Theorem 5.1, we have $\mathcal{P}_O(K) \simeq E_{t-3}$. Then, K is a Pólya field if and only if $t = 3$.

(i) We suppose $(d_i, d_j) \equiv (1,1) \pmod 4$. Thus, by K.S. Williams [15], we get $d_K = (p_1p_2p_3)^2$. So, K is a Pólya field if and only if $d_i = p_1 \quad d_j = p_2p_3$.

(ii) Now we assume $(d_i, d_j) \equiv (1,2) \pmod 4$. Then, K is a Pólya field if and only if $d_i = p_1 \quad d_j = 2p_2$.

(iii) And when $(d_i, d_j) \equiv (2,1) \pmod 4$ we get that $d_i = 2 \quad d_j = p_1p_2$.

2. Assuming $N\epsilon_1 = N\epsilon_2 = 1$ and $N\epsilon_3 = -1$, so we have $\mathcal{P}_O(K) \simeq E_{t-4}$. Then, K is a Pólya field if and only if $t = 4$. So, we get either $d_i = p_1p_2 \quad d_j = p_3p_4$, or $d_i = 2p_1 \quad d_j = p_2p_3$. $\square$

THEOREM 6.3. *Let* $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$*, where* d_1 *and* d_2 *are two square-free integers such that* $(d_1, d_2) = 1$*. Put* $i \neq j = 1, 2$*. We assume that* $N\epsilon_1 = N\epsilon_2 = -1$ *and* $N\epsilon_3 = 1$*. When* $\sqrt{\epsilon_3} \in K$*, then* K *is a Pólya field if and only if one of the following conditions holds:*
(i) $d_i = p_1 \quad d_j = p_2$*,* *(ii)* $d_i = p_1 \quad d_j = 2$*.*
And when $\sqrt{\epsilon_3} \notin K$*, then* K *is a Pólya field if and only if one of the following conditions holds:*
(i) $d_i = p_1p_2 \quad d_j = p_3$*,* *(ii)* $d_i = p_1p_2 \quad d_j = 2$*,* *(iii)* $d_i = p_1 \quad d_j = 2p_2$*.*

Proof. As $N\epsilon_1 = N\epsilon_2 = -1$ and $N\epsilon_3 = 1$. Then, we have to distinguish the following cases.
1. If $\sqrt{\epsilon_3} \in K$, then according to Theorem 5.1, we get that $\mathcal{P}_O(K) \simeq E_{t-2}$. Thence, K is a Pólya field if and only if $t = 2$.

(i) We assume $(d_i, d_j) \equiv (1,1) \pmod 4$. Then, we get that $d_i = p_1 \quad d_j = p_2$.

(ii) Now we suppose that $(d_i, d_j) \equiv (1,2) \pmod 4$. Then, $d_i = p_1 \quad d_j = 2$.

2. Otherwise i.e. $\sqrt{\epsilon_3} \notin K$ then as stated in Theorem 5.1, we get that $\mathcal{P}_O(K) \simeq E_{t-3}$. So, K is a Pólya field if and only if $t = 3$.

(i) Assuming $(d_i, d_j) \equiv (1,1) \pmod 4$, so we get that $d_i = p_1p_2 \quad d_j = p_3$.

(ii) When $(d_i, d_j) \equiv (1,2) \pmod 4$. Then, K is a Pólya field if and only if either $d_i = p_1p_2 \quad d_j = 2$ or $d_i = p_1 \quad d_j = 2p_2$. □

Remark 6.4. Building upon the results proven in the three previous theorems, we note that we give the Pólya fields in each case without mentioning that $e_2 \neq 4$. Moreover, all the cases studied were $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$, $N\epsilon_1 = N\epsilon_2 = -1 \neq N\epsilon_3 = 1$, $N\epsilon_1 = 1 \neq N\epsilon_2 = N\epsilon_3 = -1$, $N\epsilon_2 = 1 \neq N\epsilon_1 = N\epsilon_3 = -1$, and $N\epsilon_1 = N\epsilon_2 = 1 \neq N\epsilon_3 = -1$. We mention that, there is no need to add the condition of $e_2 \neq 4$ since it is implicitly we have that $e_2 \neq 4$, which means that the prime 2 is not totally ramified in $K/\mathbb{Q}$ in all mentioned cases above. We recommend the reader to refer to the beginning of the preliminaries section, as well as Remark 3.1, for further details.

On the other hand, we would like to mention that in the upcoming theorems, we are going to determine the Pólya fields in the following cases: $N\epsilon_j \neq N\epsilon_i = N\epsilon_3 = 1$ where $i \neq j = 1,2$ and $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$. Note that in these cases we can have either $e_2 = 4$ or $e_2 \neq 4$. As we are specifically interested in the case where the prime 2 is not totally ramified in $K/\mathbb{Q}$, so it is necessary to add the condition $e_2 \neq 4$.

Theorem 6.5. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where d_1 and d_2 are two square-free integers such that $(d_1, d_2) = 1$, and let $N\epsilon_j \neq N\epsilon_i = N\epsilon_3 = 1$ where $e_2 \neq 4$ and $i \neq j = 1,2$.*

Assuming either $\sqrt{\epsilon_3} \in K$ or $\sqrt{\epsilon_i\epsilon_3} \in K$. So, K is a Pólya field if and only if one of the following assertions is satisfied:

(i) $d_i = q \quad d_j = p$,

(ii) $d_i = p_1p_2, q_1q_2 \quad d_j = p$,

(iii) $d_i = p_1p_2,\ q_1q_2 \quad d_j = 2$,

(iv) $d_i = 2p_1,\ 2q_1 \quad d_j = p$.

Now we assume $\sqrt{\epsilon_3} \notin K$ and $\sqrt{\epsilon_i\epsilon_3} \notin K$. Then, K is a Pólya field if and only if one of the following conditions holds:

(i) $d_i = q_1p_1\ d_j = p_2$,

(ii) $d_i = q_1\ d_j = p_1p_2$,

(iii) $d_i = p_1p_2, q_1q_2\ d_j = pp'$,

(iv) $d_i = p_1p_2p_3, q_1q_2p'\ d_j = p$,

(v) $d_i = p_1p_2,\ q_1q_2\ d_j = 2p$,

(vi) $d_i = p_1p_2p_3,\ q_1q_2p\ d_j = 2$,

(vii) $d_i = 2p_1p_2,\ 2p_1q_1,\ 2q_1q_2\ d_j = p$,

(viii) $d_i = 2p,\ 2q\ d_j = p_1p_2$.

Proof. As $N\epsilon_j \neq N\epsilon_i = N\epsilon_3 = 1$ and $e_2 \neq 4$ with $i \neq j = 1,2$. Then, we have the two following cases.

1. When either$\sqrt{\epsilon_3} \in K$ or $\sqrt{\epsilon_i\epsilon_3} \in K$ for $i \in \{1,2\}$, so according to the Theorem 5.1, we get that $\mathcal{P}_O(K) \simeq E_{t-3}$. Thence, K is a Pólya field if and only if $t = 3$. Therefore, we have the following cases:

(i) $(d_i, d_j){\equiv}(1,1) \pmod 4$, then K is a Pólya field if and only if $d_i{=}p_1p_2, q_1q_2\ d_j{=}p$.

(ii) $(d_i, d_j){\equiv}(1,2) \pmod 4$. So, $d_i{=}p_1p_2,\ q_1q_2\ d_j{=}2$.

(iii) $(d_i, d_j){\equiv}(2,1) \pmod 4$. Then, we get that $d_i{=}2p_1,\ 2q_1\ d_j{=}p$.

(iv) $(d_i, d_j){\equiv}(3,1) \pmod 4$. Therefore, K is a Pólya field if and only if $d_i{=}q\ d_j{=}p$.

2. And when both $\sqrt{\epsilon_3}$ and $\sqrt{\epsilon_i\epsilon_3} \notin K$ with $i \neq j{=}1,2$. By the Theorem 5.1, we get that $\mathcal{P}_O(K) \simeq E_{t-4}$. So, K is a Pólya field if and only if $t{=}4$.

(i) We put $(d_i, d_j){\equiv}(1,1) \pmod 4$. Then, K is a Pólya field if and only if either $d_i{=}p_1p_2, q_1q_2\ d_j{=}pp'$ or $d_i{=}p_1p_2p_3, q_1q_2p'\ d_j{=}p$.

(ii) When $(d_i, d_j){\equiv}(1,2) \pmod 4$. Thus, we get that either $d_i{=}p_1p_2,\ q_1q_2\ d_j{=}2p$ or $d_i{=}p_1p_2p_3,\ q_1q_2p\ d_j{=}2$.

(iii) Let $(d_i, d_j){\equiv}(2,1) \pmod 4$. Then, we have either $d_i{=}2p_1p_2,\ 2p_1q_1,\ 2q_1q_2\ d_j{=}p$ or $d_i{=}2p,\ 2q\ d_j{=}p_1p_2$.

(iv) When $(d_i, d_j){\equiv}(3,1) \pmod 4$, so we get that K is a Pólya field if and only if either $d_i{=}q_1p_1\ d_j{=}p_2$, or $d_i{=}q_1\ d_j{=}p_1p_2$. □

In the following theorem we give the Pólya fields of K such that $N\epsilon_1{=}N\epsilon_2{=}N\epsilon_3{=}1$ and $\sqrt{\epsilon_1\epsilon_2} \in K$ and $\sqrt{\epsilon_1\epsilon_3} \in K$ and $\sqrt{\epsilon_2\epsilon_3} \in K$ (i.e. $E_K{=}\langle -1, \sqrt{\epsilon_1\epsilon_2}, \sqrt{\epsilon_2\epsilon_3}, \sqrt{\epsilon_1\epsilon_3}\rangle$) where $e_2 \neq 4$.

Theorem 6.6. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, d_1 and d_2 be two square-free integers such that $(d_1, d_2) = 1$ and let $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_1\epsilon_2} \in K$ and $\sqrt{\epsilon_1\epsilon_3} \in K$ and $\sqrt{\epsilon_2\epsilon_3} \in K$ where $e_2 \neq 4$. Then, K is a Pólya field if and only if $d_1 = q_1$ and $d_2 = q_2$.*

Proof. Since $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_1\epsilon_2} \in K$ and $\sqrt{\epsilon_2\epsilon_3} \in K$ and $\sqrt{\epsilon_1\epsilon_3} \in K$ such that $e_2 \neq 4$, so by Theorem 5.1 we have $\mathcal{P}_O(K) \simeq E_{t-3}$. Therefore, K is a field of Pólya if and only if $t = 3$. If $(d_i, d_j) \equiv (3,3) \pmod 4$ with $i \neq j = 1,2$ therefore $d_K = (4d_1d_2)^2$ then we find that $d_1 = q_1$ and $d_2 = q_2$. If $(d_i, d_j) \equiv (1,1) \pmod 4$, we know that $d_K = (d_1d_2)^2$ and since we have $t = 3$ and $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ then we find that this case can not occur. Similarly for the cases of $(d_i, d_j) \equiv (1,2) \pmod 4$ and $(d_i, d_j) \equiv (1,3) \pmod 4$ with $i \neq j \in \{1.2\}$. □

In the following theorem we give the Pólya fields of K in the two following cases:
1. $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_3} \in K$ or $\sqrt{\epsilon_1\epsilon_2} \in K$ or $\sqrt{\epsilon_2\epsilon_3} \in K$ or $\sqrt{\epsilon_1\epsilon_3} \in K$ or $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ where $e_2 \neq 4$, in other words $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $E_K = \langle -1, \epsilon_1, \epsilon_2, \sqrt{\epsilon_3}\rangle$ or $E_K = \langle -1, \sqrt{\epsilon_1\epsilon_2}, \epsilon_2, \epsilon_3\rangle$ or $E_K = \langle -1, \epsilon_1, \epsilon_2, \sqrt{\epsilon_2\epsilon_3}\rangle$ or $E_K = \langle -1, \epsilon_1, \epsilon_2, \sqrt{\epsilon_1\epsilon_3}\rangle$ or $E_K = \langle -1, \epsilon_1, \epsilon_2, \sqrt{\epsilon_1\epsilon_2\epsilon_3}\rangle$ respectively.

2. $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_3} \notin K$ and $\sqrt{\epsilon_1\epsilon_2} \notin K$ and $\sqrt{\epsilon_1\epsilon_3} \notin K$ and $\sqrt{\epsilon_2\epsilon_3} \notin K$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$, i.e. $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $E_K = \langle -1, \epsilon_1, \epsilon_2, \epsilon_3\rangle$.

Theorem 6.7. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, d_1 and d_2 be two square-free integers such that $(d_1, d_2) = 1$ and let $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and then $e_2 \neq 4$.*

We suppose either $\sqrt{\epsilon_3} \in K$ or $\sqrt{\epsilon_1\epsilon_2} \in K$ or $\sqrt{\epsilon_1\epsilon_3} \in K$ or $\sqrt{\epsilon_2\epsilon_3} \in K$ or $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$. Then, K is a Pólya field if and only if one of the following conditions holds:

1. $d_i = p_1q_1 \quad d_j = q_2$
2. $d_i = p_1p_2 \quad d_j = q_1$
3. $d_i = p_1p_2 \quad d_j = p_3p_4, q_1q_2$
4. $d_i = q_1q_2 \quad d_j = p_1p_2, q_3q_4$
5. $d_i = p_1p_2 \quad d_j = 2p, 2q$
6. $d_i = q_1q_2 \quad d_j = 2p, 2q.$

Now we assume that $\sqrt{\epsilon_3} \notin K$ and $\sqrt{\epsilon_1\epsilon_2} \notin K$ and $\sqrt{\epsilon_1\epsilon_3} \notin K$ and $\sqrt{\epsilon_2\epsilon_3} \notin K$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$. So, K is a Pólya field if and only if one of the following conditions holds:

1. $d_i = p_1p_2 \quad d_j = p_3p_4p_5, q_1q_2p$
2. $d_i = q_1q_2 \quad d_j = p_1p_2p_3, q_3q_4p$
3. $d_i = p_1p_2 \quad d_j = 2p_3p_4, 2q_1q_2, 2pq$
4. $d_i = q_1q_2 \quad d_j = 2p_1p_2, 2q_3q_4, 2pq$
5. $d_i = p_1p_2p_3 \quad d_j = 2p, 2q$
6. $d_i = q_1q_2p_1 \quad d_j = 2p, 2q$
7. $d_i = p_1q_1 \quad d_j = p_2q_2$
8. $d_i = q_1 \quad d_j = p_1p_2q_2, q_2q_3q_4$
9. $d_i = p_1p_2, q_1q_2 \quad d_j = pq$
10. $d_i = p_1p_2p_3, q_1q_2p \quad d_j = q.$

Proof. As $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $e_2 \neq 4$. Then, we have to distinguish the two following cases :
1. We assume either $\sqrt{\epsilon_3} \in K$, $\sqrt{\epsilon_1\epsilon_2} \in K$ or $\sqrt{\epsilon_1\epsilon_3} \in K$ or $\sqrt{\epsilon_2\epsilon_3} \in K$ or $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$. So, as stated in Theorem 5.1, we get that $\mathcal{P}_O(K) \simeq E_{t-4}$. Then, K is a Pólya field if and only if $t = 4$. Then, we have the following cases :

(i) $(d_i, d_j) \equiv (1, 1) \pmod 4$. Then, K is a Pólya field if and only if either $d_i = q_1q_2 \quad d_j = p_1p_2, q_3q_4$ or the third item.

(ii) $(d_i, d_j) \equiv (1, 2) \pmod 4$. Thus, we get the items 5. and 6.

(iii) $(d_i, d_j) \equiv (3, 3) \pmod 4$. Then, we have $d_K = (4d_id_j)^2$, so $d_i = p_1q_1 \quad d_j = q_2$.

(iv) $(d_i, d_j) \equiv (1, 3) \pmod 4$, then $d_K = (4d_id_j)^2$. Consequently, we get that K is a Pólya field if and only if $d_i = p_1p_2 \quad d_j = q_1$.

2. Now we assume $\sqrt{\epsilon_3} \notin K$ and $\sqrt{\epsilon_1\epsilon_2} \notin K$ and $\sqrt{\epsilon_1\epsilon_3} \notin K$ and $\sqrt{\epsilon_2\epsilon_3} \notin K$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$. Again, by Theorem 5.1, we get that $\mathcal{P}_O(K) \simeq E_{t-5}$. Thus, K is a Pólya field if and only if $t = 5$. We distinguish the following cases.

(i) We suppose that $(d_i, d_j) \equiv (1, 1) \pmod 4$. Then, K is a Pólya field if and only if $d_i = p_1p_2 \quad d_j = p_3p_4p_5, q_1q_2p$, or $d_i = q_1q_2 \quad d_j = p_1p_2p_3, q_3q_4p$

(ii) When $(d_i, d_j) \equiv (1, 2) \pmod 4$. Thus, we get either $d_i = q_1q_2$ $d_j = 2p_1p_2, 2q_3q_4, 2pq$ or the items 3., 5. and 6.

(iii) We assume $(d_i, d_j) \equiv (3, 3) \pmod 4$. So, we get that K is a Pólya field if and only if either $d_i = p_1q_1 \quad d_j = p_2q_2$ or $d_i = q_1 \quad d_j = p_1p_2q_2, q_2q_3q_4$.

(iv) If $(d_i, d_j) \equiv (1, 3) \pmod 4$, then $d_K = (4d_id_j)^2$. Therefore, we get K is a Pólya field if and only if either $d_i = p_1p_2, q_1q_2 \quad d_j = pq$ or $d_i = p_1p_2p_3, q_1q_2p \quad d_j = q$. □

7. Conclusion

Let $K = \mathbb{Q}(\sqrt{d}_1, \sqrt{d}_2)$, where $d_1 > 1$ and $d_2 > 1$ are two square-free integers with $(d_1, d_2) = 1$ and $d_3 = d_1 d_2$. Let $k_1 = \mathbb{Q}(\sqrt{d_1})$, $k_2 = \mathbb{Q}(\sqrt{d_2})$ and $k_3 = \mathbb{Q}(\sqrt{d_3})$ be three quadratic subfields of $K = k_1 k_2 = \mathbb{Q}(\sqrt{d_1})\mathbb{Q}(\sqrt{d_2}) = \mathbb{Q}(\sqrt{d}_1, \sqrt{d}_2)$.

As a conclusion, we can say that in each case where k_1 and k_2 are Pólya fields, then $K = k_1 k_2$ is a Pólya field taking into account that, it is not necessary that k_3 must be a Pólya field.

As an example: $k_1 = \mathbb{Q}(\sqrt{p_1 p_2})$ and $k_2 = \mathbb{Q}(\sqrt{q_1})$ where $N\epsilon_1 = N\epsilon_2 = 1$ are Pólya fields (see Proposition 3.9), but $k_3 = \mathbb{Q}(\sqrt{p_1 p_2 q_1})$ with $N\epsilon_3 = 1$ is not a Pólya field. From the head of the prevoius theorem it follows that $K = \mathbb{Q}(\sqrt{p_1 p_2}, \sqrt{q_1})$ such that $d_1 = p_1 p_2$ and $d_2 = q_1$ with $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ is a Pólya field.

We mention that we can establish that k_1 or k_2 is not a Pólya field, and that both k_1 and k_2 are not Pólya fields, but $K = k_1 k_2$ is a Pólya field.

As an example: $k_1 = \mathbb{Q}(\sqrt{p_1 q_1})$ is not a Pólya field and $k_2 = \mathbb{Q}(\sqrt{q_2})$ is a Pólya field (see Proposition 3.9). According to the above theorem, $K = \mathbb{Q}(\sqrt{p_1 q_1}, \sqrt{q_2})$ such that $d_1 = p_1 q_1$ and $d_2 = q_2$ with $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ is a Pólya field.

Another example: $k_1 = \mathbb{Q}(\sqrt{p_1 q_1})$ and $k_2 = \mathbb{Q}(\sqrt{p_2 q_2})$ are not Pólya fields. But it follows from the above theorem that $K = \mathbb{Q}(\sqrt{p_1 q_1}, \sqrt{p_2 q_2})$ such that $d_1 = p_1 q_1$ and $d_2 = q_2$ with $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ is a Pólya field.

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