MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
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GEOMETRY OF LINEAR AND NONLINEAR GEODESICS IN THE PROPER GROMOV–HAUSDORFF CLASS

Anton Vikhrov

Abstract. This paper investigates the proper class of all metric spaces considered up to isometry, equipped with the Gromov–Hausdorff distance. There constructed a pair of complete metric spaces, X and Y such that they have no metric spaces at zero distance, no optimal correspondence between X and Y, and therefore no linear geodesics joining them, but there exists a geodesic between them of a different type. There also described everywhere dense subclass of the Gromov–Hausdorff class such that any two points at finite distance within this subclass can be connected by a linear geodesic.

1. Introduction

A symmetric mapping $d : X \times X \to [0, \infty]$, equal zero on the diagonal and satisfying the triangle inequality, is called a *generalized pseudometric*. If, in addition, the function d vanishes only on the diagonal, it is called a *generalized metric*, and if it does not take infinite values, it is called a *metric*.

The Gromov–Hausdorff distance measures the degree of difference between two metric spaces. This distance was introduced by Gromov in 1981 [8] and was defined as the smallest Hausdorff distance between isometric images of the considered spaces. Later, an equivalent definition of this distance was given using correspondences.

In this work, we use the system of axioms introduced by von Neumann, Bernays, and Gödel, within which classes and proper classes are considered, generalizing the concept of a set. The proper class consisting of all metric spaces considered up to isometry is denoted as $\mathcal{GH}$. The notion of generalized pseudometric is naturally defined on this proper class.

In the work [11], the optimal correspondence between finite metric spaces was used to construct a geodesic between arbitrary compact metric spaces. Later, almost

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simultaneously in [7] and [10], the existence of optimal correspondence between compact metric spaces was proved, and as a consequence, a geodesic between these spaces generated by the optimal correspondence. Such geodesics are called linear ones. However, it is still unknown whether any pair of metric spaces at a finite distance from each other can be connected by some geodesic.

In [12], a special class of spaces called spaces in general position was studied, and it was shown that for any metric space S from this class, there exists a neighborhood $U_{\varepsilon(S)}(S) \subset \mathcal{GH}$ such that for any $Y \in U_{\varepsilon(S)}(S)$, there exists an optimal correspondence $R \in \mathcal{R}(S, Y)$ and, as a result, a linear geodesic joining S and Y. Such spaces in general position are everywhere dense in $\mathcal{GH}$, as demonstrated in [13]. The both results hold for more wide class (see Remark 2.22) of spaces in generalized general position (see Definition 2.20). Thus, the possibility of connecting every space in generalized general position with sufficiently close metric space by a linear geodesic is shown. Is it true that there is a linear geodesic between each space in generalized general position and any other metric space?

We construct in this work a pair of a complete metric spaces X and Y in generalized general position such that they have no metric spaces at zero distance; there is no optimal correspondence between X and Y, and hence no linear geodesic. However, we found a geodesic of a different type between these metric spaces.

The similar result was obtained by Hansen, see [9]. However, his spaces X, Y have spaces X' and Y' on zero distance from the former ones ($\mathrm{d_{GH}}(X, X') = \mathrm{d_{GH}}(Y, Y') = 0$) with a linear geodesic between X' and Y'.

Moreover, we construct a subclass of metric spaces in generalized general position that is everywhere dense in $\mathcal{GH}$ and possesses the following property: for any two metric spaces A, B from this class at finite distance from each other, there exists an optimal correspondence $R \in \mathcal{R}(A, B)$ and, therefore, a linear geodesic.

Finally, in this paper we construct a pair of proper non-isometric bounded metric spaces at zero Gromov–Hausdorff distance. The similar result but for unbounded metric space was obtained in [1].

2. Main definitions and preliminary results

First we introduce some basic notation. We denote by $\mathbb{R}_{\geq 0}$ the set of non-negative real numbers, and by $\mathbb{R}_+$ the set of positive real numbers. Let (X, ρ) be an arbitrary metric space, and $x, y \in X$. The distance between the points x and y is denoted by $|xy| = \rho(x, y) = d_X(x, y)$. Let $U_\varepsilon(a)$ be an open ball with center a of radius ε, and $U_\varepsilon(A) = \bigcup_{a \in A} U_\varepsilon(a)$ be a ε-neighborhood of a non-empty subset A, and $S_\varepsilon(a)$ is a sphere of radius ε centered at the point a. We denote by $\#X$ the cardinality of X, and for any $a \in \mathbb{R}_{\geq 0}$ and metric space X we put $a\,X = (X, a\,d_X)$.

Definition 2.1. Let A, B be non-empty subsets of a metric space X. *The Hausdorff distance* is the value $\mathrm{d_H}(A, B) = \inf\Big\{r : A \subset U_r(B) \,\&\, B \subset U_r(A)\Big\}$.

Definition 2.2. Let A, B, X be metric spaces. If A is isometric to $\tilde{A}$ and B is isometric to $\tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are subspaces of X, then we call the triple $(\tilde{A}, \tilde{B}, X)$ *a realization of the pair* (A, B).

Definition 2.3. *The Gromov–Hausdorff distance* between two metric spaces A, B is the infimum of the Hausdorff distances among all realizations of the pair (A, B). In other words,

$$\mathrm{d_{GH}}(A, B) = \inf\big\{r : \text{there is a realization } (\tilde{A}, \tilde{B}, X) \text{ of the pair } (A, B) \\ \text{such that } d_H(\tilde{A}, \tilde{B}) \le r\big\}.$$

Definition 2.4. *A correspondence* between two sets A and B is a subset $R \subset A \times B$ such that for any $a \in A$ and $b \in B$ there exist $\tilde{a} \in A$ and $\tilde{b} \in B$ for which $(a, \tilde{b})$, $(\tilde{a}, b)$ belong to R.

Further, aRb means that a and b are in correspondence R, and the set of all correspondences between metric spaces A, B is denoted as $\mathcal{R}(A, B)$.

Definition 2.5. Let R be a correspondence between metric spaces A, B. Its *distortion* is given by: $\operatorname{dis} R = \sup\Big\{\big|d_X(a, a') - d_Y(b, b')\big| : aRb \text{ and } a'Rb'\Big\}$.

Proposition 2.6 ([5]). *For any metric spaces A and B, the following equality holds:* $2\,\mathrm{d_{GH}}(A, B) = \inf_{R \in \mathcal{R}(A,B)} \operatorname{dis} R$.

Remark 2.7. If we define the distance between pseudometric spaces A' and B' in the same way as before: $2\,\mathrm{d_{GH}}(A', B') = \inf_{R \in \mathcal{R}(A',B')} \operatorname{dis} R$, then it coincides with the distance between metric spaces A, B obtained by factoring the spaces A', B' with respect to zero distances.

Remark 2.8. In what follows, when we deal with a pseudometric space, we automatically identify this space with the metric one by factorizing the space with respect to zero distances.

Definition 2.9. If the distortion of the correspondence R is minimal in terms of inclusion among all correspondences $\mathcal{R}(A, B)$, then it is called *irreducible.* If $\operatorname{dis}(R) = 2\,\mathrm{d_{GH}}(A, B) < \infty$, then such a correspondence is called *optimal.* The set of optimal correspondences between metric spaces A, B is denoted as $\mathcal{R}_{\mathrm{opt}}(A, B)$.

Definition 2.10. Let $R_1 \in \mathcal{R}(A, B)$ and $R_2 \in \mathcal{R}(B, C)$. Then $R_2 \circ R_1 = \{(a, c) \mid \text{there exists } b \text{ such that } aR_1b, bR_2c\}$.

Lemma 2.11. *The distortion of the composition of correspondences is no more than the sum of their distortions.*

Indeed, for arbitrary (a, c) and $(a', c') \in R_2 \circ R_1$, we found arbitrary b and $b' \in B$ such that aR_1b, $a'R_1b'$, bR_2c, $b'R_2c'$: $\big||aa'| - |cc'|\big| \le \big||aa'| - |bb'|\big| + \big||bb'| - |cc'|\big| \le \operatorname{dis}(R_1) + \operatorname{dis}(R_2)$. Taking the supremum of both sides yields the desired result.

We call the shortest curves geodesics.

THEOREM 2.12 ([10]). *If R is an optimal correspondence between metric spaces A and B, then the curve $\gamma : [0,1] \to \mathcal{GH}$, where $\gamma(t) = (R, d_t)$ and $d_t((a,b),(a',b')) = (1-t)d_A(a,a') + td_B(b,b')$, is a geodesic connecting metric spaces A, B.*

We call such geodesics as *linear*.

DEFINITION 2.13. The *Gromov–Hausdorff class $\mathcal{GH}$* is the proper class (in the sense of von Neumann–Bernays–Gödel set theory) of all metric spaces considered up to isometry.

THEOREM 2.14 ([5]). *The Gromov–Hausdorff distance is a generalized pseudometric on $\mathcal{GH}$.*

Denote by $\triangle_n$ an n-point *simplex*, that is, a metric space of cardinality n such that the distances between its different points are equal to 1. The diameter of a metric space X is given by $\operatorname{diam}(X) = \sup_{x,x' \in X} d_X(x,x')$.

DEFINITION 2.15. Metric space Z is called a 0-*modification* of a metric space X if $d_{\mathrm{GH}}(Z,X) = 0$ and $X \neq Z$.

The following result, presented as an exercise in [5], we provide its proof here for completeness.

THEOREM 2.16. *A space Y is a 0-modification of a compact metric space X if and only if $\overline{Y} = X$, where $\overline{Y}$ is the completion of the metric space Y.*

Proof. The only if statement is trivial. Choose arbitrary finite $\varepsilon/2$-net $K_{\varepsilon/2} \subseteq X$ and let $R \in \mathcal{R}(X,Y)$ with $\operatorname{dis}(R) < \varepsilon/2$. For each $x \in K_{\varepsilon/2}$, select one y from $R(x)$ and denote the resulting set as K'. We will prove K' is an ε-net for the space Y.

For any $y \in Y$, we find some $x \in X$ such that xRy. Let $k_x \in K_{\varepsilon/2}$ and $k_y \in K'$ such that $k_x R k_y$ and $|xk_x| \le \varepsilon/2$. Then $|yk_y| \le \big||yk_y| - |xk_x|\big| + |xk_x| \le \operatorname{dis}(R) + |xk_x| < \varepsilon$.

Therefore, Y is pre-compact metric space, than the completeness of Y is compact metric space. Distance d_{GH} is metric on the collection of metric spaces considered up to isometry (see [5]), so $\overline{Y} = X$. Thus, the proposition is proven. □

Note that if the completion of a metric space X is finite, then the space X is also finite.

COROLLARY 2.17. *Finite metric spaces have no 0-modifications.*

DEFINITION 2.18. Let $X \in \mathcal{GH}$. Let $S(X)$ denote the set of all bijective mappings from X to itself. We introduce the following notations:

$$\mathrm{s}(X) = \inf\left\{|xx'| \mid x \neq x';\ x,x' \in X\right\},$$
$$\mathrm{t}(X) = \inf\left\{|xx'| + |x'x''| - |xx''| \mid x \neq x' \neq x'' \neq x; x,x',x'' \in X\right\},$$
$$\mathrm{e}(X) = \inf\left\{\operatorname{dis}(f) \mid f \in S(X), f \neq \mathrm{id}\right\},$$
$$\mathrm{e}'(X) = \inf\left\{\operatorname{dis}(f) \mid f \in S(X) \setminus \mathrm{ISO}(X)\right\}.$$

Definition 2.19. A *space in general position* is a metric space X in which $\mathrm{s}(X)$, $\mathrm{t}(X)$, $\mathrm{e}(X)$ are positive.

Definition 2.20. A *space in generalized general position* is a metric space X in which $\mathrm{s}(X)$, $\mathrm{e}'(X)$ are positive.

Theorem 2.21 ([12]). *If a space M satisfies $\mathrm{e}(M) > 0$ and $\mathrm{s}(M) > 0$, then choose $\varepsilon > 0$ such that $\varepsilon < \mathrm{s}(M)/4$, $\varepsilon < \mathrm{e}(M)/4$, and if a metric space X and a correspondence $R \in \mathcal{R}(M, X)$ satisfy $\mathrm{dis}(R) < 2\varepsilon$, then R is an optimal correspondence.*

Remark 2.22. Theorem 2.21 remains true if we consider $e'(M)$ instead of $e(M)$. To prove this, it is sufficient to notice that correspondences differ by an isometry of the space M have the same distortion. Thus, the condition of being separated from the identity for all non-identity mappings can be replaced by the condition of being separated from an isometry, which coincides with the condition $\mathrm{e}'(M) > 0$.

Definition 2.23. The *spectrum* of a metric space is the set of all distances between points (including zero).

2.1 Canonical projection

Recall [2] how to construct a pseudometric space from a connected weighted graph. Everywhere below, graphs are assumed to be simple, connected, and weighted, and the edge weight function ω (given on the edges of the graph) is non-negative. The set of the vertices of the graphs and the set of edges can be infinite. The vertices of the graphs are sometimes called their points. The edge connecting x and y is denoted by xy or $\{x, y\}$. Everywhere below it is assumed that $\omega(xx) = 0$ for every graph's vertice, despite there are no edges xx.

Definition 2.24. A *generalized path* L in a graph G connecting its points x and y is a finite sequence $x_1x_2\ldots x_N$ with $x_1 = x$ and $x_N = y$ such that either x_ix_{i+1} is an edge or $x_i = x_{i+1}$, all edges are distinct and if two points from this sequence coincides, then all intermediate points coincides with them. An *edge of a generalized path* is an edge connecting consecutive distinct points of this path. The *length of the path* L is defined as $\omega(L) = \sum_{i=1}^{N-1} \omega(x_ix_{i+1})$. The *set of generalized paths* connecting x and y is denoted by $\mathrm{Ł}(x, y)$.

In this paper, **a generalized path is simply called a path.**

Definition 2.25. For every weighted graph (V, E, ω), we define a metric d_ω on V by $d_\omega(y_1, y_2) = \inf\big\{\omega(L) \mid L \in \mathrm{Ł}(y_1, y_2)\big\}$. This distance is called the weighted path metric (see [2]). The mapping $\pi : (V, E, \omega) \to (V, d_\omega)$ is called *the canonical projection.*

Since the weight function need not satisfy the triangle inequality, one can obtain $d_\omega(y_1, y_2) < \omega(y_1y_2)$, in particular, if the length of some $L \in \mathrm{Ł}(y_1, y_2)$ is less than $\omega(y_1y_2)$. We say that the projection π *preserves the edge weights* if $\omega(xy) = d_\omega(x, y)$ for any $xy \in E$.

It is well known (see [2]), that (Y, d_ω) is a pseudometric space.

Lemma 2.26. *Let $X = (V, E, \omega)$ be a graph. If there exists $C > 0$ such that $\omega(e) \geq C$ for all* e $\in E$*, then $\pi(X)$ is a metric space.*

Definition 2.27. Let X be a weighted graph and z_1z_2 its edge. Then *the polygon inequality for the lower base z_1z_2* and the path $L \in Ł(z_1, z_2)$ is the inequality $\omega(z_1z_2) \leq \omega(L)$.

Lemma 2.28. *The canonical projection preserves edge weights if and only if all polygon inequalities hold for all lower bases $xy \in E$ and any $L \in Ł(x, y)$.*

Proof. Note that $\inf\{\omega(L) : L \in Ł(x, y)\} \leq \omega(xy)$ because $xy \in Ł(x, y)$. Due to the polygon inequality with the lower base xy, we have $\omega(xy) \leq \omega(L)$ for every $L \in Ł(x, y)$.

If the polygon inequality does not hold for at least one pair of points x, y, i.e., there is a path $L \in Ł(x, y)$ such that $\omega(xy) > \omega(L)$, then $d_\omega(x, y) < \omega(xy)$. □

2.2 Subdivision of a metric space

Here is a generalization of the notion of graph subdivision.

Construction 2.29. *Consider an arbitrary metric space X as a weighted complete graph G' with the weight function ω equal to the distance between the points. For each pair of points $\{u, v\}$, assign arbitrary index set $\mathcal{I}(u, v)$ (this set might be empty) and add points $\alpha_i^{u,v}$, $i \in \mathcal{I}(u, v)$. Connect each $\alpha_i^{u,v}$ to each $\alpha_j^{u,v}$, and connect the points u, v to all $\alpha_i^{u,v}$. To the added edges we assign arbitrarily weights in such a way that the triangle inequalities hold in all the subgraphs $G_{u,v}$ generated by $\{u, v, \alpha_i^{u,v} : i \in \mathcal{I}(u, v)\}$ (actually, $G_{u,v}$ is a pseudometric space, if you consider these weights as distances). We denote the obtained graph by G.*

We put $Z = \pi(G)$, and the points obtained from X is denoted in the same way as in X. Here we write some properties of the space Z.

Lemma 2.30. *For connected weighted graph $G = (U, V, \omega)$ obtained by Construction* 2.29, *we have*

1. The projection π preserves the weights of all edges.

2. The d_Z distance between points x, y located in $G_{u,v}$ and $G_{u',v'}$, respectively, where $uv \neq u'v'$ and x, $y \notin X$, is equal to the minimal length of the following paths:

(a) $L_1 = xuu'y$, (b) $L_2 = xuv'y$, (c) $L_3 = xvu'y$, (d) $L_4 = xvv'y$.

3. The distance from a point x of $G_{u,v}$, where $v \neq x \neq u$, to $u' \in X$, where $v \neq u' \neq u$ is equal to the minimal length of the following paths:

(a) $L_1 = xuu'$, (b) $L_2 = xvu'$.

In this work, we consider the simplest case $\#I = 1$.

2.3 Metric preserving functions

Metric preserving functions are studied in [4] and their applications to the Gromov–Hausdorff class are studied in [6]. Here are some necessary properties and new theorem of their connection with Gromov–Hausdorff distance.

DEFINITION 2.31. We call $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *metric preserving* if and only if for every metric space (X, d_X), the space $(X, f \circ d_X)$ is metric.

THEOREM 2.32 ([4]). *A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is metric preserving if and only if $f(x) = 0$ for only $x = 0$, and for arbitrary non-negative a, b, c such that $|b - c| \leq a \leq b + c$, the inequality $f(a) \leq f(b) + f(c)$ holds.*

Applying the definition twice, we get that the composition of metric preserving functions is metric preserving as well. By $f(X)$ denote the metric space $(X, f \circ d_X)$. For a non-empty subset $A \subset \mathbb{R}_{\geq 0}$, we put $||f||_A = \sup\bigl\{|f(x)| \ : x \in A\bigr\}$.

THEOREM 2.33 ([4]). *A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is metric preserving if and only if $f(x) = 0$ only for $x = 0$, and for any non-negative a, b, c satisfying $|b-c| \leq a \leq b+c$, the inequality $f(a) \leq f(b) + f(c)$ holds.*

Applying the definition twice, we get that the composition of metric preserving functions is metric preserving as well. By $f(X)$ denote the metric space $(X, f \circ d_X)$. For a non-empty subset $A \subset \mathbb{R}_{\geq 0}$, we put $||f||_A = \sup\{|f(x)| \mid x \in A\}$.

LEMMA 2.34 ([13]). *For any metric space X and metric preserving function f, the inequality $2\,\mathrm{d}_{\mathrm{GH}}\bigl(X, f(X)\bigr) \leq ||\,\mathrm{id} - f||_A$ holds, where $A = \bigl[\mathrm{s}(X), \mathrm{diam}(X)\bigr]$ if $\mathrm{diam}(X) < \infty$ and $A = \bigl[\mathrm{s}(X), \infty\bigr)$ otherwise.*

LEMMA 2.35. *Let X, Y be from $\mathcal{GH}$ and $\mathrm{d}_{\mathrm{GH}}(X, Y) < \infty$, and f is a metric preserving function. Then $\mathrm{d}_{\mathrm{GH}}(f(X), f(Y)) \leq \liminf\limits_{r \to \mathrm{d}_{\mathrm{GH}}(X,Y)+} f(r)$.*

Moreover, if there exists an optimal correspondence between metric spaces X and Y, then the inequality holds $\mathrm{d}_{\mathrm{GH}}(f(X), f(Y)) \leq f(\mathrm{d}_{\mathrm{GH}}(X, Y))$.

Proof. We prove the first point. For any $r > \mathrm{d}_{\mathrm{GH}}(X, Y)$, find a correspondence $R \in \mathcal{R}(X, Y)$ such that $\mathrm{dis}(R)/2 < r$. Construct a weighted graph $Z = (X \sqcup Y, U, d_Z)$, where the weight function on X coincides with the distance function on X, and on Y it coincides with the metric on Y. On the edges connecting vertices x and y such that xRy, set $d_Z(a, b) = r$ if aRb. Since $r \geq \mathrm{dis}(R)/2$, all polygon inequalities hold, and the projection π preserves distances due to Lemma 2.28. Applying a metrically convex function f to this constructed space, we get $\mathrm{d}_{\mathrm{GH}}\bigl(f(X), f(Y)\bigr) \leq \mathrm{d}_{\mathrm{H}}\bigl(f(X), f(Y)\bigr) = f(r)$, where $\mathrm{d}_{\mathrm{H}}\bigl(f(X), f(Y)\bigr)$ is computed within $f\bigl(\pi(Z)\bigr)$. Taking the limit from the condition, we obtain the desired result.

For the second point, it is sufficient to set $r = \mathrm{d}_{\mathrm{GH}}(X, Y)$ and, by choosing a correspondence R such that $\mathrm{dis}(R)/2 = r$, complete the remaining construction from the first point to obtain the required result. ☐

This lemma yields following remark.

Remark 2.36. Suppose that for given spaces X and Y, we have found a metrically convex function f such that $f\big(\mathrm{d_{GH}}(X,Y)\big) < \mathrm{d_{GH}}\big(f(X), f(Y)\big) \le \liminf_{r\to \mathrm{d_{GH}}(X,Y)+} f(r)$. Then there is no optimal correspondence between X and Y.

3. Relation between existence of certain geodesics and optimal correspondence

Definition 3.1. A *piecewise-linear geodesic* $\gamma : [0,1] \to \mathcal{GH}$ is a geodesic with affine-natural parameterization (differ from the natural parameterization by multiplication by a constant) if there exist a finite partition $\Delta = (t_0, \dots, t_n)$ of the interval $[0,1]$ and correspondences $R_i \in \mathcal{R}_{\mathrm{opt}}\big(\gamma(t_{i-1}), \gamma(t_i)\big)$, $i \in \{1, 2, \dots n\}$ such that the geodesic γ is a linear geodesic on each subinterval $[t_i, t_{i+1}]$, constructed using the correspondence R_i.

Proposition 3.2. *If there exists a piecewise-linear geodesic between metric spaces A and B, then the set $\mathcal{R}_{\mathrm{opt}}(A,B)$ is not empty.*

Proof. It is sufficient to consider a correspondence $R = \bigcirc_{i=n}^{1} R_i$ (the composition of correspondences R_i), for which we have due to Lemma 2.11:

$$\mathrm{dis}(R) \le \sum_{i=1}^{n} \mathrm{dis}(R_i) = \sum_{i=1}^{n} 2\,\mathrm{d_{GH}}\big(\gamma(t_{i-1}), \gamma(t_i)\big) = 2\,\mathrm{d_{GH}}(A,B). \qquad \square$$

Corollary 3.3. *If there exist a piecewise-linear geodesic between metric spaces then one can find a linear geodesic.*

Construction 3.4 ([3]). *Let X_n be a fundamental sequence in $\mathcal{GH}$ such that $2\,\mathrm{d_{GH}}(X_n, X_{n+1}) < 2^{-n}$. Choose correspondences $R_n \in \mathcal{R}(X_n, X_{n+1})$ such that $\mathrm{dis}(R_n) < 2^{-n}$. Consider the set $\tilde{X} = \prod_{i=1}^{\infty} X_i$. A* thread *is a sequence of points $x_n \in X_n$ such that $x_n R_n x_{n+1}$ for all n. Denote the set of threads as $\tilde{X}$, totally order this set, and denote the i-th thread as $\{x_n^i\}$. We define a pseudo-metric on $\tilde{X}$.*

Theorem 3.5 ([3]). *The function $d_{\tilde{X}}\big(\{x_n^i\}, \{x_n^j\}\big) = \lim_{m\to\infty} d_{X_m}(x_m^i, x_m^j)$ is a pseudo-metric on $\tilde{X}$, and $\sup_{i,j}\Big|d_{\tilde{X}}\big(\{x_n^i\}, \{x_n^j\}\big) - d_{X_m}\big(x_m^i, x_m^j\big)\Big| \to 0$, $m \to \infty$. In particular, $\mathrm{d_{GH}}(\tilde{X}, X_n) \to 0$, $n \to \infty$.*

Proposition 3.6. *Let $\gamma : [0,1] \to \mathcal{GH}$ be a geodesic with an affine-natural parameterization that connects metric spaces A and B. Suppose there exists monotonically increasing sequence t_i of the interval $[0,1]$ that tends to 1, and there exists an optimal correspondence between $\gamma(t_i)$ and $\gamma(t_{i+1})$. Then there exists a 0-modification $\tilde{B}$ of space B such that there is an optimal correspondence between A and $\tilde{B}$.*

Proof. We construct a linear geodesic between each pair of neighbor $\gamma(t_i)$. Without loss of generality we assume that γ is such a geodesic. Note that for any positive

δ, between any two spaces from the image of the interval $[0, 1-\delta]$ there exists an optimal correspondence according to Proposition 3.2. Let X_n be the sequence of points $\gamma(1 - 1/2^{n+K})$ for some $K \in \mathbb{N}$. Choose K such that $\mathrm{d_{GH}}(X_n, X_{n+1}) \leq \mathrm{d_{GH}}(A, B) * 1/2^{n+K} < 1/2^{n+1}$. Using Construction 3.4 and Theorem 3.5, construct the limit space $\tilde{B}'$. Due to the choice of X_n and Theorem 3.5, $\mathrm{d_{GH}}(B, \tilde{B}') \leq \mathrm{d_{GH}}(B, X_n) + \mathrm{d_{GH}}(\tilde{B}', X_n) \to 0$ as $n \to \infty$. Therefore, the metric space $\tilde{B}$, obtained by factoring $\tilde{B}'$ by zero distances is a 0-modification of the metric space B. Notice that there exists an optimal correspondence between A and $\tilde{B}$ if and only if there exists an optimal correspondence between A and $\tilde{B}'$. Now we construct an optimal correspondence between A and $\tilde{B}'$. Recall the structure of the limit metric space and consider the correspondence $R \in \mathcal{R}(A, \tilde{B}') = \Big\{\big(a, \{x_n^i\}\big) : aR_0x_1^i\Big\}$ and $\tilde{R}_n \in \mathcal{R}(A, X_n') = \Big\{\big(a, x_n^i\big) : aR_0x_1^i\Big\}$ where $R_0 \in \mathcal{R}_{\mathrm{opt}}(A, X_1)$. Note that $\mathrm{dis}(\tilde{R}_n) \to 2\,\mathrm{d_{GH}}(A, \tilde{B}')$ as $n \to \infty$ due to the construction. Now we show, that $\mathrm{dis}(R) \leq \lim\limits_{n\to\infty} \mathrm{dis}(\tilde{R}_n)$. The distortion of R is given by

$$\begin{aligned}
\mathrm{dis}(R) &= \sup_{(a,\{x_n^i\}),(a',\{x_n^j\})} \Big| d_A(a,a') - d_{\tilde{B}'}\big(\{x_n^i\},\{x_n^j\}\big)\Big| \\
&= \sup_{(a,\{x_n^i\}),(a',\{x_n^j\})} \Big| d_A(a,a') - \lim_{n\to\infty} d_{X_n}(x_n^i, x_n^j)\Big| \\
&\leq \lim_{n\to\infty} \sup_{(a,\{x_n^i\}),(a',\{x_n^j\})} \Big| d_A(a,a') - d_{X_n}(x_n^i, x_n^j)\Big| = \lim_{n\to\infty} \mathrm{dis}(\tilde{R}_n) = 2\,\mathrm{d_{GH}}(A, \tilde{B}').
\end{aligned}$$

Thus, R is optimal correspondence. □

Corollary 3.7. *Let $\gamma : [0,1] \to \mathcal{GH}$ be a geodesic in the affine-natural parameterization connecting metric spaces A and B. Suppose there exists $t : \mathbb{Z} \to [0,1]$, $t(i) = t_i$ of the interval $[0,1]$ such that 0 and 1 are the only limit points, and there exists an optimal correspondence between neighbor points. Then there exist 0-modifications $\tilde{A}$ and $\tilde{B}$ of spaces A and B, respectively such that there exists an optimal correspondence between $\tilde{A}$ and $\tilde{B}$.*

Proof. By the analogue with Proposition 3.6, we assume that the geodesic between neighboring points is linear. Take an arbitrary point t_0 from the interval $(0,1)$. According to Proposition 3.6, there exist 0-modifications of spaces $\tilde{A}$ and $\tilde{B}$ such that $R_1 \in \mathcal{R}_{\mathrm{opt}}\big(\tilde{A}, \gamma(t_0)\big)$ and $R_2 \in \mathcal{R}_{\mathrm{opt}}\big(\gamma(t_0), \tilde{B}\big)$. Since γ is a geodesic, we have $\mathrm{d_{GH}}(A, B) = \mathrm{d_{GH}}\big(A, \gamma(t_0)\big) + \mathrm{d_{GH}}(\gamma(t_0), B) = \mathrm{d_{GH}}\big(\tilde{A}, \gamma(t_0)\big) + \mathrm{d_{GH}}\big(\gamma(t_0), \tilde{B}\big) = 1/2\big(\mathrm{dis}(R_1) + \mathrm{dis}(R_2)\big)$, therefore $R_1 \circ R_2$ is an optimal correspondence between $\tilde{A}$ and $\tilde{B}$. □

Definition 3.8. We denote such geodesics as *exhaustive piecewise-linear geodesics.*

4. Example of metric spaces with an empty set of optimal correspondences

In [1], the similar result, but for unbounded case, was obtained. We would use the following example to prove that there are no piecewise-linear geodesics.

Example 4.1. We provide an example of complete metric spaces X and Y such that $\mathrm{d_{GH}}(X,Y) > 0$, and $\mathcal{R}_{\mathrm{opt}}(X,Y)$ is an empty set.

To do this, consider $X = \triangle_2$ and $Z = (\mathbb{N}, d_Z)$, where

$$d_Z(i,j) = \begin{cases} 0, & i = j, \\ 1/4 - 1/2^{\max(i,j)+2}, & i \neq j. \end{cases}$$

Note that all distances between distinct points are greater than or equal to $1/4 - 1/2^4$. Note that all distances are in the interval $[1/8, 1/4]$. Space Z is actually a metric space because for any $p_1, p_2, p_3 \in Z$, $p_i \neq p_j$ for $i \neq j$, we have $d(p_1,p_2) \leq 1/4 \leq 1/8 + 1/8 \leq d(p_1,p_3) + d(p_3,p_2)$.

Subdivide metric space Z as follows: for each ordered pair of points $(z_1, z_2) \in Z \times Z$, where $z_1 < z_2$, add a new point z_3 and denote the new graph as G.

1. The set of points of the graph G lying in Z we denote by OLD.
2. The set of remaining points of the graph G we denote by NEW.

We connect each point $z_3 \in$ NEW with edges to z_1 and z_2 and assign weights to these edges as follows: $\omega(z_3,z_2) = \delta = 1/2023$, $\omega(z_1,z_3) = \omega(z_1,z_2) - \delta$. We define the functions:

3. $\underline{\mathrm{left}}$: NEW $\to$ OLD, $\underline{\mathrm{right}}$: NEW $\to$ OLD: $\underline{\mathrm{left}}(z_3) = z_1$ and $\underline{\mathrm{right}}(z_3) = z_2$, where z_3 added for the pair $z_1 \prec z_2$;
4. $\underline{\mathrm{nearest}}$, where $\underline{\mathrm{nearest}}(z_3) = \{\underline{\mathrm{left}}(z_3), \underline{\mathrm{right}}(z_3)\}$ for $z_3 \in$ NEW;
5. $\underline{\mathrm{far}}(z_3) = \mathrm{OLD} \setminus \underline{\mathrm{nearest}}(z_3)$ for each $z_3 \in$ NEW.

Here and below, z_3 and y_3 denote arbitrary points from NEW, $z_2 = \underline{\mathrm{right}}(z_3)$, $z_1 = \underline{\mathrm{left}}(z_3)$, $y_2 = \underline{\mathrm{right}}(y_3)$, $y_1 = \underline{\mathrm{left}}(y_3)$.

We put $Y = \pi(G)$ and continue to denote by Z the image of Z. Since the graph G was obtained by Construction 2.29, we apply Lemma 2.30 (it is valid because the subgraphs G_{z_1,z_2} satisfy the triangle inequalities), and obtain that the projection π preserves the weights. By Lemma 2.26, the space Y is metric. We will also consider Y as a weighted graph, where the weight function is the distance. We are going to describe what other distances look like in the new space.

Lemma 4.2. *The distance from $z_3 \in$ NEW to $y \in \underline{\mathrm{far}}(z_3)$ is equal to the length of the two-edge path $L = z_3uy$ for $u = \underline{\mathrm{right}}(z_3) = z_2$, moreover $|L| = \omega(z_2y) + \delta = d_{\max(z_2,y)} + \delta < 1/2$.*

Proof. According to item 3. of Lemma 2.30, the distance is computed as the length of the path $L = z_3uy$ for some $u \in \underline{\mathrm{nearest}}(z_3)$, and due to weight structure of graph G, it is equal to

1. $d_k - \delta + d_m$ for some $2 \leq k, m \in \mathbb{N}$, if L passes through $z_1 = \underline{\mathrm{left}}(z_3)$, or

2. $\delta + d_l$ for some $l \in \mathbb{N}$, if L passes through $z_2 = \underline{\text{right}}(z_3)$.
Note that

$$\delta + d_l = \delta + 1/4 - 1/2^{l+2} < 1/4 + \delta < 1/4 + 1/8 - \delta = 1/4 - 1/16 - \delta + 1/4 - 1/16$$
$$\leq 1/4 - 1/2^{k+2} - \delta + 1/4 - 1/2^{m+2} = d_k - \delta + d_m.$$

Thus, the distance is computed as the length of the path $L = z_3uy$ for $u = z_2 = \underline{\text{right}}(z_3)$, and its length is $\delta + \omega(z_2y)$. □

Lemma 4.3. *Let for $z_3, y_3 \in$ NEW the conditions hold*

$$z_1 = \underline{\text{left}}(z_3) \neq \underline{\text{right}}(y_3) = y_2, \quad z_2 = \underline{\text{right}}(z_3) \neq \underline{\text{left}}(y_3) = y_1.$$

Then the distance from z_3 to y_3 is equal to the length of the three-edge path z_3uvy_3, where $u = \underline{\text{right}}(z_3)$, $v = \underline{\text{right}}(y_3)$, and equals $\omega(z_2y_2) + 2\delta = d_{\max(y_2,z_2)} + 2\delta < 1/2$.

Proof. According to item 2. of Lemma 2.30, the shortest curve has the form z_3uvy_3, where $u = \underline{\text{nearest}}(z_3)$, $v = \underline{\text{nearest}}(y_3)$, and its length is
1. $(d_m - \delta) + (d_k) + (d_p - \delta)$, if $u = \underline{\text{left}}(z_3)$ and $v = \underline{\text{left}}(y_3)$,

2. $(\delta) + (d_{k'}) + (\delta)$, if $u = \underline{\text{right}}(z_3)$ and $v = \underline{\text{right}}(y_3)$,

3. $(d_m - \delta) + (d_{k''}) + (\delta)$, if $u = \underline{\text{left}}(z_3)$ and $v = \underline{\text{right}}(y_3)$,

4. $(\delta) + (d_{k'''}) + (d_p - \delta)$, if $u = \underline{\text{right}}(z_3)$ and $v = \underline{\text{left}}(y_3)$,
where $d_m = \omega(z_1z_2)$, $d_p = \omega(y_1y_2)$, and k, k', k'', k''' are non-negative integers, and $2 < m, p \in \mathbb{N}$ due to the conditions. Note that

$$\delta + d_{k'} + \delta < 1/4 + 2\delta < 1/2 - 1/8 - 2\delta$$
$$= 1/4 - 1/16 + 1/4 - 1/16 - 2\delta \leq d_m - \delta + d_{k''} + \delta$$

and $$\delta + d_{k'} + \delta < \delta + d_{k'''} + d_p - \delta.$$

Due to inequality $2\delta + d_n < 1/2$, we have $d(z_3, y_3) < 1/2$. □

Lemma 4.4. *Let for distinct z_3, $y_3 \in$ NEW, $\underline{\text{right}}(y_3) = \underline{\text{left}}(z_3)$. Then $d(z_3, y_3) = \omega(z_1z_2) = d_{z_2} < 1/2$.*

Proof. According to item 2. of Lemma 2.30, the shortest curve has the form z_3uvy_3, where $u = \underline{\text{nearest}}(z_3)$, $v = \underline{\text{nearest}}(y_3)$, and its length is
1. $L_1 = (d_m - \delta) + (d_k) + (d_p - \delta)$, if $u = \underline{\text{left}}(z_3)$ and $v = \underline{\text{left}}(y_3)$,

2. $L_2 = (\delta) + (d_{k'}) + (\delta)$, if $u = \underline{\text{right}}(z_3)$ and $v = \underline{\text{right}}(y_3) = \underline{\text{left}}(z_3)$,

3. $L_3 = (d_m - \delta) + (d_{k''}) + (\delta)$, if $u = \underline{\text{left}}(z_3) = v = \underline{\text{right}}(y_3)$,

4. $L_4 = (\delta) + (d_{k'''}) + (d_p - \delta)$, if $u = \underline{\text{right}}(z_3)$ and $v = \underline{\text{left}}(y_3)$,
where $d_m = \omega(z_1z_2)$, $d_p = \omega(y_1y_2)$, and k'', k''' are non-negative integers, and k, k' are greater than or equal to 2. The case of $d_{k''} = d(\underline{\text{right}}(z_3), \underline{\text{left}}(z_3)) = 0$ means $L_3 = d_m < 1/4$. Note that $L_1 \geq d_m + (d_p - 2\delta)$, and the expression in parentheses is positive, so $L_1 > L_3$. Note that $d_{k'} = d(\underline{\text{right}}(z_3), \underline{\text{left}}(z_3)) = d(z_1, z_2) = d_m$ and $L_2 = d_m + 2\delta > L_1$. Finally, conditions $\underline{\text{right}}(z_3) = \underline{\text{left}}(y_3)$ and $\underline{\text{right}}(y_3) = \underline{\text{left}}(z_3)$ are mutually exclusive for $z_3 \neq y_3$, so $d_{k'''} > 0$ and $L_4 = d_{k'''} + d_p \geq 1/4 - 1/16 + 1/4 - 1/16 > 1/4 > d_m = L_3$. □

Lemma 4.5. *For spaces X and Y, we have $2\,\mathrm{d_{GH}}(X,Y)=3/4+\delta$ and $R_{\mathrm{opt}}(X,Y)=\emptyset$.*

Proof. Any irreducible correspondence between spaces X,Y has the form $R=\big\{\{x_1\}\times R(x_1),\ \{x_2\}\times R(x_2)\big\}$, where $R(x_1)\cap R(x_2)=\emptyset$.

We provide an example of a sequence of correspondences $R_N\in\mathcal{R}(X,Y)$ whose distortion approaches to $3/4+\delta$. Let $A_1=\{n\in Z\mid n\le N\}$, $A_2=\{n\in Z\mid n>N\}$, then

$$R_N(x_1)=A_1\bigcup\{u\in Y\mid \exists a\in A_1 \text{ such that } d_Z(u,a)=\delta\},$$
$$R_N(x_2)=A_2\bigcup\{u\in Y\mid \exists a\in A_2 \text{ such that } d_Z(u,a)=\delta\}.$$

The union of $R_N(x_1)$ and $R_N(x_2)$ contains all of Z, as well as all of NEW (for each z_3 in NEW, there exists $\underline{\mathrm{right}}(z_3)$ at the distance δ), and $R_N(x_1)\cap R_N(x_2)=\emptyset$, since each point z_3 in NEW has exactly one OLD point at a distance of δ, and points at a distance of δ can only be from $z_3\in$ NEW and its $\underline{\mathrm{right}}(z_3)$.

The diameter of this partition is less than $1/2$ because $\mathrm{diam}(Y)$ is less than $1/2$. The distances between points from different elements of the partition have the following types:
1. the distance from a point in OLD $\cap R_N(x_2)$ to a point in OLD $\cap R_N(x_1)$ is of type d_M,

2. from a point in OLD $\cap R_N(x_2)$ to a point in NEW $\cap R_N(x_1)$ is of type $d_M+\delta$ (they are $\underline{\mathrm{far}}$ for these NEW),

3. from a point in NEW $\cap R_N(x_2)$ to a point in OLD $\cap R_N(x_1)$ is either $d_M-\delta$ or $d_M+\delta$,

4. from a point in NEW $\cap R_N(x_2)$ to a point in NEW $\cap R_N(x_1)$ is either $d_M+2\delta$ or d_M (see Lemma 4.3 and 4.4),

where M is a positive integer.

Each of the distances described above, except, maybe, d_M in Item 4, is calculated along some path that goes through an old point $z>N$, so $M>N$.

The value of d_M in Item 4 is the distance from $z_3\in R_N(x_2)\cap$ NEW, which was added for $z_1<z_2$, where $z_2>N$, to some $y_3\in R_N(x_1)\cap$NEW, where $y_1=\underline{\mathrm{left}}(y_3)=z_2=\underline{\mathrm{right}}(z_3)$, or $y_2=\underline{\mathrm{right}}(y_3)=z_1=\underline{\mathrm{left}}(z_3)$ according to Lemma 4.4. The first case is impossible as y_3 could only be added for $y_1<y_2\le N$, while z_3 was added for $z_1<z_2>N$. Hence, $y_2=\underline{\mathrm{right}}(y_3)=z_1=\underline{\mathrm{left}}(z_3)$. According to Lemma 4.4, the distance between z_3 and y_3 is $d(z_1,z_2)=d_M=d_{\max(z_1,z_2)}$, which means that $M>N$ in this case as well.

In accordance with the distances between points in $R(x_1)$ and $R(x_2)$, the minimum distance between points $y\in R_N(x_1)$ and $y'\in R_N(x_2)$ is equal to $d_{N+1}-\delta=1/4-1/2^{N+3}-\delta$, that is the distance between z_3 (added for $1<N+1$) and 1. Since $\max\Big(\mathrm{diam}\big(R_N(x_1)\big),\mathrm{diam}\big(R_N(x_2)\big)\Big)<1/2$, then

$$\mathrm{dis}(R_N)=\max\big(1-1/4+1/2^{N+3}+\delta,\mathrm{diam}\big(R_N(x_1)\big),\mathrm{diam}\big(R_N(x_2)\big)\big)\to 3/4+\delta.$$

Now let us consider an arbitrary irreducible correspondence $R\in\mathcal{R}(X,Y)$ with $\mathrm{dis}(R)<1-\delta$. We will prove that its distortion is strictly greater than $3/4+\delta$.

Consider an arbitrary point z_3 from NEW, suppose it belongs to $R(x_1)$. If $\underline{\text{right}}(z_3)$ belongs to $R(x_2)$, then the distortion of such correspondence is no less than $1-\delta$. Therefore, each $z_3 \in$ NEW belongs to the same partition element as its $\underline{\text{right}}(z_3)$. Since each $R(x_1)$ and $R(x_2)$ is a non-empty subset, each of them contains a point from Z (if it contains $z_3 \in$ NEW, it also contains its $\underline{\text{right}}(z_3)$). Let's find the minimum n such that $n \in R(x_1)$ (without loss of generality), and $n+1 \in R(x_2)$. Then $\operatorname{dis}(R) \geq \big|1-(1/4+1/2^{n+3}+\delta)\big|$, because $z_3 \in$ NEW added to the pair $n, n+1$, belongs to $R(x_2)$. Therefore, $2\,\mathrm{d_{GH}}(X,Y) = 3/4+\delta$, and $\mathcal{R}_{\mathrm{opt}}(X,Y)=\emptyset$. □

5. Exhaustive piecewise linear geodesics are insufficient to describe all geodesics in the Gromov–Hausdorff space

In this section we prove that the metric spaces X and Y from Example 4.1 do not have 0-modifications, but could be connected by some geodesics of other kind.

Lemma 5.1. *Let* $\mathrm{d_{GH}}(A,B)=0$*, then* $\overline{\sigma(B)} = \overline{\sigma(A)}$.

Proof. We prove that $\sigma(B) \subseteq \overline{\sigma(A)}$. Consider an arbitrary $|yy'| \in \sigma(B)$ and $R_n \in \mathcal{R}(A,B)$ such that $\operatorname{dis}(R_n) \to 0$. Then for any n, there exist a_n, a'_n such that $\big||bb'| - |a_n a'_n|\big| \leq \operatorname{dis}(R_n) \to 0$, which implies that $|bb'|$ is a limit point of $\sigma(A)$ and thus lies in its closure. Similarly, we obtain $\sigma(A) \subseteq \overline{\sigma(B)}$, which implies the desired result. □

Lemma 5.2. *Let a metric space A have positive* $\mathrm{s}(A)$ *and* $\mathrm{e}(A)$*. Then A does not have* 0*-modifications.*

Proof. Let $\tilde{A}$ be a 0-modification of A. Choose $\varepsilon < \min(\mathrm{e}(A), \mathrm{s}(A))/8$ and $R \in \mathcal{R}(A,\tilde{A})$ such that $\operatorname{dis}(R) < 2\varepsilon$. Then, by Theorem 2.21, the correspondence R is optimal. This implies $\operatorname{dis}(R)=0$, and thus R is an isometry. □

Lemma 5.3. *Let $\sigma(A)$ be a finite subset of the real line. Then A does not have* 0*-modifications.*

Proof. Consider B such that $\mathrm{d_{GH}}(A,B)=0$. Due to Lemma 5.1, its spectrum is also a finite subset of the real line. By Proposition 6.1, the distance between them is achieved by some correspondence R, meaning R is an isometry. □

Proposition 5.4. *Let A, B be metric spaces, and let* $U = \big\{\gamma \in \mathbb{R}_{\geq 0} : \exists \alpha \in \sigma(A)$ *and* $\beta \in \sigma(B)$ *such that* $\gamma = |\alpha-\beta|\big\}$ *be discrete, namely it has no limit points (as a subset of the real line). Then there exists an optimal correspondence between metric spaces A and B.*

Proof. Indeed, the distortions of all correspondences are in U, so any decreasing sequence of distortion values attains its infimum on some correspondence. □

Consider the spaces X and Y from Example 4.1.

Lemma 5.5. *The spaces X and Y do not have* 0*-modifications.*

Proof. The space X has no 0-modifications due to Corollary 2.17.

We prove that $\mathrm{s}(Y)$ and $\mathrm{e}(Y)$ are strictly positive. By construction, $\mathrm{s}(Y) = \delta$. Let $\phi : Y \to Y$ be a bijection such that $\mathrm{dis}(\phi) < \delta$. Consider Y as a weighted graph G with weights corresponding to distances between points. Let G_δ be a subgraph of G, where the vertices set of the graph G_δ coincides with the set of vertices of the graph G, and the edges of G_δ are all edges of G of the weight δ. Each point i has exactly $i-1$ points at the distance δ. Each of these $i-1$ points is from NEW and has exactly one point at the distance δ, namely, the point i. Therefore, graph G_δ has a countable number of connected components, each finite, and all consisting of different number of points. The mapping ϕ takes points at the distance δ to points at distance δ, because $\sigma(Y) \cap (0, 2\delta) = \{\delta\}$ and $\mathrm{dis}(\phi) < \delta$. This implies $\phi(G_\delta) \subseteq G_\delta$ and $\phi^{-1}(G_\delta) \subseteq G_\delta$ (because $\mathrm{dis}(\phi^{-1}) = \mathrm{dis}(\phi) < \delta$). Therefore, $\phi(G_\delta) = G_\delta$, which means that ϕ is an isomorphism of the graph G_δ. In particular, $\phi(i) = i$ for $i = 1$ or $i > 2$, because such points i have degree $i-1$, and there are no other points of degree $i-1$ in the graph G_δ. We denote the unique point added for the pair $1 < 2$ by $3/2$. Since connected components are mapped to connected components, the vertex 2 of the graph G is mapped to the vertex $3/2$ or 2 from G. If $\phi(2) = 3/2$, then $\mathrm{dis}(\phi) \geq \big|d(\phi(2), \phi(1)) - d(2,1)\big| = \big|d(3/2, 1) - d(2,1)\big| = \delta$. A contradiction. Hence, $\phi(i) = i$ for all positive integer i. We prove that the mapping ϕ is the identical on NEW. Let z_3 be added for $k < i$, and $y_3 := \phi(z_3) \neq z_3$. Since ϕ is an isomorphism of the graph G_δ and ϕ is constant on i, then $\phi(S_\delta(i)) = S_\delta(\phi(i)) = S_\delta(i)$. Thus, y_3 is added for $l < i$, $k \neq l$ ($d(y_3, i) = \delta$, hence, $i = \underline{\mathrm{right}}(y_3)$). But $d(z_3, k) = d_i - \delta$ by definition and $d(y_3, k) = d_i + \delta$ according to Lemma 4.4 because $k \in \underline{\mathrm{far}}(y_3)$. Therefore, $\big|d(z_3, k) - d(\phi(z_3), \phi(k))\big| = \big|d(z_3, k) - d(y_3, k)\big| = 2\delta$, a contradiction. Thus, $\phi = \mathrm{id}$ and $\mathrm{e}(Y) \geq \delta$.

Therefore, we have $\min(\mathrm{s}(Y), \mathrm{e}(Y)) > 0$. Due to Lemma 5.2, Y has no 0-modifications. □

Theorem 5.6. *There exists a geodesic between the spaces X and Y from Example* 4.1.

Proof. Consider two spaces, $Z = (\mathbb{N}, d_Z)$ and $Z' = (\mathbb{N} \cup \infty, d_{Z'})$, where the distances are defined as follows:

$$d_Z(i,j) = \begin{cases} 0, & i = j, \\ 1/4 - 1/2^{\max(i,j)+2} + 2\delta, & i \neq j, \end{cases}$$

and

$$d_{Z'}(i,j) = \begin{cases} 0, & i = j, \\ 1/4 - 1/2^{\max(i,j)+2} + 2\delta, & i, j < \infty \text{ and } i \neq j, \\ 1/4 + 2\delta, & \max(i,j) = \infty \text{ and } i \neq j \end{cases}$$

These distances are metrics because all their nonzero distances belong to the interval $[1/7, 2/7]$.

Lemma 5.7. *For the metric spaces Z and Z', the equality* $\mathrm{d_{GH}}(Z, Z') = 0$ *holds.*

Proof. To show this, consider the correspondences $R_i = \Big\{(j,j) : j \in \mathbb{N}\Big\} \cup \Big\{(i, \infty)\Big\}$.

Their distortion are

$$\begin{aligned}\operatorname{dis}(R_i) &= \max\Big(\sup_{j,k<\infty}\big|d_Z(j,k) - d_{Z'}(j,k)\big|,\ \sup_j\big|d_Z(j,i) - d_{Z'}(j,\infty)\big|\Big)\\ &= \sup_j\big|d_Z(j,i) - d_{Z'}(j,\infty)\big| = \sup_j 1/2^{\max(i,j)+2} = 1/2^{i+2}.\end{aligned}$$

To complete the proof, we note these distortions tend to zero as $i \to \infty$. □

Now, we construct geodesics between $X = \triangle_2 = (p_1, p_2)$ and Z' as well as between Z and Y. Consider $R = \big\{(p_1, i) : i \in \mathbb{N}\big\} \cup \big\{(p_2, \infty)\big\} \in \mathcal{R}(X, Z')$. The distortion $\operatorname{dis}(R)$ is

$$\begin{aligned}&\max\Big(\operatorname{diam}\big(R(p_1)\big), \operatorname{diam}\big(R(p_2)\big), \sup_j\Big\{\big|d_X(p_1,p_2) - d_{Z'}(j,\infty)\big|\Big\}\Big)\\ =\ &\max\Big(1/4 + 2\delta, \sup_j\big(|1 - 1/4 - 2\delta|\big)\Big) = 3/4 - 2\delta.\end{aligned}$$

Let $R' = \bigcup\limits_{i<\infty} \{i\} \times B_\delta(i) \in \mathcal{R}(Z, Y)$. The distortion $\operatorname{dis}(R')$ is

$$\begin{aligned}&\max\Big(\sup_i\Big\{\big|d_i + 2\delta - d_i\big|\Big\}, \sup_i\Big\{\big|d_i + 2\delta - (d_i + \delta)\big|\Big\},\\ &\quad \sup_i\Big\{\big|d_i + 2\delta - (d_i - \delta)\big|\Big\}, \sup_i\Big\{\big|d_i + 2\delta - (d_i + 2\delta)\big|\Big\}, 2\delta\Big) = 3\delta.\end{aligned}$$

According to Lemma 4.5, we have $3/4+\delta = 2\,\mathrm{d_{GH}}(X,Y)$, thus $3/4+\delta = 2\,\mathrm{d_{GH}}(X,Y) \le 2\,\mathrm{d_{GH}}(X,Z') + 2\,\mathrm{d_{GH}}(Z',Z) + 2\,\mathrm{d_{GH}}(Z,Y) \le \operatorname{dis}(R) + \operatorname{dis}(R') = 3/4 + \delta$.

Therefore, R and R' are optimal correspondences, so there exist linear geodesics between the spaces X, Z, and Z', Y. Since $\mathrm{d_{GH}}(X,Y) = \mathrm{d_{GH}}(X,Z) + \mathrm{d_{GH}}(Z,Z') + \mathrm{d_{GH}}(Z',Y)$ and $\mathrm{d_{GH}}(Z,Z') = 0$, then metric spaces X, Y can also be connected by a geodesic concatenated form the two geodesics constructed above. □

Notice that there is no piecewise-linear geodesic or geodesic exhaustible by piecewise-linear geodesics between the spaces X and Y as well as their 0-modifications, because there is no optimal correspondence between X, Y and the spaces X, Y do not have 0-modifications.

Remark 5.8. The spaces Z and Z' are not isometric to each other and $\mathrm{d_{GH}}(Z,Z') = 0$ because their spectra are different.

6. Constructing a dense subclass of $\mathcal{GH}$ consisting of metric spaces with optimal correspondence between each pair

Lemma 6.1. *Let $X, Y \in \mathcal{B}$ be metric spaces with finite spectra $\sigma(X)$ and $\sigma(Y)$. Then the distance between these metric spaces is achieved by some correspondence.*

Proof. It is enough to notice that the distortions of all correspondences also form a finite subset of the real line, then the infimum of distortions is achieved. □

Recall the ceiling function $\lceil x \rceil = \min\{n \in N, x \leq n\}$.

Definition 6.2. The function $l_\varepsilon(x) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $l_\varepsilon(x) = \varepsilon\lceil x/\varepsilon \rceil$ is called the *ε-ladder.*

Lemma 6.3 ([13]). *The ε-ladder is the distance-preserving function.*

Corollary 6.4. *In $\mathcal{GH}$, there exists a dense class of metric spaces D such that for any pair $X, Y \in D$, there exists an optimal correspondence.*

Proof. Consider class D obtained by applying $l_{2^{-n}}$ to each metric space. Then the difference between any two distances from metric space $A_n = l_{2^{-n}}(X)$ and metric space $B_m = l_{2^{-m}}(Y)$ is $k2^{-\max(m,n)}$. By Lemma 5.4, there is an optimal correspondence between the spaces A_n and B_m. Due to Lemma 2.34, the spaces A_n tends to X as $n \to \infty$. □

Corollary 6.5. *For any $X \in \mathcal{GH}$, we have $\mathrm{s}\big(l_{2^{-n}}(X)\big) \geq 2^{-n} > 0$ and $\mathrm{e}'\big(l_{2^{-n}}(X)\big) \geq 2^{-n} > 0$, hence class D is contained in the class of metric spaces in generalized general position.*

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Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Moscow, Russia
E-mail: Vihrov.09@gmail.com
ORCID iD: https://orcid.org/0009-0008-9852-2152