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ON P_p–STATISTICAL EXHAUSTIVENESS

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Abstract. In this paper we study with statistical convergence in the sense of the power series method which is not comparable with statistical convergence. Using this notion, we introduce the concepts of P_p–statistical exhaustiveness and weak P_p–statistical exhaustiveness. Also, we study several types of convergence of sequences of functions between two metric spaces and we obtain more general results from the concepts of exhaustiveness and the strong uniform convergence on a bornology.

1. Introduction

In 2008, V. Gregoriades and N. Papanastassiou [8] defined the concept of exhaustiveness and study relations with certain types of convergence. Using the notion of statistical convergence which is stronger than usual convergence, Caserta and Kočinac [4] introduced the statistical version of that notion. Also, Beer and Levi [2] defined a new topology called topology of strong uniform convergence on a bornology $\mathfrak{B}$ on X. In this paper we use the concept of power series method which was defined by Boos [3]. Using this notion, we introduce the concepts of P_p–statistical exhaustiveness and weak P_p–statistical exhaustiveness. Also, we study several types of convergence of sequences of functions between two metric spaces. Before giving the main results, we give some basic concepts.

The natural density of $E \subset \mathbb{N}$ which is defined as

$$d(E) = \lim_n \frac{1}{n} card \{k \in E : k \leq n\}$$

provided that the limit exists is crucial to defining statistical convergence.

It is obvious that, if the natural density of E exists then $d(E)$ provide the inequality $0 \leq d(E) \leq 1$ and $d(\mathbb{N} \setminus E) = 1 - d(E)$. Also, a set $E \subset \mathbb{N}$ is said to be statistically dense if $d(E) = 1$.

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With the help of natural density, the concept of statistical convergence was introduced as follows.

A sequence $\xi = (\xi_k)$ is said to be statistically convergent to η if the set of $B(\varepsilon) := \{k \in \mathbb{N} : |\xi_k - \eta| \geq \varepsilon\}$ has natural density zero; that is,

$$d(B(\varepsilon)) = \lim_n \frac{1}{n} card\{k \leq n : |\xi_k - \eta| \geq \varepsilon\} = 0$$

(see [7,15]). Then we write $st - \lim \xi = \eta$.

Also, the definition of statistical convergence in topological spaces can be given as follows.

A sequence $\xi = (\xi_k)$ in a topological space X is said to be statistically convergent to $\alpha \in X$ if for every neighborhood U of α, $d(\{k \in \mathbb{N} : \xi_k \notin U\}) = 0$ [13]. In this case we write $(\xi_k) \overset{st-\tau}{\to} \alpha$ where τ is a topology on X.

In recent years, the concept of power series method has become an important research area for researchers [5, 6, 16]. Now, we recall the concept of power series methods.

Let (p_n) be a non-negative real sequence such that $p_0 > 0$ and the corresponding power series

$$p(r) = \sum_{n=0}^{\infty} p_n r^n$$

has radius of convergence R with $0 < R < \infty$. If the limit

$$\lim_{r \to R^-} \frac{1}{p(r)} \sum_{n=0}^{\infty} x_n p_n r^n$$

exists for all $0 < r < R$, then we say $x = (x_n)$ is convergent in the sense of power series method P_p [3, 12, 14]. If the above limit exists and is L then we denoted by

$$P_p - \lim x = \lim_{r \to R^-} \frac{1}{p(r)} \sum_{n=0}^{\infty} x_n p_n r^n = L.$$

A power series method is regular if $P_p - \lim x = L$ provided that $\lim x = L$. Also, the following theorem characterize the regularity of a power series method.

Theorem 1.1 ([3]). *A power series method P_p is regular if and only if for any $n \in \mathbb{N}_0$ and* $0 < r < R$, $\lim_{r \to R^-} \frac{p_n r^n}{p(r)} = 0$.

The concept of P_p–density have been recently introduced by Ünver and Orhan [16]. They also defined the concept of statistical convergence in the sense of power series method. Now, we recall this definition.

Definition 1.2 ([16]). Let P_p be a regular power series method and let $K \subset \mathbb{N}_0$. If the limit

$$d_{P_p}(K) := \lim_{r \to R^-} \frac{1}{p(r)} \sum_{n \in K} p_n r^n$$

exists for all $0 < r < R$, then $d_{P_p}(K)$ is called the P_p–density of K.

It is obvious that, if the P_p–density of K exists then $d_{P_p}(K)$ provide the inequality $0 \leq d_{P_p}(K) \leq 1$. Also, a set $K \subset \mathbb{N}$ is said to be P_p–dense if $d_{P_p}(K) = 1$.

We can list some basic properties of P_p–density as follow.

Lemma 1.3. *Let P_p be a regular power series method. Then we have the following from the definition of P_p–density of subsets of $\mathbb{N}$:*
(i) $d_{P_p}(\mathbb{N}_0) = 1$,

(ii) if $A \subset B$ *then* $d_{P_p}(A) \leq d_{P_p}(B)$,

(iii) if A *has* P_p*–density then* $d_{P_p}(\mathbb{N}_0 \diagdown A) = 1 - d_{P_p}(A)$,

(iv) $d_{P_p}(A \cup B) \leq d_{P_p}(A) + d_{P_p}(B)$,

(v) if $d_{P_p}(A) = 1$ *and* $d_{P_p}(B) = 1$ *then* $d_{P_p}(A \cap B) = 1$.

Definition 1.4 ([16]). Let P_p be a regular power series method and $x = (x_n)$ be a real sequence. Then x is said to be P_p–statistically convergent to γ if for any $\varepsilon > 0$ and all $0 < r < R$

$$\lim_{r \to R^-} \frac{1}{p(r)} \sum_{n \in K_\varepsilon} p_n r^n = 0$$

where $K_\varepsilon = \{n \in \mathbb{N}_0 : |x_n - \gamma| \geq \varepsilon\}$, that is $d_{P_p}(K_\varepsilon) = 0$ for any $\varepsilon > 0$. Then we write $st_{P_p} - \lim x = \gamma$.

Also, we can give the following.

A sequence $x = (x_n)$ in a topological space X is said to be P_p–statistically convergent to $\gamma \in X$ if for every neighborhood U of γ,

$$d_{P_p}(\{n \in \mathbb{N} : x_n \notin U\}) = 0.$$

In this case, we write $(x_n) \overset{st_{P_p}-\tau}{\to} \gamma$ where τ is a topology on X.

The example below show us that statistical convergence and P_p–statistical convergence can not be compared.

Example 1.5. Let P_p be a power series method that is given by

$$p_n = \begin{cases} \frac{1}{2}, & n \text{ is prime,} \\ 0, & \text{otherwise,} \end{cases}$$

and consider the sequence $x = (x_n)$ defined by

$$x_n = \begin{cases} 0, & n \text{ is prime,} \\ n^2, & \text{otherwise.} \end{cases}$$

In this case we can get that

$$\lim_{r \to R^-} \frac{1}{p(r)} \sum_{n \in \{n \in \mathbb{N}_0 : |x_n| \geq \varepsilon\}} p_n r^n = 0.$$

So, $x = (x_n)$ is P_p–statistically convergent to 0. Also, we can see that x is not statistical convergent.

Now, we consider the sequence $x = (x_n)$ defined by

$$x_n = \begin{cases} n^3, & n \text{ is prime,} \\ 0, & \text{otherwise.} \end{cases}$$

In this case, we can get that

$$\lim_{r\to R^-}\frac{1}{p(r)}\sum\nolimits_{n\in\{n\in\mathbb{N}_0:|x_n|\geq\varepsilon\}}p_n r^n \neq 0.$$

So, we can see that while the sequence $x=(x_n)$ is statistically convergent to zero, it is not P_p–statistically convergent.

Before we give the our main results, we recall some basic information which is used in the paper.

2. Bornology

Let (X,ω) and (Y,ρ) be metric spaces, $x\in X$ and $\varepsilon>0$, and let $B_\omega(x,\varepsilon)=\{t\in X:\omega(x,t)<\varepsilon\}$ denote the ball with radius ε centered at x according to the metric ω. Let $A^\varepsilon:=\bigcup_{a\in A}B_\omega(a,\varepsilon)$ denote the ε–enlargement of A.

A bornology on a set X is a family $\mathfrak{B}$ of subsets of X satisfying the following axioms:

(B.1) $\mathfrak{B}$ is a covering of X, i.e. $X=\bigcup_{B\in\mathfrak{B}}B$;

(B.2) $\mathfrak{B}$ is hereditary under inclusion, i.e. if $A\in\mathfrak{B}$ and B is a subset of X contained in A, then $B\in\mathfrak{B}$;

(B.3) $\mathfrak{B}$ is closed under finite union [9,10].

A pair $(X,\mathfrak{B})$ consisting of a set X and a bornology $\mathfrak{B}$ on X is called a bornological set. A base of a bornology $\mathfrak{B}$ on X is any subfamily $\mathfrak{B}_0$ of $\mathfrak{B}$ such that every element of $\mathfrak{B}$ is contained in an element of $\mathfrak{B}_0$. A family $\mathfrak{B}_0$ of subsets of X is a base for a bornology on X if and only if $\mathfrak{B}_0$ covers X and every finite union of elements of $\mathfrak{B}_0$ is contained in a member of $\mathfrak{B}_0$. Then the collection of those subsets of X which are contained in an element of $\mathfrak{B}_0$ defines a bornology $\mathfrak{B}$ on X having $\mathfrak{B}_0$ as a base. A base is called closed (compact) if all its members are closed (compact) subsets of X. We know that, the family $\mathfrak{A}$ of all nonempty finite subsets of X is a bornology on X. Also, it is the smallest bornology on X and has a closed (compact) base. Another bornology that will be used in this article is the collection $\mathcal{K}_r$ of all nonempty relatively compact subsets (i.e. subsets with compact closure).

Beer and Levi [2] presented a new uniformizable topology on the set of all functions on Y^X which is the set of all functions from X to Y that preserves strong uniform continuity on a given bornology $\mathfrak{B}$ with closed base on X. This topology is in general finer than the classical topology of uniform convergence on $\mathfrak{B}$, but reduces to it on the class of functions that are strongly uniformly continuous on $\mathfrak{B}$.

Definition 2.1 ([2]). Let (X,ω) and (Y,ρ) be metric spaces and let $\mathfrak{B}$ be a bornology with closed base on X. Then the topology of strong uniform convergence $\tau^s_{\mathfrak{B}}$ on $\mathfrak{B}$ is determined by a uniformity on Y^X having as a base all sets of the form

$$[B;\varepsilon]^s:=\left\{(f,g):\exists\delta>0,\ \forall x\in B^\delta,\ \rho(f(x),g(x))<\varepsilon\right\}\quad(B\in\mathfrak{B},\ \varepsilon>0).$$

The topology $\tau_{\mathfrak{B}}^{s}$ is stronger than the topology of uniform convergence on elements of $\mathfrak{B}$.

Note that, if we take $\mathfrak{B} = \mathcal{F}$, we get the standard uniformity for the topology of pointwise convergence.

3. Exhaustiveness

Let (X, ω) and (Y, ρ) be metric spaces, $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions from X to Y and g be a function from X to Y. This situation is denoted by $g_n, g \in Y^X$.

In 2008, V. Gregoriades and N. Papanastassiou [8] defined the concept of exhaustiveness as follows.

Definition 3.1 ([8]). The sequence $(g_n)_{n\in\mathbb{N}}$ in Y^X is called exhaustive at a point $x \in X$, if for each $\varepsilon > 0$ there are $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $t \in B_\omega(x, \delta)$ and all $n \geq n_0$ we have $\rho(g_n(t), g_n(x)) < \varepsilon$, where $B_\omega(x, \delta)$ is the ball of radius δ centered x with the metric ω.

Then, using the concept of statistical convergence Caserta and Kočinac, defined the statistical version of this notion as follows.

Definition 3.2 ([4]). The sequence $(g_n)_{n\in\mathbb{N}}$ in Y^X is called statistically exhaustive at a point $x \in X$, if for each $\varepsilon > 0$ there are $\delta > 0$ and a statistical dense set $K \subset \mathbb{N}$ such that for each $t \in B_\omega(x, \delta)$ and each $n \in K$, we have $\rho(g_n(t), g_n(x)) < \varepsilon$. The sequence $(g_n)_{n\in\mathbb{N}}$ is statistically exhaustive if it is statistically exhaustive at every $x \in X$.

It should be point out that, every exhaustive sequence $(g_n)_{n\in\mathbb{N}}$ is also statistically exhaustive. However, the following example show that the converse is not true in general.

Example 3.3. Let $(g_n)_{n\in\mathbb{N}}$ be the sequence of functions in $\mathbb{R}^{\mathbb{R}}$ and defined by

$$g_n(x) = \begin{cases} 1/3, & \text{if } x \leq 0,\ n \text{ is a square,} \\ 1/2n, & \text{if } x \leq 0,\ n \text{ is not a square,} \\ 1, & \text{if } x > 0,\ n \text{ is a square,} \\ 1/3n, & \text{if } x > 0,\ n \text{ is not a square.} \end{cases}.$$

If we denote by D the set of all square numbers which is a subset of natural numbers, it is obvious that $d(D) = 0$. So we get $d(\mathbb{N}\backslash D) = 1$. Take $\varepsilon > 0$ and $m \in \mathbb{N}\backslash D$ such that $\frac{1}{6m} < \varepsilon$. For each $n \in (\mathbb{N}\diagdown D) \cap \{n \in \mathbb{N} : n > m\}$ and each $t \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, we get $|g_n(t) - g_n(0)| \leq \frac{1}{6n} < \frac{1}{6m} < \varepsilon$. So, the sequence $(h_n)_{n\in\mathbb{N}}$ is statistically exhaustive at zero.

On the other hand, for every $\delta > 0$, n is a square number and every $t \in (-\delta, \delta)$ we have $|g_n(t) - g_n(0)| = \frac{2}{3}$. Hence, the sequence $(h_n)_{n\in\mathbb{N}}$ is not exhaustive at zero.

In the previous example, the subsequence $(h_n)_{n\in D}$ of the sequence $(h_n)_{n\in\mathbb{N}}$ is not statistically exhaustive. However, we have the following.

Lemma 3.4 ([4]). *A sequence $(g_n)_{n\in\mathbb{N}}$ in Y^X is statistically exhaustive if and only if each of its statistical dense subsequence is statistically exhaustive.*

In this paper, we study the concept of P_p–statistically exhaustiveness. Firstly, we give this definition.

Definition 3.5. The sequence $(g_n)_{n\in\mathbb{N}}$ in Y^X is called P_p–statistically exhaustive at a point $x \in X$, if for each $\varepsilon > 0$ there are $\delta > 0$ and a P_p–dense set $K \subset \mathbb{N}$ such that for each $t \in B_w(x, \delta)$ and each $n \in K$ we have $\rho(g_n(t), g_n(x)) < \varepsilon$. The sequence $(g_n)_{n\in\mathbb{N}}$ is P_p–statistically exhaustive if it is P_p–statistically exhaustive at every $x \in X$.

It should be point out that, every exhaustive sequence $(g_n)_{n\in\mathbb{N}}$ is also P_p–statistically exhaustive. However, the following example show that the converse is not true in general. Also, from the following example we can see that P_p–statistical exhaustiveness and statistical exhaustiveness is not compared.

Example 3.6. Let $(g_n)_{n\in\mathbb{N}}$ be the sequence of functions in $\mathbb{R}^{\mathbb{R}}$ and $E = \{2k+1 : k \in \mathbb{N}\}$. Let P_p be a power series method that is given by

$$p_n = \begin{cases} 0, & n = 2k+1, \\ 1, & \text{otherwise,} \end{cases}$$

and consider the sequence $(g_n)_{n\in\mathbb{N}}$ defined by

$$g_n(x) = \begin{cases} -1, & \text{if } x \le 0,\ n \text{ is a square and } n \in E, \\ 1, & \text{if } x > 0,\ n \text{ is a square and } n \in E, \\ 1/n, & \text{if } x \le 0,\ n \text{ is a square and } n \notin E, \\ 1/2n, & \text{if } x > 0,\ n \text{ is a square and } n \notin E, \\ -1, & \text{if } x \le 0,\ n \text{ is not a square and } n \in E, \\ 1, & \text{if } x > 0,\ n \text{ is not a square and } n \in E, \\ 1/n, & \text{if } x \le 0,\ n \text{ is not a square and } n \notin E, \\ 1/2n, & \text{if } x > 0,\ n \text{ is not a square and } n \notin E. \end{cases}$$

It is obvious that $d_{P_p}(E) = \lim_{r\to R^-} \frac{1}{p(r)} \sum_{n\in E}^{\infty} p_n r^n = 0$. So we get $d_{P_p}(\mathbb{N}\backslash E) = 1$. Take $\varepsilon > 0$ and $m \in (\mathbb{N}\backslash E)$ such that $\frac{1}{2m} < \varepsilon$. For each $n \in (\mathbb{N}\backslash E) \cap \{n \in \mathbb{N} : n > m\}$ and each $t \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, we get $|g_n(t) - g_n(0)| \le \frac{1}{2n} < \frac{1}{2m} < \varepsilon$.

So, the sequence $(g_n)_{n\in\mathbb{N}}$ is P_p–statistically exhaustive at zero. We should point out that the sequence $(g_n)_{n\in\mathbb{N}}$ is not exhaustive at zero because for every $\delta > 0$ and every $t \in (-\delta, \delta)$ we have $|g_n(t) - g_n(0)| = 2$ for infinitely many n.

Now, let $H = \left\{k : k = n^2,\ n \in \mathbb{N}\right\}$. Then $d(H) = 0$ and so $d(\mathbb{N}\backslash H) = 1$. If we take $n \in (\mathbb{N}\backslash H)$ then the function sequence become

$$g_n(x) = \begin{cases} -1, & \text{if } x \le 0,\ n \in E, \\ 1, & \text{if } x > 0,\ n \in E, \\ 1/n, & \text{if } x \le 0,\ n \notin E, \\ 1/2n, & \text{if } x > 0,\ n \notin E. \end{cases}$$

Also, for every $\delta > 0$ and every $t \in (-\delta, \delta)$ we have $|g_n(t) - g_n(0)| = 2$. Hence, the sequence $(g_n)_{n\in\mathbb{N}}$ is not statistically exhaustive at zero.

Now, consider the sequence $(h_n)_{n\in\mathbb{N}}$ defined by

$$h_n(x) = \begin{cases} 1/2, & \text{if } x \le 0,\ n \text{ is square,} \\ 1/2n, & \text{if } x \le 0, n \text{ is not square,} \\ 1, & \text{if } x > 0,\ n \text{ is square} \\ 1/3n, & \text{if } x > 0,\ n \text{ is not square.} \end{cases}$$

Also, let P_p be a power series method that is given by

$$p_n = \begin{cases} 1, & n \text{ is square,} \\ 0, & \text{otherwise} \end{cases}$$

Since the set D of square natural numbers has density zero, we get $d(\mathbb{N}\backslash D) = 1$. Take $\varepsilon > 0$ and $m \in \mathbb{N}\backslash D$ such that $\frac{1}{6m} < \varepsilon$. For each $n \in (\mathbb{N}\backslash D) \cap \{n \in \mathbb{N} : n > m\}$ and each $t \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, we get $|h_n(t) - h_n(0)| \le \frac{1}{6n} < \frac{1}{6m} < \varepsilon$. So, the sequence $(g_n)_{n\in\mathbb{N}}$ is statistically exhaustive at zero. On the other hand, for every $\delta > 0$, n is a number of D and every $t \in (-\delta, \delta)$, we have $|h_n(t) - h_n(0)| = \frac{1}{2}$ and $d_{P_p}(D) = \lim_{r\to R^-} \frac{1}{p(r)} \sum_{n\in D}^{\infty} p_n r^n = 1$. Hence, the sequence $(g_n)_{n\in\mathbb{N}}$ is not P_p–statistical exhaustive at zero.

In the previous example, the subsequence $(g_n)_{n\in E}$ of the sequence $(g_n)_{n\in\mathbb{N}}$ is not P_p–statistically exhaustive. However, we have the following lemma.

Lemma 3.7. *A sequence $(g_n)_{n\in\mathbb{N}}$ is P_p–statistically exhaustive if and only if each of its P_p–dense subsequence is P_p–statistically exhaustive.*

Proof. We have only to prove that a P_p–dense subsequence $(g_{n_k})_{n_k\in\mathbb{N}}$ of the P_p–statistically exhaustive sequence $(g_n)_{n\in\mathbb{N}}$ is also P_p–statistically exhaustive. Suppose that, a P_p–dense subsequence $(g_{n_k})_{n_k\in\mathbb{N}}$ of the P_p–statistically exhaustive sequence $(g_n)_{n\in\mathbb{N}}$ is not P_p–statistically exhaustive. So, $d_{P_p}(D) = 1$ where is the set $D = \{n_k : k \in \mathbb{N}\}$. Let $x \in X$ and $\varepsilon > 0$. Hence, for each $\delta_1 > 0$ and each P_p–dense subset K of $\mathbb{N}$ there exist $t \in B_\omega(x, \delta_1)$ and $k \in K$ such that $\rho(g_k(x), g_k(t)) \ge \varepsilon$.

Because of the sequence $(g_n)_{n\in\mathbb{N}}$ is P_p–statistically exhaustive, there are $\delta > 0$ and a P_p–dense subset M of $\mathbb{N}$ such that $\rho(g_m(x), g_m(t)) < \varepsilon$ for each $t \in B_\omega(x, \delta)$ and $m \in M$. We get that $d_{P_p}(K \cap M) = 1$ from the definition of P_p–density. So, we have $\rho(g_{m_0}(x), g_{m_0}(t)) \ge \varepsilon$ for some $t \in B_\omega(x, \delta_1)$ and $m_0 \in K \cap M$. This is a contradiction. Thus the proof is obtained. □

In [8, 11] the classical notion of α–convergence was defined. Then, the statistical version of this convergence was given in [4].

Definition 3.8. A sequence $(g_n)_{n\in\mathbb{N}}$ in Y^X is P_p–statistically α–convergent to $g \in Y^X$ if for every $x \in X$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in X converging to x, the sequence $(g_n(x_n))_{n\in\mathbb{N}}$ is P_p–statistically convergent to $g(x)$. In this case we write, $(g_n) \overset{st_{P_p}-\alpha}{\rightarrow} g$.

THEOREM 3.9. *For a sequence $(g_n)_{n\in\mathbb{N}}$ in Y^X and a function $g \in Y^X$ the following are equivalent:*

(i) $(g_n) \overset{st_{P_p}-\alpha}{\to} g$;

(ii) $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$ *and* $(g_n)_{n\in\mathbb{N}}$ *is P_p–statistically exhaustive;*

(iii) g is continuous and $(g_n) \overset{st_{P_p}-\tau_{uc}}{\to} g$.

If g is locally compact, then (i)–(iii) are equivalent also to:

(iv) g is continuous and $(g_n) \overset{st_{P_p}-\tau^s_{\mathcal{K}_r}}{\to} g$.

Proof. (i)⇒(ii): Let $(g_n) \overset{st_{P_p}-\alpha}{\to} g$, then from the definition of P_p–statistically α–convergence, we have for every $x \in X$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in X converging to x, the sequence $(g_n(x_n))_{n\in\mathbb{N}}$ is P_p–statistically convergent to $g(x)$. So, we have $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$.

Now we show that, $(g_n)_{n\in\mathbb{N}}$ is P_p–statistically exhaustive. Suppose that $(g_n)_{n\in\mathbb{N}}$ is not P_p–statistically exhaustive. Then, there are $x \in X$ and $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ and a P_p–dense set $K \subset \mathbb{N}$ there exists $x_n \in B_w\left(x, \frac{1}{n}\right)$ such that

$$\rho\left(g_k(x_n), g_k(x)\right) \geq \varepsilon \tag{1}$$

for each $k \in K$.

Since $(g_n(x_n))_{n\in\mathbb{N}}$ is P_p–statistically convergent to $g(x)$,

$$d_{P_p}\left(\left\{k \in \mathbb{N}_0 : \rho\left(g_k(x_k), g(x)\right) \geq \frac{\varepsilon}{2}\right\}\right) = 0.$$

If $K_1 = \mathbb{N}\diagdown\left\{k \in \mathbb{N} : \rho\left(g_k(x_k), \quad g(x)\right) \geq \frac{\varepsilon}{2}\right\}$ then $d_{P_p}(K_1) = 1$ and we get $\rho\left(g_{k_1}(x_{k_1}), g(x)\right) < \frac{\varepsilon}{2}$ for all $k_1 \in K_1$. Also, since $(g_n)_{n\in\mathbb{N}}$ is P_p–statistically converges to g at x,

$$d_{P_p}\left(\left\{k \in \mathbb{N}_0 : \rho\left(g_k(x), g(x)\right) \geq \frac{\varepsilon}{2}\right\}\right) = 0.$$

If $K_2 = \mathbb{N}\diagdown\left\{k \in \mathbb{N} : \rho\left(g_k(x), g(x)\right) \geq \frac{\varepsilon}{2}\right\}$, then $d_{P_p}(K_2) = 1$ and we get $\rho\left(g_{k_2}(x), g(x)\right) < \frac{\varepsilon}{2}$ for all $k_2 \in K_2$. It is obvious that $d_{P_p}(K_1 \cap K_2) = 1$. Hence, for each $j \in K_1 \cap K_2$ we get $\rho\left(g_j(x_j), g_j(x)\right) \leq \rho\left(g_j(x_j), g(x)\right) + \rho\left(g(x), g_j(x)\right) < \varepsilon$.

However, this is a contradiction. Thus the proof is obtained.

(ii)⇒(iii): Let $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$ and $(g_n)_{n\in\mathbb{N}}$ is P_p–statistically exhaustive. Firstly, we show that g is continuous. Let $x \in X$ and $\varepsilon > 0$ be fixed. Since $(g_n)_{n\in\mathbb{N}}$ is P_p–statistically exhaustive at x, there is $\delta > 0$ and a P_p–dense set $K_1 \subset \mathbb{N}$ such that for every $t \in B_\omega(x, \delta)$ we get $\rho\left(g_n(x), g_n(t)\right) < \frac{\varepsilon}{3}$ for all $n \in K_1$. Now fix $y \in B_\omega(x, \delta)$. Since $(g_n(x))_{n\in\mathbb{N}}$ is P_p–statistically convergent to $g(x)$, as in the proof above, there is a P_p–dense set $K_2 \subset \mathbb{N}$ such that $\rho\left(g_n(x), g(x)\right) < \frac{\varepsilon}{3}$ for all $n \in K_2$. Similarly, $(g_n(y))_{n\in\mathbb{N}}$ is P_p–statistically convergent to $g(y)$, there is P_p–dense set $K_3 \subset \mathbb{N}$ such that $\rho\left(g_n(y), g(y)\right) < \frac{\varepsilon}{3}$ for all $n \in K_3$. It is obvious that $d_{P_p}(K_1 \cap K_2 \cap K_3) = 1$. Hence, for each $j \in K_1 \cap K_2 \cap K_3$ we get

$$\rho\left(g(x), g(y)\right) \leq \rho\left(g(x), g_j(x)\right) + \rho\left(g_j(x), g_j(y)\right) + \rho\left(g_j(y), g(y)\right) < \varepsilon.$$

So we get that, g is continuous at x.

We recall that, if a function $g \in C(X,Y)$ (continuous functions from X to Y) and K is a compact subset of X , then $f \mid K$ is uniformly continuous [2].

Now we show that, $(g_n) \overset{st_{P_p}-\tau_{uc}}{\to} g$. For this aim, let $\varepsilon > 0$ and let M be a compact subset of X. Since g is continuous for every $x \in M$ there is a δ_x such that if $\omega(x,t) < \delta_x$, then $\rho(g(x), g(t)) < \frac{\varepsilon}{3}$. Since $(g_n)_{n\in\mathbb{N}}$ is P_p–statistically exhaustive at each $x \in M$, there exist $\eta_x > \delta_x$ and P_p–dense sets $G_x \subset \mathbb{N}$ such that for each $t \in B_\omega(x, \eta_x)$ and each $n \in G_x$ we have $\rho(g_n(t), g_n(x)) < \frac{\varepsilon}{3}$. Also, we can say that from the compactness of M, there are finitely many $x_1, \ldots, x_k$ such that $M \subset \bigcup_{i=1}^{k} B_\omega(x_i, \eta_{x_i})$.

From (ii), $(g_n(x_i))_{n\in\mathbb{N}}$ P_p–statistically convergent to $g(x_i)$ for every $i \leq k$, so there are P_p–dense sets $\Omega_i \subset \mathbb{N}$, $i \leq k$ such that $\rho(g_n(x_i), g(x_i)) < \frac{\varepsilon}{3}$ for every $n \in \Omega_i$. Since g is continuous at every x_i, there are $\eta_i > 0$ such that for every $t \in B_\omega(x_i, \eta_i)$, we get $\rho(g(x_i), g(t)) < \frac{\varepsilon}{3}$. Let $H = \bigcap_{i=1}^{k} (\Omega_i \cap G_{x_i})$ and $\eta = \min\{\eta_{x_1}, \ldots, \eta_{x_k}, \eta_1, \ldots, \eta_k\}$. Let $z \in M$ be arbitrary. Then $z \in B_\omega(x_i, \eta_{x_i})$ for some $i \leq k$ and thus for every $m \in H$ we have

$$\rho(g_m(z), g(z)) \leq \rho(g_m(z), g_m(x_i)) + \rho(g_m(x_i), g(x_i)) + \rho(g(x_i), g(z)) < \varepsilon.$$

So, $(g_n(x))_{n\in\mathbb{N}}$ uniformly converges to g on M.

(iii)⇒(i): Let g is continuous and $(g_n) \overset{st_{P_p}-\tau_{uc}}{\to} g$. Now, we show that $(g_n) \overset{st_{P_p}-\alpha}{\to} g$. Let $\varepsilon > 0$ and $x \in X$. Suppose that $(x_n)_{n\in\mathbb{N}}$ be a sequence in X convergent to x. Since $H = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is a compact set in X, there exists a P_p–dense set $M_1 \subset \mathbb{N}$ and for every $z \in H$ and every $m \in M_1$, $\rho(g_m(z), g(z)) < \frac{\varepsilon}{2}$. Also, since g is continuous at x, there is $\delta > 0$ such that $\rho(g(x), g(t)) < \frac{\varepsilon}{2}$ for every $t \in B_\omega(x, \delta)$. Also, since $(x_n)_{n\in\mathbb{N}}$ converges to x there is a $n_0 \in \mathbb{N}$ such that $x_n \in B_\omega(x, \delta)$ for every $n \geq n_0$. We can see that $d_{P_p}(M_1 \cap \{n \in \mathbb{N} : n \geq n_0\}) = 1$. Then we have

$$\rho(g_m(x_m), g(x)) \leq \rho(g_m(x_m), g(x_m)) + \rho(g(x_m), g(x)) < \varepsilon$$

for each $m \in M_1 \cap \{n \in \mathbb{N} : n \geq n_0\}$. So we can see

$$\lim_{r\to R^-} \frac{1}{p(r)} \sum_{m\in\{m:\rho(g_m(x_m),g(x))\geq\varepsilon\}} p_m r^m = 0.$$

(iii)⇔(iv): It can be obtained from [2, Theorem 6.2]. □

4. Weak P_p–Statistical Exhaustiveness

Gregoriades and Papanastassiou introduced the concept of weak exhaustiveness [8]. Then, Caserta and Kočinac gave a statistical version of this notion [4].

In [8], the following general result was obtained .

Theorem 4.1 ([8]). *Let (X,ω), (Y,ρ) be metric spaces, $x \in X$ and the sequence $(g_n)_{n\in\mathbb{N}}$ in Y^X be pointwise convergent to function $g \in Y^X$. Then g is continuous at x if and only if the sequence $(g_n)_{n\in\mathbb{N}}$ is weakly exhaustive at x.*

Now, we define the notion of weak P_p–statistical exhaustiveness.

DEFINITION 4.2. The sequence $(g_n)_{n\in\mathbb{N}}$ is called weakly P_p–statistical exhaustive at a point $x \in X$, if for each $\varepsilon > 0$ there exist $\delta > 0$ and a P_p–dense set $K_t \subset \mathbb{N}$, depending on t, such that for each $t \in B_\omega(x,\delta)$ we have $\rho(g_n(t), g_n(x)) < \varepsilon$ for each $n \in K_t$. The sequence $(g_n)_{n\in\mathbb{N}}$ is weakly P_p– statistical exhaustive if it is weakly P_p–statistical exhaustive at every $x \in X$.

LEMMA 4.3. *Let* $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$*. Then* $(g_n)_{n\in\mathbb{N}}$ *is weakly* P_p*–statistical exhaustive if and only if* g *is continuous.*

Proof. Let $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$ and $(g_n)_{n\in\mathbb{N}}$ is weakly P_p–statistical exhaustive. Also, let $x \in X$ and $\varepsilon > 0$. From the definition of weakly P_p–statistical exhaustive, there are $\delta > 0$ and a P_p–dense set $K_t \subset \mathbb{N}$, depending on t, such that for each $t \in B_\omega(x,\delta)$ we have $\rho(g_n(t), g_n(x)) < \frac{\varepsilon}{3}$ for each $n \in K_t$. Since $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$, then we have $\lim_{r\to R^-} \frac{1}{p(r)} \sum_{m\in\{m:\rho(g_m(t),g(t))\geq\varepsilon\}} p_m r^m = 0$. If we define the set $M_1 = \{m : \rho(g_m(t), g(t)) \geq \varepsilon\}$, $d_{P_p}(\mathbb{N}\setminus M_1) = 1$. So, for every $m \in \mathbb{N}/M_1$ $\rho(g_m(t), g(t)) < \frac{\varepsilon}{3}$ and $\rho(g_m(x), g(x)) < \frac{\varepsilon}{3}$.

If we take arbitrary $y \in B_\omega(x,\delta)$ and any $k \in (\mathbb{N}\setminus M_1) \cap K_t$, then

$$\rho(g(y), g(x)) \leq \rho(g(y), g_k(y)) + \rho(g_k(y), g_k(x)) + \rho(g_k(x), g(x)) < \varepsilon.$$

Hence, we get g is continuous at x.

Now, let g is continuous at $x \in X$ and $\varepsilon > 0$. So, there is a $\delta > 0$ such that for every $t \in B_w(x,\delta)$ we have $\rho(g(t), g(x)) < \frac{\varepsilon}{2}$. Since $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$, there is a set $M_1 \subset \mathbb{N}$ as in the first part of the proof above. So, $d_{P_p}(\mathbb{N}\setminus M_1) = 1$ and for every $m \in \mathbb{N}\setminus M_1$ we get $\rho(g_m(x), g(x)) < \frac{\varepsilon}{4}$ and $\rho(g_m(t), g(t)) < \frac{\varepsilon}{4}$. Then, for every $m \in \mathbb{N}\setminus M_1$ and $t \in B_w(x,\delta)$, we get

$$\rho(g_m(x), g_m(t)) \leq \rho(g_m(x), g(x)) + \rho(g(x), g(t)) + \rho(g(t), g_m(t)) < \varepsilon$$

which means

$$\lim_{r\to R^-} \frac{1}{p(r)} \sum_{m\in\{m:\rho(g_m(x),g_m(t))\geq\varepsilon\}} p_m r^m = 0.$$

Hence the desired is achieved. □

PROPOSITION 4.4. *Let* $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$ *and* $(g_n)_{n\in\mathbb{N}}$ *is weakly* P_p*–statistical exhaustive. Then* $(g_n) \overset{st_{P_p}-\tau_G^s}{\to} g$*.*

Proof. Suppose that $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$ and let $(g_n)_{n\in\mathbb{N}}$ is weakly P_p–statistical exhaustive. Let $G = \{x_1, \ldots, x_k\}$ be a finite subset of X and $\varepsilon > 0$. From the weak P_p–statistical exhaustiveness of $(g_n)_{n\in\mathbb{N}}$ at every x_i, $i \leq k$, there exist $\delta_i > 0$ and a P_p–dense set $K_t \subset \mathbb{N}$, depending on t, such that for each $t \in B_\omega(x_i,\delta_i)$ we have $\rho(g_n(t), g_n(x_i)) < \frac{\varepsilon}{3}$ for each $n \in K_t$. Also, since $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$, the sequence $(g_n(x_i))_{n\in\mathbb{N}}$ P_p–statistically convergent to $g(x_i)$. So, $\lim_{r\to R^-} \frac{1}{p(r)} \sum_{m\in\{m:\rho(g_m(x_i),g(x_i))\geq\varepsilon\}} p_m r^m = 0$.

If we define the set $M_i = \{m : \rho(g_n(x_i), g(x_i)) \geq \varepsilon\}$, then $d_{P_p}(\mathbb{N}\setminus M_i) = 1$. So, if we define $M_i' = \mathbb{N}\setminus M_i$ for every $m \in M_i' \subset \mathbb{N}$ and $i \leq k$, $\rho(g_n(x_i), g(x_i)) < \frac{\varepsilon}{3}$. From

Lemma 4.3, g is continuous at every x_i. Hence, there are $\delta_i' > 0$ for $i \leq k$, such that $t \in B_\omega(x_i, \delta_i')$ we have $\rho(g(x_i), g(t)) < \frac{\varepsilon}{3}$. If we take $\delta = \min\{\delta_1, \dots, \delta_k, \delta_1', \dots \delta_k'\}$ and let $y \in G^\delta$, then $y \in B_\omega(x_j, \delta)$ for some $j \leq k$. If we take $M = K_y \cap \bigcap_{i \leq k} M_i'$ then $d_{P_p}(M) = 1$. So, for every $m \in M$, we get

$$\rho(g_m(y), g(y)) \leq \rho(g_m(y), g_m(x_j)) + \rho(g_m(x_j), g(x_j)) + \rho(g(x_j), g(y)) < \varepsilon.$$

Hence we have $\lim_{r \to R^-} \frac{1}{p(r)} \sum_{m \in \{m : \rho(g_m(y), g(y)) \geq \varepsilon\}} p_m r^m = 0$. □

PROPOSITION 4.5. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $C(X,Y)$ and $g \in Y^X$ such that $(g_n) \overset{st_{P_p}-\tau_G^s}{\to} g$. Then $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$ and $(g_n)_{n\in\mathbb{N}}$ is weakly P_p–statistical exhaustive.*

Proof. Let $(g_n) \overset{st_{P_p}-\tau_G^s}{\to} g$. By Lemma 4.3 it is sufficient to prove that g is continuous. For this aim, let $x \in X$, $\varepsilon > 0$. So, there are $\delta_x > 0$ and a P_p–dense set $K \subset \mathbb{N}$ such that for every $n \in K$ and every $t \in B_\omega(x, \delta_x)$ we have $\rho(g_n(t), g(t)) < \frac{\varepsilon}{3}$. Since, every $(g_n)_{n\in\mathbb{N}}$ be a sequence in $C(X,Y)$ then there exist $\delta_n > 0$ such that for every $t \in B_\omega(x, \delta_n)$ and $n \in K$, we get $\rho(g_n(x), g_n(t)) < \frac{\varepsilon}{3}$. Now, we take a arbitrary element k in K and let $\delta = \min\{\delta_k, \delta_x\}$. Then, for any $y \in B_\omega(x, \delta)$ we get

$$\rho(g(x), g(y)) \leq \rho(g(x), g_k(x)) + \rho(g_k(x), g_k(y)) + \rho(g_k(y), g(y)) < \varepsilon.$$

Hence the desired is achieved. □

We can give the following theorem omitting the proof.

THEOREM 4.6. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $C(X,Y)$ and $g \in Y^X$ such that $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$. Then the following are equivalent:*
(i) $(g_n)_{n\in\mathbb{N}}$ is weakly P_p–statistical exhaustive;
(ii) $(g_n) \overset{st_{P_p}-\tau_G^s}{\to} g$;
(iii) g is continuous.

The following example show that there is a weakly P_p–statistical exhaustive sequence of functions which is not P_p–statistically exhaustive.

EXAMPLE 4.7. Let P_p be a power series method that is given by

$$p_n = \begin{cases} 1, & n \text{ is prime}, \\ 0, & \text{otherwise}, \end{cases}$$

and consider the sequence of function $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^\mathbb{R}$ which is defined by

$$g_n(x) = \begin{cases} 0, & \text{if } x \in \left(-\infty, \frac{-1}{n^2}\right) \cup \{0\} \cup \left(\frac{1}{n^2}, \infty\right), \ n \text{ is a prime number}, \\ n^2 x + 1, & \text{if } x \in \left[-\frac{1}{n^2}, 0\right), \ n \text{ is a prime number}, \\ -n^2 x + 1, & \text{if } x \in \left(0, \frac{1}{n^2}\right], \ n \text{ is a prime number}, \\ 0, & \text{if } x \in \mathbb{R}, \ n \text{ is not a prime number}. \end{cases}$$

Then, for each $x \in \mathbb{R}$

$$\lim_{r \to R^-} \frac{1}{p(r)} \sum_{n \in \{n : |g_n(x) - g(x)| \geq \varepsilon\}} p_n r^n = 0,$$

where $g(x)$ is the constant zero function. So, $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$. Then we get from Lemma 4.3, the sequence $(g_n)_{n\in\mathbb{N}}$ is weakly P_p–statistical exhaustive.

Now, we show that $(g_n)_{n\in\mathbb{N}}$ is not P_p–statistically exhaustive.

Let $\varepsilon \in (0,1)$ be given and let D be any P_p–dense subset of $\mathbb{N}$ and $\delta > 0$. Pick $k \in D \cap \{n \in \mathbb{N} : n \text{ is a prime number}\}$. Let $t \in (0,\delta)$ such that $t < \frac{1-\varepsilon}{n^2}$. Hence, $|g_k(t) - g_k(0)| > \varepsilon$. So we get $(g_n)_{n\in\mathbb{N}}$ is not P_p–statistically exhaustive.

In 1948, Alexandroff [1] defined a new convergence for sequences of functions on a topological space as follows.

Definition 4.8 ([1]). Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions from a topological space X to a metric space (Y,ρ) and let $g \in Y^X$. Then, $(g_n)_{n\in\mathbb{N}}$ is said to be Alexandroff convergent to g on X, provided it pointwise converges to g and for every $\varepsilon > 0$ and integer n_0 there exist a countable open covering $\{U_0, U_{1,\dots}\}$ of X and a sequence $\{n_k\}$ of positive integers greater than n_0 such that for each $x \in U_k$ we have $\rho(g_{n_k}(x), g(x)) < \varepsilon$.

So we can give the following definition.

Definition 4.9. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C(X,Y)$. Then, $(g_n)_{n\in\mathbb{N}}$ is said to be P_p–statistically Alexandroff convergent to $g \in Y^X$, provided $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$ and for every $\varepsilon > 0$ and every P_p–dense subset $D \subset \mathbb{N}$ there exist a infinite set $K_D = \{n_1 < n_2 < \dots < n_k < \dots\} \subset D$ and an open cover $\mathcal{U} = \{U_n : n \in D\}$ such that for every $x \in U_k$ we get $\rho(g_{n_k}(x), g(x)) < \varepsilon$. In this case we write $(g_n) \overset{st_{P_p}-Al}{\to} g$.

Theorem 4.10. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C(X,Y)$ and $g \in Y^X$. If $(g_n) \overset{st_{P_p}-Al}{\to} g$, then g is continuous.*

Proof. Let $(g_n) \overset{st_{P_p}-Al}{\to} g$. So, $x \in X$ and let $(x_i)_{i\in\mathbb{N}}$ be a sequence converging to x. We show that the sequence $(g(x_i))_{i\in\mathbb{N}}$ converges to $g(x)$. Let $\varepsilon > 0$ be given. Since $(g_n) \overset{st_{P_p}-\tau_\rho}{\to} g$, there exists a P_p–dense set $K_x \subset \mathbb{N}$ such that $\rho(g_n(x), g(x)) < \frac{\varepsilon}{3}$ for every $n \in K_x$. Since $(g_n) \overset{st_{P_p}-Al}{\to} g$, there exists an infinite set $M = \{n_1 < n_2 < \dots < n_k < \dots\} \subset K_x$ and an open cover $\mathcal{U} = \{U_n : n \in K_x\}$ of X such that for every $y \in U_k$, $\rho(g_{n_k}(y), g(y)) < \frac{\varepsilon}{3}$. Let k be such that $x \in U_k$. Since g_{n_k} is continuous at x and $(x_i)_{i\in\mathbb{N}}$ converging to x, there is $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, $x_i \in U_k$ and $\rho(g_{n_k}(x_i), g_{n_k}(x)) < \frac{\varepsilon}{3}$. Hence, for $i \geq i_0$ we have

$$\rho(g(x_i), g(x)) \leq \rho(g(x_i), g_{n_k}(x_i)) + \rho(g_{n_k}(x_i), g_{n_k}(x)) + \rho(g_{n_k}(x), g(x)) < \varepsilon.$$

So we get $(g(x_i))_{i\in\mathbb{N}}$ converges to $g(x)$. Hence g is continuous. □

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