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GROMOV–HAUSDORFF DISTANCES BETWEEN NORMED SPACES

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Abstract. In the present paper we study the original Gromov–Hausdorff distance between real normed spaces. In the first part of the paper we prove that two finite-dimensional real normed spaces on a finite Gromov–Hausdorff distance are isometric to each other. We then study the properties of finite point sets in finite-dimensional normed spaces whose cardinalities exceed the equilateral dimension of an ambient space. By means of the obtained results we prove the following enhancement of the aforementioned theorem: every finite-dimensional normed space lies on an infinite Gromov–Hausdorff distance from all other non-isometric normed spaces.

1. Introduction

The Gromov–Hausdorff distance is one of the most beautiful constructions in metric geometry. It allows us to compare how similar two arbitrary metric spaces are. The concept was first introduced in [4] by D. Edwards. Later, it became famous due to M. Gromov's paper [5] (see historical details in [12]). The classical Gromov–Hausdorff distance between metric spaces X and Y is defined as the infimum of the Hausdorff distances between the images X' and Y' of the spaces X and Y under all possible isometric embeddings $\phi\colon X \to Z$ and $\psi\colon Y \to Z$ into an arbitrary metric space Z.

In most of applications the Gromov–Hausdorff distance is used to study compact metric spaces. In the case of unbounded metric spaces the punctured spaces are considered, i.e., the pairs (X, p) where X is an arbitrary metric space and p is one of its points. Traditionally, in such situations the topology generated by the Gromov–Hausdorff distance is of more interest than the distance itself. By $B_r(x)$ let us denote a closed ball of radius r centered at the point x. Then the sequence of punctured metric spaces (X_n, p_n) converges by Gromov–Hausdorff to a punctured metric space (Z, p), if and only if the following conditions hold (see details in [3,8]): for every $r > 0$ and $\varepsilon > 0$ there exists a natural number n_0 such that for every $n > n_0$, there exists a mapping $f\colon B_r(p_n) \to X$ with properties:

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(i) $f(p_n)=p$; (ii) $\operatorname{dis}(f):=\sup\Big\{\big||xx'|-|f(x)f(x')|\big|\colon x,x'\in X\Big\}<\varepsilon$;

(iii) ε-neighbourhood of the set $f(B_r(p_n))$ in X contains the ball $B_{r-\varepsilon}(p)$.

In the present paper, we study the Gromov–Hausdorff distance in its original sense. Restrictions of the Gromov–Hausdorff distance to various classes of metric spaces possess fascinating and often unexpected geometric properties. For example, let us consider the class (in the sense of NBG set theory) $\mathcal{GH}$ of representatives of all isometry classes of metric spaces. The Gromov–Hausdorff distance defines a generalised pseudometric on this class, i.e., is a symmetric, non-zero function, that satisfies the triangle inequality. This space is called *the Gromov-Hausdorff class.* The equivalence classes in $\mathcal{GH}$ under the relation: $X\sim Y$ if and only if $d_{GH}(X,Y)<\infty$ — are called *clouds.* In the monograph [6] M. Gromov announced that each cloud in $\mathcal{GH}$ is contractible and gave an example of the cloud corresponding to $\mathbb{R}^n$. However, it turned out later that there exist clouds which are not even invariant under the multiplication of all their spaces by some positive number λ (see details in paper [1]).

Here, we are studying the Gromov–Hausdorff distance between real normed spaces. Unlike in [9], we consider the global Gromov–Hausdorff distance rather than the Gromov–Hausdorff distance between the closed unit balls of normed spaces. For an arbitrary $\varepsilon>0$ let us call a mapping T between normed spaces E and F an ε-isometry, if for all $x,y\in E$, the following inequality holds: $\Big|\|x-y\|-\|T(x)-T(y)\|\Big|\leqslant\varepsilon$. In the paper [7] the following theorem is proved: if E and F are finite-dimensional normed spaces and $T\colon E\to F$ is a surjective ε-isometry for some ε, then there exists an isometry $I\colon E\to F$ such that $\|T(x)-I(x)\|\leqslant 5\varepsilon$ for all $x\in E$. It follows from this result that any two finite-dimensional normed spaces on a finite Gromov–Hausdorff distance are isometric. In Section 3 we give a simpler proof of this theorem, based on Theorem 3.1, which provides a natural sufficient condition for isometric embeddability of a given bounded metric space into a given finite-dimensional normed space.

Equilateral dimension of a metric space is the largest cardinality of such its subset that all the distances between different points of this subset are pairwise equal to each other. In the paper [11] it is proved that the equilateral dimension of an arbitrary n-dimensional normed space does not exceed 2^n. In Section 4 we study some properties of finite sets of points in normed spaces whose cardinalities exceed the equilateral dimension of the ambient space. Based on the obtained results, we prove a new lower estimate on the Gromov–Hausdorff distance between a finite-dimensional normed space and an arbitrary metric space of a larger equilateral dimension (see Theorem 4.4). From this estimate we obtain the following enhancement of the main theorem of Section 3: every finite-dimensional normed space lies on an infinite Gromov–Hausdorff distance from all other non-isometric normed spaces.

2. Main definitions and preliminary results

We start with the introduction of some basic notation.

Let X be an arbitrary metric space. We denote the distance between the points x and y of X by $|xy|$. Let $U_r(a) = \{x \in X : |ax| < r\}$, $B_r(a) = \{x \in X : |ax| \leqslant r\}$ be an open and a closed ball of radius r centered at the point a correspondingly. For an arbitrary subset $A \subset X$ we define $U_r(A) = \cup_{a\in A} U_r(a)$ — an open r-neighbourhood of A. For non-empty subsets $A \subset X$ and $B \subset X$ by $\mathrm{d}(A, B)$ we denote a simple distance between these subsets, namely, $\mathrm{d}(A, B) = \inf\bigl\{|ab| : a \in A, b \in B\bigr\}$.

DEFINITION 2.1. Let A and B be non-empty subsets of a metric space X. *The Hausdorff distance* between A and B is the value

$$\mathrm{d}_H(A, B) = \inf\{r > 0 : A \subset U_r(B), B \subset U_r(A)\}.$$

DEFINITION 2.2. Let X and Y be metric spaces. If X', Y' are subsets of a metric space Z such that X' is isometric to X and Y' is isometric to Y, then we call the triple (X', Y', Z) *a metric realization of a pair* (X, Y).

DEFINITION 2.3. *The Gromov–Hausdorff distance* $\mathrm{d}_{GH}(X, Y)$ between two metric spaces X, Y is the infimum of positive numbers r such that there exists a metric realization (X', Y', Z) of a pair (X, Y) with $\mathrm{d}_H(X', Y') \leqslant r$.

Let now X, Y be non-empty sets.

DEFINITION 2.4. Any subset $\sigma \subset X \times Y$ is called a *relation* between X and Y.

Denote the set of all non-empty relations between X and Y by $\mathcal{P}_0(X, Y)$.
Define $\pi_X : X \times Y \to X$, $\pi_X(x, y) = x$, $\pi_Y : X \times Y \to Y$, $\pi_Y(x, y) = y$.

DEFINITION 2.5. Relation $R \subset X \times Y$ is called a *correspondence*, if $\pi_X|_R$ and $\pi_Y|_R$ are surjective.

Denote the set of all correspondences between X and Y by $\mathcal{R}(X, Y)$.

DEFINITION 2.6. Let X, Y be arbitrary metric spaces. Then for every $\sigma \in \mathcal{P}_0(X, Y)$ *the distortion* of σ is defined as $\operatorname{dis}\sigma = \sup\Bigl\{\bigl||xx'| - |yy'|\bigr| : (x, y), (x', y') \in \sigma\Bigr\}$.

CLAIM 2.7 ([3]). *For arbitrary metric spaces X and Y the following equality holds*

$$2\,\mathrm{d}_{GH}(X, Y) = \inf\bigl\{\operatorname{dis} R : R \in \mathcal{R}(X, Y)\bigr\}.$$

DEFINITION 2.8. Let X be an arbitrary metric space. By $\mathcal{H}(X)$ we denote the set of all non-empty closed bounded subsets in X. The Hausdorff distance defines a finite metric on $\mathcal{H}(X)$ (see, for example, [3]). The resulting metric space is called a *hyperspace* of the space X.

THEOREM 2.9 ([3]). *Let X be an arbitrary metric space. Then X is compact iff $\mathcal{H}(X)$ is compact.*

THEOREM 2.10 ([3]). *If X is a compact metric space and Y is a complete metric space such that $\mathrm{d}_{GH}(X, Y) = 0$, then X is isometric to Y.*

Definition 2.11. The subset of an arbitrary metric space X is called *equilateral* if all the distances between its distinct points are pairwise equal to each other. The *equilateral dimension* of a metric space X is the largest cardinality of its equilateral subset. We denote the equilateral dimension of X by $\mathrm{ed}(X)$.

Theorem 2.12 ([11]). *Let V be an n-dimensional normed space. Then $\mathrm{ed}(V) \leqslant 2^n$.*

Theorem 2.13 ([2]). *Let X be an arbitrary infinite-dimensional normed space. Then for an arbitrary $m \in \mathbb{N}$ there exists a finite equilateral subset in X of m points.*

Definition 2.14. Let X be an arbitrary metric space, and $\varepsilon > 0$ an arbitrary positive number. The subset $P \subset X$ is called *ε-separated*, if for any two points p and q in P the inequality holds $|pq| \geqslant \varepsilon$.

Definition 2.15. Let V be a normed space, and m be some positive integer. We denote the infimum of positive numbers r such that there exists a 1-separated set of m points in $B_r(0)$ by $R_m(V)$.

Finally, we need a few classical results.

Theorem 2.16 ([3, 13]). *Let K be a compact metric space. Then an arbitrary isometric mapping $f\colon K \to K$ is surjective and, hence, is an isometry.*

Theorem 2.17 ([10]). *Two arbitrary normed spaces are isometric iff their closed unit balls are isometric.*

3. Finite-dimensional case

In this section we show that any two finite-dimensional normed spaces on a finite Gromov–Hausdorff distance are isometric to each other.

Theorem 3.1. *Let V be a finite-dimensional normed space and X be an arbitrary bounded metric space. Suppose there exists a sequence of positive numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0 such that for any n there exists a mapping $f_n\colon X \to V$ with $\mathrm{dis}(f_n) \leqslant \varepsilon_n$. Then the completion $\tilde{X}$ of X is compact and both X and $\tilde{X}$ can be isometrically embedded into V.*

Proof. Fix an arbitrary $x \in X$. By t_n denote a translation of V by the vector $-f_n(x)$. Let $g_n = t_n \circ f_n$. Since t_n is an isometry, we have $\mathrm{dis}\, g_n = \mathrm{dis}\, f_n$. For every $n \in \mathbb{N}$, define $X_n = \overline{g_n(X)}$ — a closure of $g_n(X)$ in the topology on V generated by the norm. By the definition of the distortion, for any points $a, b \in X$ the inequality holds $\big|\|a-b\| - \|g_n(a) - g_n(b)\|\big| \leqslant \varepsilon_n$. According to Claim 2.7, we obtain that $\mathrm{d}_{GH}\big(X, g_n(X)\big) \leqslant \frac{\varepsilon_n}{2}$. Since $\mathrm{d}_{GH}\big(g_n(X), \overline{g_n(X)}\big) = 0$, by applying the triangle inequality we obtain that

$$\mathrm{d}_{GH}(X, X_n) \leqslant \mathrm{d}_{GH}\big(X, g_n(X)\big) + \mathrm{d}_{GH}(g_n(X), X_n) \leqslant \frac{\varepsilon_n}{2}.$$

Note that for every $n \in \mathbb{N}$ the point 0 belongs to the set X_n. Also, $\operatorname{diam} X_n \leqslant \operatorname{diam} X + \varepsilon_n$. Hence, all X_n belong to the ball $B := B_{\operatorname{diam}(X)+c}(0)$, where c is some constant such that $\varepsilon_n \leqslant c$ for every n. The space V is finite-dimensional, thus, B is compact. Therefore, $\mathcal{H}(B)$ is compact by Theorem 2.9. Sets X_n are closed and bounded, so they form a sequence of points in $\mathcal{H}(B)$. Then it follows from the compactness of $\mathcal{H}(B)$ that there exists a subsequence X_{n_s} that converges to some non-empty closed and bounded subset A of B. Note that A is compact as the closed subset of a compact ball B. Then for every s the following inequalities hold

$$\mathrm{d}_{GH}(X, A) \leqslant \mathrm{d}_{GH}(X, X_{n_s}) + \mathrm{d}_{GH}(X_{n_s}, A) \leqslant \mathrm{d}_{GH}(X, X_{n_s}) + \mathrm{d}_{H}(X_{n_s}, A).$$

The right side tends to 0 when $s \to \infty$. Thus, $\mathrm{d}_{GH}(X, A) = 0$. Since $\mathrm{d}_{GH}(X, \tilde{X}) = 0$, the triangle inequality implies, that $\mathrm{d}_{GH}(\tilde{X}, A) \leqslant \mathrm{d}_{GH}(\tilde{X}, X) + \mathrm{d}_{GH}(X, A) = 0$, i.e., $\mathrm{d}_{GH}(\tilde{X}, A) = 0$. Then from Theorem 2.10 it follows that $\tilde{X}$ and A are isometric. It implies that $\tilde{X}$ is compact and can be isometrically embedded into V. In particular, X can be isometrically embedded into V. □

LEMMA 3.2. *If X, Y are normed spaces such that $\mathrm{d}_{GH}(X,Y) < \infty$, then $\mathrm{d}_{GH}(X,Y) = 0$.*

Proof. Suppose $\mathrm{d}_{GH}(X, Y) = c < \infty$. Then for an arbitrary $\lambda > 0$ the following equalities hold $c = \mathrm{d}_{GH}(X,Y) = \mathrm{d}_{GH}(\lambda X, \lambda Y) = \lambda\, \mathrm{d}_{GH}(X,Y) = \lambda c$. Hence, $c = 0$. □

THEOREM 3.3. *If V, W are arbitrary finite-dimensional normed spaces with $\mathrm{d}_{GH}(V, W) < \infty$, then V and W are isometric.*

Proof. From Lemma 3.2 we have $\mathrm{d}_{GH}(V, W) = 0$. Then, by Claim 2.7, for an arbitrary positive number $\varepsilon > 0$ there exists a correspondence R between V and W with the distortion $\operatorname{dis} R \leqslant \varepsilon$.

Let us consider the unit balls $B_1 = B_1^V(0), B_2 = B_1^W(0)$ of the spaces V, W correspondingly. It follows from Theorem 3.1 that there exist isometric maps $f_1 \colon B_1 \to B_2$ and $f_2 \colon B_2 \to B_1$. Then $f_2 \circ f_1 \colon B_1 \to B_1$ and $f_1 \circ f_2 \colon B_2 \to B_2$ are isometric embeddings. From Theorem 2.16 we conclude that the constructed mappings $f_2 \circ f_1$, $f_1 \circ f_2$ are isometries. It follows immediately that f_1 and f_2 are bijective, so the balls B_1 and B_2 are isometric to each other. From Theorem 2.17 we conclude that the spaces X and Y are isometric. □

EXAMPLE 3.4. Let us consider any two non-isometric normed spaces X and Y. For example, two copies of $\mathbb{R}^2$ one with the euclidean norm and one with the max-norm. Let us also consider any ε_1-network σ_1 in X and any ε_2-network σ_2 in Y. Then it immediately follows from the proven theorem that $\mathrm{d}_{GH}(\sigma_1, \sigma_2) = \infty$. In particular, the Gromov–Hausdorff distance between $\mathbb{Z}^2$ with metrics induced from X and Y respectively is infinite.

4. Metric imbalance

We start this section with a few results about point sets in finite-dimensional normed spaces whose cardinalities exceed the equilateral dimension of the corresponding space (which is finite due to Theorem 2.12). Firstly, we prove an important inequality that characterizes such point sets.

THEOREM 4.1. *Let V be a finite-dimensional normed space with* $\mathrm{ed}(V) = p$. *Then there exists a constant $c > 0$ such that for an arbitrary set of $m > p$ distinct points $v_1, \ldots, v_m$ in V there exist three of them $\{v_i, v_j, v_k\}$ such that*

$$\varphi(v_i, v_j, v_k) := \left| \frac{\|v_i - v_k\|}{\|v_j - v_k\|} - 1 \right| \geqslant c.$$

Proof. Arguing by contradiction, suppose that for every $\varepsilon_n = \frac{1}{n}, n \in \mathbb{N}, n \geqslant 6$ there exist distinct points $v_1^n, \ldots, v_m^n$ such that for all different i, j, k the following inequalities hold $\varphi(v_i^n, v_j^n, v_k^n) < \varepsilon_n$. For every $\lambda > 0$ we have $\varphi(\lambda v_i^n, \lambda v_j^n, \lambda v_k^n) = \varphi(v_i^n, v_j^n, v_k^n)$, so without loss of generality we assume that $\|v_1^n - v_2^n\| = 1$. Then inequalities $\varphi(v_i^n, v_2^n, v_1^n) < \varepsilon_n$ imply that $\big|\|v_1^n v_i^n\| - 1\big| \leqslant \varepsilon_n$ for each $i = 1, \ldots, m$. Hence, from $\varphi(v_j^n, v_1^n, v_i^n) < \varepsilon_n$ it now follows that

$$\big|\|v_i^n - v_j^n\| - \|v_1^n - v_i^n\|\big| < \varepsilon_n \|v_1 - v_i\| \leqslant \varepsilon_n(1 + \varepsilon_n) < 2\varepsilon_n.$$

By triangle inequality,

$$\big|\|v_i^n - v_j^n\| - 1\big| \leqslant \big|\|v_i^n - v_j^n\| - \|v_1^n - v_i^n\|\big| + \big|\|v_1^n - v_i^n\| - 1\big| < 3\varepsilon_n.$$

Thus, for every distinct i and j the following equality holds $\|v_i^n - v_j^n\| = 1 + \delta_n$ where $|\delta_n| < 3\varepsilon_n$. In particular, $\|v_i^n - v_j^n\| \geqslant \frac{1}{2}$ for $i \neq j$.

Let us put $\xi^n = (0, v_2^n - v_1^n \ldots, v_m^n - v_1^n) \in V^m$ and consider V^m with the norm $\|(v_1, v_2, \ldots, v_m)\| = \max_q \|v_q\|$. Note that the sequence ξ^n is bounded due to already established inequalities $\big|\|v_i^n - v_1^n\| - 1\big| \leqslant \varepsilon_n$, $i = 1, \ldots, m$. Consider the subset $P = \big\{(v_1, \ldots, v_m) \colon \|v_i - v_j\| \geqslant \frac{1}{2}\ \forall i \neq j\big\} \subset V^m$ with the metric induced from V^m. Since $\|v_i^n - v_j^n\| \geqslant \frac{1}{2}$ for $i \neq j$, the sequence ξ^n lies in P. The space V^m is finite-dimensional so there exists a subsequence ξ^{n_s} that converges to some $\xi = (w_1, \ldots, w_m)$, which belongs to P due to the closeness of P in V^m. Define a function $\tilde{\varphi} \colon V^m \to \mathbb{R}$ by the formula $\tilde{\varphi}(v_1, \ldots, v_m) = \max_{i \neq j \neq k \neq i} \varphi(v_i, v_j, v_k)$. Note that $\tilde{\varphi}$ is continuous on P. Since $\varphi(v_i^n, v_j^n, v_k^n) = \varphi(v_i^n - v_1^n, v_j^n - v_1^n, v_k^n - v_1^n)$, it follows from the inequalities $\varphi(v_i^n, v_j^n, v_k^n) < \varepsilon_n$ that $\varphi(v_i^n - v_1^n, v_j^n - v_1^n, v_k^n - v_1^n) < \varepsilon_n$. The latter inequalities are equivalent to $\tilde{\varphi}(0, v_2^n - v_1^n, \ldots, v_m^n - v_1^n) < \varepsilon_n$. Hence, from the continuity of $\tilde{\varphi}$ on P we conclude that $\tilde{\varphi}(w_1, \ldots, w_m) = 0$. Thus, for every different i, j, k we have $\varphi(w_i, w_j, w_k) = 0$ which is equivalent to $\|w_i - w_k\| = \|w_j - w_k\|$. Therefore, $(w_1, \ldots, w_m)$ is an equilateral set in V of cardinality $m > \mathrm{ed}(V) = p$, which is a contradiction. □

Let V be an arbitrary finite-dimensional normed space. Let us also fix a positive integer m. We define a *metric imbalance* of V of order m as follows

$$c_m(V) = \sup\left\{c \colon \forall (v_1, \ldots, v_m) \in V^m \exists i \neq j, j \neq k, k \neq i \colon \left|\frac{|v_i v_k|}{|v_j v_k|} - 1\right| \geqslant c\right\}.$$

Note that in the definition of metric imbalance we consider m-tuples of necessarily distinct points, so that $v_i \neq v_j$ if $i \neq j$.

We conclude immediately from this definition that metric imbalance is a non-decreasing function of m. Besides, Theorem 4.1 implies that if $m > \mathrm{ed}(V)$ then $c_m(V) > 0$.

THEOREM 4.2. *Let V be a finite-dimensional normed space. Denote its metric imbalance by $c_m := c_m(V)$ and $R_m := R_m(V)$. Then the following inequalities hold $2R_m + 1 \geqslant c_m \geqslant R_m - 2$.*

Proof. Let us put $t_n = c_m + \frac{1}{n}$, $n \in \mathbb{N}$. By definition of c_m there exists such set of distinct points $(x_1^n, \dots, x_m^n)$ that for all different i, j, k the following inequalities hold $\varphi(x_i^n, x_j^n, x_k^n) \leqslant t_n$.

Note that for every $\lambda > 0$ it holds $\varphi(x_i^n, x_j^n, x_k^n) = \varphi(\lambda x_i^n, \lambda x_j^n, \lambda x_k^n)$. Hence, by putting $\lambda = \min_{i\neq j} |x_i^n x_j^n|$ and $y_i^n = \frac{x_i^n}{\lambda}$, we obtain such set of points $(y_1^n, \dots, y_m^n)$ that for all different i, j, k we have $\varphi(y_i^n, y_j^n, y_k^n) \leqslant t_n$, and also $\min_{i\neq j} |y_i^n y_j^n| = 1$. By renumerating the points, without loss of generality we can assume that $\min_{i\neq j} |y_i^n y_j^n| = |y_1^n y_2^n|$. Let us put $z_j^n = y_j^n - y_1^n$. Since $\varphi(y_i^n, y_j^n, y_k^n) = \varphi(y_i^n - y_1^n, y_j^n - y_1^n, y_k^n - y_1^n) = \varphi(z_i^n, z_j^n, z_k^n)$, a set of points $(z_1^n, \dots, z_m^n)$ has the following properties: for every different i, j, k the inequalities hold $\varphi(z_i^n, z_j^n, z_k^n) \leqslant t_n$; $\min_{i\neq j} |z_i^n z_j^n| = |z_1^n z_2^n| = 1$, and, finally, $z_1^n = 0$.

For each k distinct from 1 and 2 the inequality holds $\varphi(z_k^n, z_2^n, z_1^n) \leqslant t_n$, therefore, $\left|\frac{|z_k^n z_1^n|}{|z_2^n z_1^n|} - 1\right| = \left||z_1^n z_k^n| - 1\right| \leqslant t_n$ from which it follows that $|z_1^n z_k^n| \leqslant t_n + 1$. Hence, for every n all the points z_k^n belong to the ball $B_{t_n+1}(0)$. Besides, $\min_{i\neq j} |z_i^n z_j^n| = 1$. Then by definition of R_m for every n the inequality holds $R_m \leqslant c_m + 1 + \frac{1}{n}$. Therefore, $R_m \leqslant c_m + 1$.

Let us prove the second inequality. Consider an arbitrary 1-separated set $a_1, \dots, a_m$ in V. By definition of the metric imbalance there exist such different indices i, j, k that $\left|\frac{|a_i a_k|}{|a_j a_k|} - 1\right| \geqslant c_m$. Then, $|a_i a_k| \geqslant (c_m - 1)|a_j a_k| \geqslant c_m - 1$, which implies that it is impossible for a 1-separated set of m points to lie in a ball of a radius less than $\frac{c_m - 1}{2}$. Hence, $R_m \geqslant \frac{c_m - 1}{2}$ which is equivalent to $2R_m + 1 \geqslant c_m$. □

COROLLARY 4.3. *If V is a finite-dimensional normed space then $\lim\limits_{m\to\infty} c_m(V) = \infty$.*

Proof. Inequalities from Theorem 4.2 imply that it suffices to prove that $\lim_{m\to\infty} R_m = \infty$. Suppose that is not true. From the definition of R_m we conclude that it is a non-decreasing function of m. Then it follows from our assumption that there exists a constant C such that an inequality $C \geqslant R_m$ holds for all m. But this inequality would mean that it is possible to find an arbitrarily large 1-separated set in $B_C(0)$, which contradicts its compactness. □

THEOREM 4.4. *Let Y be a finite-dimensional normed space with $\mathrm{ed}(Y) = n$. Also let X be an arbitrary metric space with an equilateral subset $\{x_1, \dots, x_m\}$ of diameter d, where $m > n$. Denote $c_m := c_m(Y)$. Then $\mathrm{d}_{GH}(X, Y) \geqslant \frac{1}{2}\min\left\{\frac{d}{2}, \frac{dc_m}{2+c_m}\right\} > 0$.*

Proof. Let R be an arbitrary correspondence between X, Y with the distortion $\operatorname{dis} R = t$. At first, we consider the case $t < \frac{d}{2}$. Choose an arbitrary $y_i \in R(x_i)$. In the considered case all y_i are automatically different. By the definition of distortion $\big||y_iy_j| - |x_ix_j|\big| \leqslant t$. Hence, the following inequalities hold

$$\frac{-2t}{d+t} = \frac{|x_ix_k| - t}{|x_jx_k| + t} - 1 \leqslant \frac{|y_iy_k|}{|y_jy_k|} - 1 \leqslant \frac{|x_ix_k| + t}{|x_jx_k| - t} - 1 = \frac{2t}{d-t}.$$

According to Theorem 4.1 there exist such indices i, j, k that $\left|\frac{|y_iy_k|}{|y_jy_k|} - 1\right| \geqslant c_m$. Then $\frac{2t}{d-t} \geqslant c_m$, which implies $t \geqslant \frac{dc_m}{2+c_m}$. Thus, for an arbitrary correspondence $R \in \mathcal{R}(X, Y)$ its distortion is either not less than $\frac{d}{2}$, or not less than $\frac{dc_m}{2+c_m}$. It follows that $\operatorname{dis} R \geqslant \min\left\{\frac{d}{2}, \frac{dc_m}{2+c_m}\right\}$. Hence, by Claim 2.7 we obtain that $\mathrm{d}_{GH}(X, Y) \geqslant \frac{1}{2}\min\left\{\frac{d}{2}, \frac{dc_m}{2+c_m}\right\}$. □

Note that Corollary 4.3 implies that $\lim\limits_{m\to\infty} \frac{dc_m}{2+c_m} = d$. So for sufficiently large m the estimate from Theorem 4.4 turns into $\mathrm{d}_{GH}(X, Y) \geqslant \frac{d}{4}$.

Theorem 4.5. *Let X, Y be normed spaces such as* $\dim(Y) < \infty$ *and* $\mathrm{d}_{GH}(X, Y) < \infty$. *Then X is isometric to Y.*

Proof. According to Theorem 3.3, it suffices to prove that X is finite-dimensional. Suppose $\dim(X) = \infty$. Theorem 2.12 implies that the equilateral dimension of Y is finite. Denote $\mathrm{ed}(Y) = n$. By Theorem 2.13 there exists an equilateral set $x_1, \ldots, x_m \in X$ of $m > n$ points. Denote its diameter by d. Then for an arbitrary $\lambda > 0$ points $\lambda x_1, \ldots, \lambda x_m$ form an equilateral set in X of diameter λd. Hence, there are equilateral sets of $m > n$ points and sufficiently large diameters in X. Now, Theorem 4.4 implies that $\mathrm{d}_{GH}(X, Y) = \infty$, which is a contradiction. □

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