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GROWTH OF SOLUTIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH SOLUTIONS OF ANOTHER EQUATION AS COEFFICIENTS

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Abstract. In this paper, we study the growth of higher order linear differential equations, where some coefficients are non-trivial solutions of certain second order linear differential equations. Some conditions guaranteeing that every non-trivial solution of the equation is of infinite order are obtained, in which the notion of accumulation rays of the zero sequence of an entire function is used.

1. Introduction and main results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (see [8, 10, 26]). In addition, we use the notations $\sigma(f)$ and $\lambda(f)$ to denote respectively the order of growth and exponent of convergence of zeros of a meromorphic function $f(z)$.

Consider the second order linear differential equation:

$$f'' + A(z)f' + B(z)f = 0, \tag{1}$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. It is well-known that each solution of the equation (1) is an entire function. If $B(z)$ is transcendental and f_1, f_2 are two linearly independent solutions of the equation (1), then at least one of f_1, f_2 must have infinite order. Hence, most solutions of the equation (1) will have infinite order. On the other hand, there are equations of the form (1) that possess a solution $f \not\equiv 0$ of finite order, for example, $f(z) = e^{-z}$ satisfies $f'' + e^z f' + (e^z - 1)f = 0$. Therefore, one may ask: What conditions on $A(z)$ and $B(z)$ will guarantee that every solution $f \not\equiv 0$ of equation (1) has an infinite order?. There are many results in the literature about the order of growth of solutions of (1), see for example [10, 11, 13, 16].

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Recently, this problem was studied by using a new idea that a coefficient of (1) is a solution of the following equation

$$\omega'' + P(z)\omega = 0, \tag{2}$$

where $P(z) = a_n z^n + \ldots + a_0, a_n \neq 0$, (see for example [14, 15, 22]). The following result shows that the idea is viable.

THEOREM 1.1. *[23] Let $A(z)$ be a non-trivial solution of (2) and $B(z)$ be a transcendental entire function with $\sigma(B) < \frac{1}{2}$. Then every non-trivial solution of (1) is of infinite order.*

By Bank and Laine's result [1, Theorem 1], we know that $\sigma(A) = \frac{n+2}{2}$, and then $\sigma(A) > \sigma(B)$ in Theorem 1.1. The fact that $A(z)$ satisfies an equation of the form $\omega''+P(z)\omega = 0$ makes $A(z)$ a special function. In the particular cases when $P(z) = -z$ or $P(z) = -z^n$, the solution $A(z)$ is known as the Airy integral or generalization of the Airy integral [6]. Another special case is the Weber-Hermite function, which is a solution in the case $P(z) = v + \frac{1}{2} - \frac{z^2}{4}$, where v is a constant. In the case when $P(z)$ is an arbitrary polynomial, Hille's classical method on asymptotic integration will become available, the consequences are summarized in Lemma 2.2 of the Section 2.

Now a new idea is used to study the growth of solutions of (1), in which two coefficients of equation (1) are solutions of equation (2). To that end, we recall the concept of accumulation rays of zeros sequence of a meromorphic function f, which can be found in [18, 19, 24, 25].

DEFINITION 1.2. Let $g(z)$ be a meromorphic function in $\mathbb{C}$, and let $\arg z = \theta \in R$ be a ray from the origin. We denote, for each $\varepsilon > 0$, the exponent of convergence of the zero sequence of $g(z)$ at the ray $\arg z = \theta$ by $\lambda_{\theta,\varepsilon}(g)$ and by $\lambda_\theta(g) = \lim_{\varepsilon\to 0^+} \lambda_{\theta,\varepsilon}(g)$, where

$$\lambda_\theta(g) = \lim_{\varepsilon\to 0^+} \limsup_{r\to\infty} \frac{\log^+ n_{\theta-\varepsilon,\theta+\varepsilon}(r, 0, g)}{\log r},$$

here $n_{\alpha,\beta}(r, 0, g)$ is the number of zeros of g counting multiplicity in $\{z : \alpha < \arg z < \beta\} \bigcap \{z : |z| < r\}$.

We call the ray $\arg z = \theta$ which has the property $\lambda_\theta(g) = \sigma(g)$ an accumulation ray of the zero sequence of g.

We need also to the following definition.

DEFINITION 1.3. Let $\omega(z)$ be a non-trivial solution of (2), where $P(z) = a_n z^n + \ldots + a_0, a_n \neq 0$. We denote by $p(\omega)$ the number of rays $\arg z = \theta_j$ which are not accumulation rays of the zero sequence of $\omega(z)$, where $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$, $j = 0, 1, \ldots, n+1$.

Recently, the authors in [17] studied equation (1) in the case where the coefficients are non-trivial solutions of (2) and proved the following result:

THEOREM 1.4. *Suppose that $A(z)$ and $B(z)$ are two linearly independent solutions of (2), where $P(z) = a_n z^n + \ldots + a_0, a_n \neq 0$. If the number of accumulation rays of* *the zero sequence of $A(z)$ is less than $n+2$, then every non-trivial solution of* (1) *is of infinite order.*

The next result in [17] shows that two coefficients of (2) are non-trivial solutions of (3) and (4) respectively

$$\omega'' + Q_1(z)\omega = 0 \tag{3}$$

$$\omega'' + Q_2(z)\omega = 0, \tag{4}$$

where $Q_1(z) = a_n z^n + \ldots + a_0, a_n \neq 0$, $Q_2(z) = b_m z^m + \ldots + b_0, b_m \neq 0$.

Theorem 1.5. *Suppose that $A(z)$ and $B(z)$ are non-trivial solutions of* (3) *and* (4) *respectively. Suppose that $A(z)$ and $B(z)$ satisfy one of the following conditions:*
(i) $m > n$;

(ii) $m < n$;

(iii) $m = n$, $\arg a_n \neq \arg b_m$, the number of accumulation rays of the zero sequence of $A(z)$ is less than $n+2$;

(iv) $m = n$, and $a_n = cb_m$, where $0 < c < 1$.
Then every non-trivial solution of (1) *is of infinite order.*

Consider the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = 0, \tag{5}$$

where $k \geq 2$ is an integer and $A_j(z)(j = 0, \ldots, k-1)$ are entire functions with $A_0(z) \not\equiv 0$. It is well known that all solutions of (5) are entire, for the case of polynomial coefficients, a classical result due to Wittich [21] is the following: The coefficients $A_0(z), \ldots, A_{k-1}(z)$ in (5) are polynomials in the complex plane if and only if all solutions of (5) are entire functions of finite order of growth. In [3], M. Frei extended the above result by assuming that $A_j(z)$ is the last transcendental entire coefficient while the coefficients $A_{j+1}(z), \ldots, A_{k-1}(z)$ are polynomials and resulting in that (5) possesses at most j linearly independent entire solutions of finite order. Thus it can be deduced that "most" of the solutions of (5) with at least one $A_i(z)$ transcendental have infinite order. On the other hand, there exist equations of the form (5) that possess one or more non-trivial solutions of finite order. For example: (a) $f(z) = -z$ solves $f'' - ze^z f' + e^z f = 0$, (b) $f(z) = c_1 \sin z + c_2 \cos z$ solves $f''' + e^z f'' + f' + e^z f = 0$, where c_1, c_2 are arbitrary constants. The question which arises is: what conditions on $A_0(z), \ldots, A_{k-1}(z)$ will guarantee that every solution $f \not\equiv 0$ of (5) is of infinite order? In this paper we continue to consider this question. We will prove the following two results which extend the above results:

Theorem 1.6. *Let $k \geq 2$ be an integer and let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be entire functions. Suppose that there exists $s \in \{1, \ldots, k-1\}$ such that $A_0(z)$ and $A_s(z)$ are two linearly independent solutions of* (2), *and for $j \neq 0, s$, $\sigma(A_j) < \sigma(A_0)$. If the number of accumulation rays of the zero sequence of $A_s(z)$ is less than $n+2$, then every non-trivial solution of* (5) *is of infinite order.*

THEOREM 1.7. *Let $k \geq 2$ be an integer and let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be entire functions. Suppose that there exist $s, d \in \{1, \ldots, k-1\}$ such that $A_s(z)$ and $A_d(z)$ are two linearly independent solutions of* (3) *and $A_0(z)$ is a non-trivial solution of* (4) *such that $\max\{\sigma(A_j) : j \neq 0, s, d\} < \sigma(A_0)$. Suppose that $A_0(z)$ and $A_s(z)$ satisfy one of the following conditions:*

(i) $m > n$;

(ii) $m < n$;

(iii) $m = n$, $\arg a_n \neq \arg b_m$, the number of accumulation rays of the zero sequence of $A_s(z)$ is less than $n + 2$;

(iv) $m = n$, $a_n = cb_m$, where $0 < c < 1$.
Then every transcendental solution of (5) *is of infinite order.*

2. Auxiliary results

LEMMA 2.1 ([5]). *Let f be a transcendental meromorphic function of finite order $\sigma(f)$, Let $\varepsilon > 0$ be a given real constant and let k and j be two integers such that $k > j \geqslant 0$. Then the following statements hold.*
(i) There exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi) - E_1$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$,

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma(f)-1+\varepsilon)}. \tag{6}$$

(ii) There exists a set $E_2 \subset (1, \infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E_2 \cup [0, 1]$, the inequality (6) *holds.*

Asymptotic properties of solutions of $\omega'' + P(z)\omega = 0$ play an important role in proving our results, where $P(z)$ is a non constant-polynomial. Next some notation are stated. Let $\alpha < \beta$ be such that $\beta - \alpha < 2\pi$, and let $r > 0$. Denote

$$S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$$
$$S(\alpha, \beta, r) = \{z : \alpha < \arg z < \beta\} \cap \{z : |z| < r\}$$

Let $\bar{F}$ denote the closure of F. Let A be an entire function of order $\sigma(A) \in (0, \infty)$.

For simplicity, set $\sigma = \sigma(A)$ and $S = S(\alpha, \beta)$. We say that A blows up exponentially in $\bar{S}$ if for any $\theta \in (\alpha, \beta)$ the relation

$$\lim_{r \to \infty} \frac{\log \log |A(re^{i\theta})|}{\log r} = \sigma$$

holds. We also say that A decays to zero exponentially in $\bar{S}$ if for any $\theta \in (\alpha, \beta)$, the next relation holds

$$\lim_{r \to \infty} \frac{\log \log |A(re^{i\theta})|^{-1}}{\log r} = \sigma.$$

Lemma 2.2 ([7, 9, 20]). *Let A be a non-trivial solution of $\omega'' + P(z)\omega = 0$, where $P(z) = a_n z^n + \ldots + a_0, a_n \neq 0$. Set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = S(\theta_j, \theta_{j+1})$, where $j = 0, 1, \ldots, n+1$ and $\theta_{n+2} = \theta_0 + 2\pi$. Then A has the following properties:*
(i) In each sector S_j, A either blows up or decays to zero exponentially.

(ii) If, for some j, A decays to zero in S_j, then it must blow up in S_{j-1} and S_{j+1}. However it is possible for A to below up in many adjacent sectors.

(iii) If A decays to to zero in S_j, then A has at most finitely many zeros in any closed sub-sector within $S_{j-1} \cup \bar{S}_j \cup S_{j+1}$.

(iv) If A blows up in S_{j-1} and S_j then for each $\varepsilon > 0$, A has infinitely many zeros in each sector $\bar{S}(\theta_j - \varepsilon, \theta_j + \varepsilon)$ and furthermore, as $r \to \infty$,

$$n(\bar{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, A) = (1 + o(1)) \frac{2\sqrt{|a_n|}}{\pi(n+2)} r^{\frac{n+2}{2}},$$

where $n(\bar{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, A)$ is the number of zeros of A in the region $\bar{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r)$ counting multiplicity.

Lemma 2.3 ([4]). *Let A be defined as in Lemma 2.2. Then the following equality holds:*

$$\log M(r, A) = (1 + o(1)) \frac{2\sqrt{|a_n|}}{(n+2)} r^{\frac{n+2}{2}}, \quad \text{as } r \to \infty$$

Lemma 2.4 ([12]). *Let $\theta_1 < \theta_2$ be given to fix a sector $S(0) : \theta_1 \leq \arg z \leq \theta_2$, let $k \geq 2$ be a natural number, and let $\delta > 0$ be any real number such that $k\delta < 1$. Suppose that $A_0(z), \ldots, A_{k-1}(z)$ with $A_0(z) \not\equiv 0$ are entire functions such that for real constants $\alpha > 0$, $\beta > 0$, we have, for any some $s = 1, \ldots, k-1$, $|A_s(z)| \geq \exp((1+\delta)\alpha |z|^\beta)$, $|A_j(z)| \leq \exp(\delta \alpha |z|^\beta)$ for all $j = 0, \ldots, s-1, s+1, \ldots, k-1$ whenever $|z| = r \geq r_\delta$ in the sector $S(0)$.*

Given $\varepsilon > 0$ small enough, if f is a transcendental solution of finite order $\sigma < \infty$ of the linear differential equation (5). Then the following conditions hold:
(i) There exists $t \in \{0, \ldots, s-1\}$ and a complex constant $b_t \neq 0$ such that $f^{(t)} \to b_t$ as $z \to \infty$ in the sector $S(\varepsilon) : \theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$. More precisely, $|f^{(t)}(z) - b_t| \leq \exp(-(1 - k\delta)\alpha |z|^\beta)$ in $S(\varepsilon)$, provided $|z|$ is large enough.

(ii) For each integer $q \geq t+1$, $|f^{(q)}(z)| \leq \exp(-(1 - k\delta)\alpha |z|^\beta)$ in $S(3\varepsilon)$, for all $|z|$ large enough.

Lemma 2.5 ([12]). *Suppose that $f(z)$ is an entire function, and that $|f^{(k)}(z)|$ is unbounded on a ray $\arg z = \theta$. Then there exists a sequence $z_n = r_n e^{i\theta}$ tending to infinity such that $f^(k)(z) \to \infty$ and that*

$$\Big| \frac{f^{(i)}(z_n)}{f^{(k)}(z_n)} \Big| \leq \frac{1}{(k-i)!} (1 + o(1)) |z|^{k-i}$$

provided $i < k$.

3. Proofs of theorems

Proof (Proof of Theorem 1.6). Suppose on the contrary to the assertion that there exists a non-trivial solution f of (5) with $\sigma(f) < \infty$, we aim for a contradiction. Using Lemma 2.2 and the condition of Theorem 1.6, set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = \{z : \theta_j < \arg z < \theta_{j+1}\}$, $j = 0, \dots, n+1$, $\theta_{n+2} = \theta_0 + 2\pi$.

By the condition of Theorem 1.6 and the definition of accumulation rays of the zero sequence of meromorphic functions, we know that $p(A_s) \geq 2$. It follows from Lemma 2.2 that there exists at least one sector of the $n+2$ sectors, such that A_s decays to zero exponentially, without loss of generality, say $S_{j_0} = \{z : \theta_{j_0} < \arg z < \theta_{j_0+1}\}$, $0 \leq j_0 \leq n+1$. That is for any $\theta \in (\theta_{j_0}, \theta_{j_0+1})$

$$\lim_{r\to\infty} \frac{\log\log \frac{1}{|A_s(re^{i\theta})|}}{\log r} = \frac{n+2}{2}. \tag{7}$$

Next we claim that it is impossible that both A_s and A_0 decay to zero exponentially in a common sector. To prove our claim, without loss of generality, we suppose that A_s and A_0 decay to zero exponentially in S_0. Set $h = \frac{A_s}{A_0}$. It follows from [4, Lemma 3], that as $r \to \infty$,

$$N(r, \frac{1}{h-b}) = (1+o(1))T(r,h) = (1+o(1))\frac{2\sqrt{|a_n|}}{\pi\alpha} r^{\alpha},$$

holds for any $b \in \mathbb{C}$, with at most finitely many exceptions, where $\alpha = \frac{n+2}{2}$. Set $\omega = A_s - bA_0$. It is easy to see that ω is a solution of (2). It follows from [4, Theorem 3], that

$$N(r, \frac{1}{h-b}) = N(r, \frac{1}{\omega}) = (1+o(1))\frac{2\alpha - p(\omega)}{\pi\alpha^2}\sqrt{|a_n|} r^{\alpha}$$

as $r \to \infty$. Combining the two equalities mentioned above, we get $p(\omega) = 0$. This implies that ω blows up exponentially in every sector S_j, $j = 0, 1, \dots, n+1$. This contradicts the assumption that ω decays to zero exponentially in S_0, Hence A_0 blows up exponentially in S_{j_0}, that is, for any $\theta \in (\theta_{j_0}, \theta_{j_0+1})$,

$$\lim_{r\to\infty} \frac{\log\log |A_0(re^{i\theta})|}{\log r} = \frac{n+2}{2}. \tag{8}$$

By Lemma 2.1, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E_1$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$,

$$\left|\frac{f^{(i)}(z)}{f(z)}\right| \leq |z|^{k\sigma(f)}, \ i = 1, \dots, k. \tag{9}$$

Let $\varepsilon \in (0, \sigma(A_0)/2)$ be a given constant. Since $\sigma(A_i) < \sigma(A_0)$ for all $i \neq 0, s$ and $0 \leq i \leq k-1$, then there exists an $R_1 > 1$ such that

$$|A_i(z)| < \exp(r^{\frac{n+2}{2} - 2\varepsilon}), \tag{10}$$

for all $|z| = r > R_1$.

Thus, there exists a sequence of points $z_l = r_l e^{i\theta}$, where $r_l \to +\infty$ as $l \to +\infty$ and

$\theta \in (\theta_{j_0}, \theta_{j_0+1}) - E_1$, such that (7), (8) and (9), (10) hold. Combining (7)-(9), (10) and (5), for any $l > l_0$,

$$\begin{aligned}\exp(r_l^{\frac{n+2}{2}-\varepsilon}) &\le |A_0(r_l e^{i\theta})| \\ &\le \Big|\frac{f^{(k)}(r_l e^{i\theta})}{f(r_l e^{i\theta})}\Big| + \sum_{j=1}^{k-1} |A_j(r_l e^{i\theta})| \Big|\frac{f^{(j)}(r_l e^{i\theta})}{f(r_l e^{i\theta})}\Big| \\ &\le r_l^{k\sigma(f)}\Big(1 + \frac{1}{\exp\big(r_l^{\frac{n+2}{2}-\varepsilon}\big)} + (k-2)\exp\big(r_l^{\frac{n+2}{2}-2\varepsilon}\big)\Big).\end{aligned}$$

Obviously, that is a contradiction for sufficiently large l and for any given $\varepsilon > 0$. Hence, the conclusion of Theorem 1.6 holds. □

Proof (Proof of Theorem 1.7). Suppose the contrary to the assertion, that there exists a transcendental solution f of (5) with $\sigma(f) < \infty$, we aim for contradiction. It follows from [1] that $\sigma(A_s) = \frac{n+2}{2}$ and $\sigma(A_0) = \frac{m+2}{2}$.
1) Suppose that the condition (i) holds. Then $\max\{\sigma(A_i) : i = 1, \dots, k-1\} < \sigma(A_0)$. Therefore, the conclusion of Theorem 1.7 is deduced from [2].

2) Suppose that the condition (ii) holds. Set

$$F_{A_0} = \Big\{\theta \in [0, 2\pi) : \theta = \frac{2j\pi - \arg(b_m)}{m+2}\Big\}, \quad j = 0, 1, \dots, m+1,$$

and

$$F_{A_s} = \Big\{\theta \in [0, 2\pi) : \theta = \frac{2j\pi - \arg(a_n)}{n+2}\Big\}, \quad j = 0, 1, \dots, n+1.$$

By Lemma 2.1, there exists a set $F \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - F$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \ge R_0$, (9) holds. Set $E = F \cup F_{A_0} \cup F_{A_s}$. Then for any $\theta \in [0, 2\pi) - E$, A_0, A_s have four possible growth types on the ray $\arg z = \theta$:

(a) $A_s(re^{i\theta})$ satisfies

$$\lim_{r\to\infty} \frac{\log\log|A_s(re^{i\theta})|^{-1}}{\log r} = \frac{n+2}{2} \tag{11}$$

and $A_0(re^{i\theta})$ satisfies

$$\lim_{r\to\infty} \frac{\log\log|A_0(re^{i\theta})|}{\log r} = \frac{m+2}{2} \tag{12}$$

(b) $A_s(re^{i\theta})$ satisfies (11) and $A_0(re^{i\theta})$ satisfies

$$\lim_{r\to\infty} \frac{\log\log|A_0(re^{i\theta})|^{-1}}{\log r} = \frac{m+2}{2} \tag{13}$$

(c) $A_s(re^{i\theta})$ satisfies

$$\lim_{r\to\infty} \frac{\log\log|A_s(re^{i\theta})|}{\log r} = \frac{n+2}{2} \tag{14}$$

and $A_0(re^{i\theta})$ satisfies (12).

(d) $A_s(re^{i\theta})$ satisfies (14) and $A_0(re^{i\theta})$ satisfies (13).

(a') If $A_s(re^{i\theta})$ and $A_0(re^{i\theta})$ satisfy the growth type (a), then using similar reasoning as in the proof of Theorem 1.6, we get a contradiction.

(b') Suppose that $A_s(re^{i\theta})$ and $A_0(re^{i\theta})$ satisfy the growth type (b). Suppose that $|f^{(d)}(z)|$ is unbounded on the ray $\arg z = \theta$. Using Lemma 2.5, there exists an infinite sequence of points $z_l = r_l e^{i\theta}$ tending to infinity such that $f^{(d)}(z_l) \to \infty$ and

$$\Big|\frac{f^{(i)}(z_l)}{f^{(d)}(z_l)}\Big| \le \frac{1}{(d-i)!}(1+o(1))|z_l|^{d-i}, \quad i = 0, 1, \ldots, d-1 \tag{15}$$

as $l \to \infty$.

It follows from the proof of Theorem 1.6 that $A_d(z)$ blows up exponentially in E, that is on the ray $\arg z = \theta$, we have

$$\lim_{r\to\infty} \frac{\log\log|A_d(re^{i\theta})|}{\log r} = \frac{n+2}{2}.$$

It follows from (5), (9), (10) and (15) that

$$\begin{aligned}
\exp\{r_l^{\frac{n+2}{2}-\varepsilon}\} &\le |A_d(r_l e^{i\theta})| \\
&\le \Big|\frac{f^{(k)}(z_l)}{f(z_l)}\Big|\Big|\frac{f(z_l)}{f^{(d)}(z_l)}\Big| + |A_{k-1}(r_l e^{i\theta})|\Big|\frac{f^{(k-1)}(z_l)}{f(z_l)}\Big|\Big|\frac{f(z_l)}{f^{(d)}(z_l)}\Big| \\
&\quad + \ldots + |A_s(r_l e^{i\theta})|\Big|\frac{f^{(s)}(z_l)}{f(z_l)}\Big|\Big|\frac{f(z_l)}{f^{(d)}(z_l)}\Big| + \ldots + |A_0(r_l e^{i\theta})|\Big|\frac{f(z_l)}{f^{(d)}(z_l)}\Big| \\
&\le M_1 r_l^{d+k\sigma(f)}\Big(1+\frac{1}{\exp\{r_l^{\sigma(A_0)-\varepsilon}\}}+\frac{1}{\exp\{r_l^{\frac{n+2}{2}-\varepsilon}\}}+(k-3)\exp\{r_l^{\frac{n+2}{2}-2\varepsilon}\}\Big),
\end{aligned}$$

where $M_1 > 0$ is a constant. That is a contradiction for sufficiently large l and for $\varepsilon \in \big(0, \frac{\sigma(A_d)}{2}\big)$. Hence $|f^{(d)}(z)|$ must be bounded in the whole complex plane by Phragmén-Lindelöf principle.

(c') Suppose that $A_s(re^{i\theta})$ and $A_0(re^{i\theta})$ satisfy the growth type (c). From Bank and Laine's results [1, Theorem 1], we get $\sigma(A_s) = \frac{n+2}{2} > \frac{m+2}{2} = \sigma(A_0)$, there exists a real number $\gamma > 0$ such that $\sigma(A_s) = \frac{n+2}{2} > \frac{m+2+\gamma}{2} > \frac{m+2}{2} = \sigma(A_0)$. Then for any given $\varepsilon \in (0, \frac{\pi}{8\sigma(A_s)})$ and $\eta \in (0, \frac{\sigma(A_s)-\sigma(A_0)}{4})$, we have

$$|A_s(z)| \ge \exp\Big\{(1+\delta)\alpha|z|^{\frac{n+2}{2}-\eta}\Big\}$$

and

$$|A_0(z)| \le \exp\Big\{|z|^{\sigma(A_0)+\eta}\Big\} \le \exp\Big\{|z|^{\frac{n+2}{2}-2\eta}\Big\} \le \exp\Big\{\delta\alpha|z|^{\frac{n+2}{2}-\eta}\Big\}$$

as $z \to \infty$ in $\bar{S}(\frac{\varepsilon}{2}) = \{z : \theta - \frac{\varepsilon}{2} < \arg z < \theta + \frac{\varepsilon}{2}\}$, where α and δ are positive constants satisfying $\delta k < 1$.

On the other hand, it follows from the proof of Theorem 1.6 that $A_d(z)$ decays to zero exponentially, that is for the $\arg z = \theta$, we have

$$\lim_{r\to\infty} \frac{\log\log\frac{1}{|A_d(re^{i\theta})|}}{\log r} = \frac{n+2}{2}$$

Hence

$$|A_d(re^{i\theta})| \le \frac{1}{\exp\Big\{|z|^{\frac{n+2}{2}-\varepsilon}\Big\}} \le \exp\Big\{\delta\alpha|z|^{\frac{n+2}{2}-\varepsilon}\Big\}$$

we have also

$$|A_i(z)| \le \exp\Big\{|z|^{\sigma(A_0)+\eta}\Big\} \le \exp\Big\{|z|^{\frac{n+2}{2}-2\eta}\Big\} \le \exp\Big\{\delta\alpha|z|^{\frac{n+2}{2}-\varepsilon}\Big\}, \quad i \ne 0, s, d$$

By Lemma 2.4 , there exists $t \in \{1, 2, \dots, s-1\}$ and $b_t \ne 0$ such that

$$|f^{(t)}(z) - b_t| \le \exp\Big\{-(1-k\delta)\alpha|z|^{\frac{n+2}{2}-\eta}\Big\}$$

as $z \to \infty$ in $\bar{S}(\varepsilon)$. For each integer $i \ge t+1$,

$$|f^{(i)}(z)| \le \exp\Big\{-(1-k\delta)\alpha|z|^{\frac{n+2}{2}-\eta}\Big\}$$

as $z \to \infty$ in $\bar{S}=(\frac{3\varepsilon}{2})$. Hence $|f^{(s)}(z)|$ must be bounded in the whole complex plane by Phragmén-Lindelöf principle.

(d') Suppose that $A_s(re^{i\theta})$ and $A_0(re^{i\theta})$ satisfy the growth type (d)). Suppose that $|f^{(s)}(z)|$ is unbounded on the ray $\arg z = \theta$. Using Lemma 2.5, there exists an infinite sequence of points $z_l = r_l e^{i\theta}$ tending to infinity such that $f^{(s)}(z_l) \to \infty$ and

$$\Big|\frac{f^{(i)}(z_l)}{f^{(s)}(z_l)}\Big| \le \frac{1}{(s-i)!}(1+o(1))|z_l|^{s-i}, \quad i = 0, 1, \dots, s-1 \tag{16}$$

as $l \to \infty$.

It follows from the proof of Theorem 1.6 that $A_d(z)$ decays to zero exponentially, that is on the ray $\arg z = \theta$, we have

$$\lim_{r\to\infty} \frac{\log\log|A_d(z)|^{-1}}{\log r} = \frac{n+2}{2}.$$

It follows from (5), (9), (10) and (16) that

$$\begin{aligned}
\exp\{r_l^{\frac{n+2}{2}-\varepsilon}\} &\le |A_s(r_l e^{i\theta})| \\
&\le \Big|\frac{f^{(k)}(z_l)}{f(z_l)}\Big|\Big|\frac{f(z_l)}{f^{(s)}(z_l)}\Big| + |A_{k-1}(r_l e^{i\theta})|\Big|\frac{f^{(k-1)}(z_l)}{f(z_l)}\Big|\Big|\frac{f(z_l)}{f^{(s)}(z_l)}\Big| \\
&\quad + \dots + |A_s(r_l e^{i\theta})|\Big|\frac{f^{(d)}(z_l)}{f(z_l)}\Big|\Big|\frac{f(z_l)}{f^{(s)}(z_l)}\Big| + \dots + |A_0(r_l e^{i\theta})|\Big|\frac{f(z_l)}{f^{(s)}(z_l)}\Big| \\
&\le M_2 r_l^{s+k\sigma(f)}\Big(1+\frac{1}{\exp\{r_l^{\sigma(A_0)-\varepsilon}\}}+\frac{1}{\exp\{r_l^{\frac{n+2}{2}-\varepsilon}\}}+(k-3)\exp\{r_l^{\frac{n+2}{2}-2\varepsilon}\}\Big)
\end{aligned}$$

as $l \to +\infty$, where M_2 is a positive constant.

Obviously, this is a contradiction for sufficiently large l and for $\varepsilon \in (0, \frac{\sigma(A_s)}{2})$. Hence $|f^{(s)}(z)|$ must be bounded in the whole complex plane by Phragmén-Lindelöf principle. Combining the case of (b')-(d'), by the Liouville Theorem, f has to be a polynomial. This is contradicts with the fact that f is transcendental.

3) Suppose that the condition (iii) holds. This implies that the set of accumulation

rays of the zero sequence of $A_s(z)$ and $A_0(z)$ are not the same. Then there exists a sector $S(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}$, such that for any $\theta\in(\alpha,\beta)$ (7) and (8) hold. Then using similar reasoning as in the proof of Theorem 1.6, we get a contradiction, and then the conclusion is obtained.

4) Suppose that the condition (iv) holds. By Lemma 2.1, there exists a set $E_2\subset(1,+\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z|\notin[0,1]\cup E_2$, (9) holds.

Since $A_s(z)$ and $A_0(z)$ are non-trivial solutions of (2) and (3), respectively, by Lemma 2.3, as $r\to\infty$, the following equalities hold,

$$\log M(r,A_s)=(1+o(1))\frac{\sqrt{|a_n|}}{\alpha}r^{\alpha}\quad\text{and}\quad \log M(r,A_0)=(1+o(1))\frac{\sqrt{|b_m|}}{\alpha}r^{\alpha},\tag{17}$$

where $\alpha=\frac{n+2}{2}$, we choose a sequence of points $\{z_l\}$ tending to infinity, $|z_l|=r_l\in(1,+\infty)-E_2$, such that

$$|A_0(z_l)|=M(r_l,A_0).\tag{18}$$

Combining (5), (9), (17),(18), as $l\to\infty$, we get

$$\exp\left\{(1+o(1))\frac{\sqrt{|b_m|}}{\alpha}r^{\alpha}\right\}=M(r_l,A_0)=|A_0(z_l)|\le|z_l|^{k\sigma(f)}\Big(1+\sum_{j=1}^{k-1}|A_j(z_l)|\Big)$$

$$\le|z_l|^{k\sigma(f)}\Big(1+(k-2)\exp\left\{r_l^{\alpha-\varepsilon}\right\}+\exp\left\{(1+o(1))\frac{\sqrt{|a_n|}}{\alpha}r_l^{\alpha}\right\}\Big).$$

This is a contradiction. The conclusion of Theorem 1.7 holds. □

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