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ON b REPDIGITS AS PRODUCT OR SUM OF FIBONACCI AND TRIBONACCI NUMBERS

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Abstract. Let $b \geq 2$ be an integer. In this paper we study the base b repdigits that can be expressed as sums or products of Fibonacci and Tribonacci numbers. As a corollary, it is shown that the numbers 1 and 7 are the only Mersenne numbers which can be expressed respectively as product and sum of Fibonacci and Tribonacci numbers. This is done using linear forms in logarithms of algebraic numbers (Baker's method) and the Baker-Davenport reduction method (the Dujella-Pethő's version).

1. Introduction

For an integer $b \geq 2$, a positive integer N is called a base b repdigit if it has only one digit in its base b representation. That is,

$$N = d\left(\frac{b^\ell - 1}{b-1}\right),$$

for some integers $\ell \geq 1$ and $d \in \{1, \ldots, b-1\}$. When $b = 10$, one usually omits to mention b and simply call these numbers as repdigits. The sequence of numbers with repeated digits is included in Sloane's *On-Line Encyclopedia of Integer Sequences* (OEIS) [16] as sequence A010785. The Fibonacci sequence [10] $\{F_n\}_{n\geq 0}$, is the binary recurrence sequence given by $F_0 = 0$, $F_1 = 1$ and the recurrence formula

$$F_{n+2} = F_{n+1} + F_n, \text{ for all } n \geq 0.$$

First few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots$$

Moreover, the Tribonacci sequence $\{T_n\}_{n\geq 0}$ is defined by the recurrence formula

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \text{ for all } n \geq 0,$$

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with $T_0 = 0$ and $T_1 = T_2 = 1$ (see [18]). Its first terms are

$$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768, \ldots$$

Note that F_n and T_n are called n-th Fibonacci number and n-th Tribonacci number, respectively. The Fibonacci and Tribonacci sequences are included in the (OEIS) [16] as sequences A000045 and A000073 respectively. Diophantine equations involving repdigits, Fibonacci and Tribonacci numbers have been considered in various papers in recent years (see [2,4,6–8,11,13,15]). We point out that Luca [11] and Marques [12] proved that the largest repdigits in the Fibonacci and Tribonacci sequences are $F_{10} = 55$ and $T_8 = 44$, respectively. Recently, Bednařík and Trojovský [2] found all the repdigits that can be written as a product of Fibonacci and Tribonacci numbers with the same index while Trojovský in [17] found all repdigits that can be written as sum of Fibonacci and Tribonacci numbers with the same index. Motivated by the work of the authors from [2,17], we devote this study to fully solve the following two Diophantine equations

$$F_nT_n = d\left(\frac{b^\ell - 1}{b-1}\right) \quad \text{and} \quad F_n + T_n = d\left(\frac{b^\ell - 1}{b-1}\right) \tag{1}$$

in positive integers b, n, ℓ, d with $b \geq 2$ and $d \in \{1, \ldots, b-1\}$.

We organize this paper as follows. In Section 2, we recall some elementary properties of Fibonacci and Tribonacci numbers, a result due to Matveev on lower bounds of linear forms in logarithms of algebraic numbers, and a result on reduction method due to Dujella and Pethő. The proofs of our main results are given in Section 3.

2. Useful tools

In this section, we gather the tools we need to prove Theorems 3.1 and 3.6.

2.1 Linear forms in logarithms

Let η be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\eta = \eta^{(1)}, \ldots, \eta^{(d)}$ denote its conjugates. The logarithmic height of α is defined by

$$h(\eta) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d} \log\max\left(1, \left|\eta^{(j)}\right|\right)\right).$$

Paricularly, if $\eta = p/q \in \mathbb{Q}$ is a rational number in reduced form (so, $q \geq 1$), then the above definition reduces to $h(\eta) = \log\max\{|p|, q\}$. Now, let us give some basic properties of this height. For η_1, η_2 algebraic numbers and $m \in \mathbb{Z}$ we have

$$h(\eta_1 \pm \eta_2) \leq h(\eta_1) + h(\eta_2) + \log 2, \quad h(\eta_1\eta_2^{\pm}) \leq h(\eta_1) + h(\eta_2), \quad h(\eta_1^m) = |m|h(\eta_1).$$

Let $\mathbb{L}$ be a real number field of degree $d_{\mathbb{L}}$, $\gamma_1, \ldots, \gamma_s \in \mathbb{L}$ and $b_1, \ldots, b_s \in \mathbb{Z} \setminus \{0\}$. Let $B \geq \max\{|b_1|, \ldots, |b_s|\}$ and $\Gamma = \gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1$. Now, let $A_1, \ldots, A_s$ be real numbers with $A_i \geq \max\{d_{\mathbb{L}}h(\gamma_i), |\log\gamma_i|, 0.16\}$, $i = 1, 2, \ldots, s$.

The first tool that we need is the following result due to Matveev [14]. Here, we use the version of Bugeaud, Mignotte, and Siksek [3, Theorem 9.4].

THEOREM 2.1. *Assume that* $\Gamma \neq 0$. *Then*

$$\log|\Gamma| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{L}}^2 \cdot (1+\log d_{\mathbb{L}}) \cdot (1+\log B) \cdot A_1 \cdots A_s.$$

Also, we will need the following lemma due to Guzmán and Luca.

LEMMA 2.2 ([9, Lemma 7]). *If* $l \geq 1$, $H > \left(4l^2\right)^l$ *and* $H > L/(\log L)^l$, *then* $L < 2^l H(\log H)^l$.

2.2 Reduction method

Our next tool is a version of the reduction method of Baker and Davenport [1]. We use a slight variant of the version given by Dujella and Pethő [5]. For a real number x, we write $\|x\|$ for the distance from x to the nearest integer.

LEMMA 2.3. *Let* M *be a positive integer,* p/q *be a convergent of the continued fraction expansion of the irrational number* τ *such that* $q > 6M$, *and* A, B, μ *be some real numbers with* $A > 0$ *and* $B > 1$. *Furthermore, let* $\varepsilon := \|\mu q\| - M \cdot \|\tau q\|$.

If $\varepsilon > 0$, *then there is no solution to the inequality* $0 < |u\tau - v + \mu| < AB^{-w}$ *in positive integers* u, v *and* w *with* $u \leq M$ *and* $w \geq \frac{\log(Aq/\varepsilon)}{\log B}$.

2.3 The Fibonacci or Tribonacci sequences

We recall here some useful properties of Fibonacci and Tribonacci sequences. We recall a well-known non-recursive formula for generating Fibonacci numbers. Binet's formula asserts that

$$F_n = \frac{\lambda^n - (-\lambda)^n}{\sqrt{5}}, \quad \text{for} \quad n \geq 0,$$

where $\lambda = (1+\sqrt{5})/2$. With this formula, we can deduce that

$$\lambda^{n-2} \leq F_n \leq \lambda^{n-1}, \quad \text{for} \quad n \geq 1. \tag{2}$$

It is also possible to infer that

$$F_n = \frac{\lambda^n}{\sqrt{5}} + \nu \quad \text{with} \quad |\nu| \leq \frac{1}{\sqrt{5}}, \quad \text{for} \quad n \geq 1. \tag{3}$$

Next, the characteristic equation for Tribonacci sequence is $\psi(x) := x^3 - x^2 - x - 1 = 0$, and has one real root α and two complex roots β and $\gamma = \bar{\beta}$. More precisely, we have

$$\alpha = \frac{1}{3}(1 + r_1 + r_2), \quad \beta = \frac{1}{6}\left(2 - (r_1 + r_2) + (r_1 - r_2)\sqrt{-3}\right),$$

with $$r_1 = \sqrt[3]{19 + 3\sqrt{33}} \quad \text{and} \quad r_2 = \sqrt[3]{19 - 3\sqrt{33}}.$$

Moreover, Binet's formula for the general terms of the Tribonacci sequence is given by $T_n = a\alpha^n + b'\beta^n + c\gamma^n$, for $n \geq 0$, where

$$a = \frac{1}{(\alpha-\beta)(\alpha-\gamma)}, \; b' = \frac{1}{(\beta-\alpha)(\beta-\gamma)} \text{ and } c = \frac{1}{(\gamma-\alpha)(\gamma-\beta)} = \overline{b'}.$$

Furthermore, one can observe that $a = \frac{\alpha}{\alpha^2+2\alpha+3}$ and its minimal polynomial over the integers is given by $44x^3 + 4x - 1$, and has zeros a, b', c with $|a|, |b'|, |c| < 1$. Numerically, the following estimates hold

$$\begin{aligned} &1.83 < \alpha < 1.84, && 0.73 < |\beta| = |\gamma| = \alpha^{-\frac{1}{2}} < 0.74, \\ &0.18 < a < 0.19, && 0.35 < |b'| = |c| < 0.36. \end{aligned}$$

Also, setting $e_n = T_n - a\alpha^n = b'\beta^n + c\gamma^n$, we can show that

$$T_n = a\alpha^n + e_n, \quad \text{with} \quad |e_n| < \frac{1}{\alpha^{n/2}}, \tag{4}$$

holds for all $n \geq 1$. Furthermore, by induction, one can prove that

$$\alpha^{n-2} \leq T_n \leq \alpha^{n-1}, \quad \text{for} \quad n \geq 1. \tag{5}$$

Let $\mathbb{K} := \mathbb{Q}(\alpha, \beta, \gamma) = \mathbb{Q}(\alpha, \beta)$ be the splitting field of the polynomial ψ over $\mathbb{Q}$. Then, $[\mathbb{K} : \mathbb{Q}] = 6$. Furthermore, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. The Galois group of $\mathbb{K}/\mathbb{Q}$ is given by

$$\mathcal{G} = Gal(\mathbb{K}/\mathbb{Q}) = \{(1), (\alpha\beta), (\alpha\gamma), (\beta\gamma), (\alpha\beta\gamma), (\alpha\gamma\beta)\} \cong S_3.$$

Thus, we identify the automorphisms of $\mathcal{G}$ with the permutations of the zeros of the polynomial ψ. For example, the permutation $(\alpha\beta)$ corresponds to the automorphism $\sigma : \alpha \to \beta,\ \beta \to \alpha, \gamma \to \gamma$.

3. Main results

3.1 On b repdigits as product of Fibonacci and Tribonacci numbers

In this subsection, we will prove the following result.

Theorem 3.1. *Let $b \geq 2$ be an integer. Then, the Diophantine equation*

$$F_n T_n = d\left(\frac{b^\ell - 1}{b - 1}\right) \tag{6}$$

has only finitely many solutions in integers (n, b, d, ℓ) such that $n, \ell \geq 1$ and $1 \leq d \leq b - 1$. Moreover, we have $n < 3.6 \times 10^{18} \log^3 b$ and $\ell < 9 \times 10^{18} \log^3 b$.

Note that if $n = 1$, then all solutions of equation (6) are of the form $(b, n, \ell, d) = (b, 1, 1, 1)$ with $b \geq 2$. For the remaining proof, we consider $n \geq 2$. The following result will be useful in proving Theorem 3.1, which gives a relation between the variables ℓ, n, and b of equation (6).

Lemma 3.2. *All solutions of Diophantine equation (6) satisfy*

$$(\ell - 1)\frac{\log b}{\log \lambda\alpha} + 1 < n < \ell\frac{\log b}{\log \alpha\lambda} + 2.$$

Proof. From inequalities (2) and (5), we get

$$\lambda^{n-2}\alpha^{n-2} < F_n T_n = d\left(\frac{b^\ell - 1}{b - 1}\right) < b^\ell. \tag{7}$$

Taking the logarithm on both sides of (7), we get

$$n \log \lambda\alpha < \ell \log b + 2 \log \lambda\alpha. \tag{8}$$

For the lower bound, from (2) and (5), we have

$$b^{\ell-1} < d\left(\frac{b^\ell - 1}{b-1}\right) = F_n T_n < \lambda^{n-1}\alpha^{n-1}.$$

Taking the logarithm on both sides, we get

$$(\ell - 1)\log b < (n-1)\log\lambda + (n-1)\log\alpha,$$

which leads to

$$(\ell - 1)\log b + \log\lambda\alpha < n\log\lambda\alpha. \tag{9}$$

Combining (8) and (9) we obtain the desired inequalities. □

Now, we will complete the proof of Theorem 6. Substituting (3) and (4) in (6), we have

$$\frac{db^\ell}{b-1} - \frac{d}{b-1} = \left(\frac{\lambda^n}{\sqrt{5}} + \nu\right)(a\alpha^n + e_n) = \frac{a(\lambda\alpha)^n}{\sqrt{5}} + \frac{e_n\lambda^n}{\sqrt{5}} + a\nu\alpha^n + \nu e_n,$$

which leads to

$$\frac{a(\lambda\alpha)^n}{\sqrt{5}} - \frac{db^\ell}{b-1} = -\frac{d}{b-1} - \frac{e_n\lambda^n}{\sqrt{5}} - a\nu\alpha^n - \nu e_n. \tag{10}$$

Taking the absolute value of both sides of (10), we get for $n \geq 1$

$$\begin{aligned}\left|\frac{a(\lambda\alpha)^n}{\sqrt{5}} - \frac{db^\ell}{b-1}\right| &\leq \frac{d}{b-1} + \frac{\lambda^n}{\sqrt{5}}|e_n| + |a\nu|\,|\alpha|^n + |\nu e_n| \\ &< 1 + \frac{1}{\sqrt{5}}\left(\frac{\lambda}{\sqrt{\alpha}}\right)^n + \frac{0.19}{\sqrt{5}}(\alpha)^n + \frac{1}{\sqrt{5}}\left(\frac{1}{\sqrt{\alpha}}\right)^n \\ &= \alpha^n\left[\frac{1}{\alpha^n} + \frac{0.19}{\sqrt{5}} + \frac{1}{\sqrt{5}}\left(\frac{\lambda}{\alpha\sqrt{\alpha}}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1}{\alpha\sqrt{\alpha}}\right)^n\right] < 1.1\cdot\alpha^n.\end{aligned}$$

Thus, we see that

$$\left|\frac{a(\lambda\alpha)^n}{\sqrt{5}} - \frac{db^\ell}{b-1}\right| < 1.1\cdot\alpha^n. \tag{11}$$

Dividing both sides of inequality (11) by $a(\lambda\alpha)^n/\sqrt{5}$, we get

$$\left|1 - \frac{db^\ell}{b-1}\cdot\frac{\sqrt{5}}{a(\lambda\alpha)^n}\right| < \frac{1.1\sqrt{5}}{a\lambda^n},$$

which becomes

$$\left|b^\ell\cdot(\lambda\alpha)^{-n}\cdot\frac{d\sqrt{5}}{a(b-1)} - 1\right| < \frac{13.7}{\lambda^n}. \tag{12}$$

Put $\Gamma_1 := b^\ell\cdot(\lambda\alpha)^{-n}\cdot\dfrac{d\sqrt{5}}{a(b-1)} - 1$.

Next, we have to apply Theorem 2.1 to Γ_1. First, we need to check that $\Gamma_1 \neq 0$.

If it were not, then we would get that $\dfrac{d\sqrt{5}}{a(b-1)}b^{\ell} = (\alpha\lambda)^n$ and so $\lambda^{2n} \in \mathbb{Q}(\alpha)$.

Since $[\mathbb{Q}(\alpha):\mathbb{Q}] = 3$, then λ^{2n} is either a rational or a 3-degree algebraic number. However, λ is a quadratic algebraic number and since $\mathbb{Q}(\lambda^n) \subseteq \mathbb{Q}(\lambda)$, then the degree of λ^{2n} is either 1 or 2. So, we conclude that $\lambda^{2n} \in \mathbb{Q}$, which is an absurdity since $\lambda^{2n} = A_n + B_n\sqrt{5}$, for some positive rational numbers A_n and B_n. Therefore, we have that $\Gamma_1 \neq 0$.

Now, let us apply Theorem 2.1 to Γ_1 with $s := 3$ and $(\gamma_1, b_1) := (b, \ell)$, $(\gamma_2, b_2) := (\lambda\alpha, -n)$, $(\gamma_3, b_3) := \left(\dfrac{d\sqrt{5}}{a(b-1)}, 1\right)$.

Observe that $\mathbb{L} := \mathbb{Q}(\gamma_1, \gamma_2, \gamma_3) = \mathbb{Q}(\alpha, \lambda)$, so $d_{\mathbb{L}} := 6$. Moreover, we have $h(\gamma_1) = \log b$ and $h(\gamma_2) = h(\alpha\lambda) \leq h(\alpha) + h(\lambda) = \dfrac{1}{2}\log\lambda + \dfrac{1}{3}\log\alpha$. Furthermore, we get

$$\begin{aligned} h(\gamma_3) = h\left(\frac{d\sqrt{5}}{a(b-1)}\right) &\leq h\left(\frac{d}{b-1}\right) + h(\sqrt{5}) + h(a) \\ &= \log(\max\{b-1, d\}) + \frac{1}{2}\log 5 + \frac{1}{3}\log 44 \leq \log(b-1) + \frac{1}{2}\log 5 + \frac{1}{3}\log 44. \end{aligned}$$

Thus, we can take $A_1 := 6\log b$, $A_2 := 2.7$ and $A_3 := 6\log(b-1) + 12.4$. As $n \geq 2$ and $B \geq \max\{|b_1|, |b_2|, |b_3|\} = \max\{\ell, n, 1\}$, then we can take $B := \max\{\ell, n\}$. Using the previous data, Theorem 2.1 tells us that

$$\log|\Gamma_1| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2(1 + \log 6)(1 + \log B) \cdot A_1 \cdot A_2 \cdot A_3, \tag{13}$$

where

$$A_1 \cdot A_2 \cdot A_3 = 16.2\log b \cdot (6\log(b-1) + 12.4) < 291.6\log^2 b. \tag{14}$$

In the above inequality, we have used the fact that $6\log(b-1) + 12.4 < 18\log b$, which holds for all $b \geq 2$. Combining (13) and (14), we get

$$\log|\Gamma_1| > -4.19 \times 10^{15} \cdot \log^2 b \cdot (1 + \log B). \tag{15}$$

Case 1: $\boldsymbol{B = n}$. Then, from (12) and (15), it follows that

$$n\log\lambda - \log 13.7 < 4.19 \times 10^{15} \cdot \log^2 b \cdot (1 + \log n)$$

and then $n < 2.2 \times 10^{16} \cdot \log^2 b \cdot \log n$ holds for $n \geq 2$.

Case 2: $\boldsymbol{B = \ell}$. Then, from (12) and (15), we get

$$n\log\lambda - \log 13.7 < 4.19 \times 10^{15} \cdot \log^2 b \cdot (1 + \log\ell). \tag{16}$$

By Lemma 3.2, it is easy to see that $\ell < 2.5n$. Using this with (16), we get $n < 3.3 \times 10^{16} \cdot \log^2 b \cdot \log n$.

In all cases, we see that $n < 3.3 \times 10^{16} \cdot \log^2 b \cdot \log n$. To get an upper bound of n in term of b, we have to apply Lemma 2.2 with $l = 1$, $L = n$ and $H = 3.3 \times 10^{16} \cdot \log^2 b$. Therefore, $n < 3.6 \times 10^{18} \cdot \log^3 b$ and $\ell < 9 \times 10^{18} \cdot \log^3 b$.

This completes the proof of Theorem 3.1.

Remark 3.3. The inequalities from Theorem 3.1 allows to compute all the solutions to equation (6), for every fixed b.

Now, as an illustration, we will solve equation (6) for $2 \leq b \leq 9$. Therefore, we have the following result.

THEOREM 3.4. *The only solutions* (b, n, ℓ, d) *of Diophantine equation* (6) *are in*

$$\left\{\begin{matrix} (2,1,1,1), & (2,2,1,1), & (3,1,1,1), & (3,2,1,1), & (4,1,1,1), & (4,2,1,1), \\ (5,1,1,1), & (5,2,1,1), & (5,3,1,4), & (6,1,1,1), & (6,2,1,1), & (6,3,1,4), \\ (7,1,1,1), & (7,2,1,1), & (7,3,1,4), & (8,1,1,1), & (8,2,1,1), & (8,3,1,4), \\ (9,1,1,1), & (9,2,1,1), & (9,3,1,4), & (3,3,2,1), & (5,4,2,2), & (6,5,2,5), \\ & & & (5,7,4,2) & & \end{matrix}\right\}.$$

Note that in the case $b = 2$, we have to consider the following equation $F_n T_n = 2^\ell - 1$, which allows to find all Mersenne numbers that are products of Fibonacci and Tribonacci numbers. Thus, we have the following result, which is a consequence of Theorem 3.4.

COROLLARY 3.5. *The number* 1 *is the only Mersenne number which is a product of Fibonacci and Tribonacci numbers. Namely, we have* $F_1 T_1 = 2^1 - 1$ *and* $F_2 T_2 = 2^1 - 1$.

Proof (of Theorem 3.4). When $2 \leq b \leq 9$, the bounds on n and ℓ become $n < 4 \times 10^{19}$ and $\ell < 10^{20}$. To lower these bounds, we return to inequality (12) by putting

$$\Lambda_1 := \log(\Gamma_1 + 1) = \ell \log b - n \log \lambda\alpha + \log\left(\frac{d\sqrt{5}}{a(b-1)}\right).$$

Inequality (12) can be written as $\left|e^{\Lambda_1} - 1\right| < \dfrac{13.7}{\lambda^n}$.

For $n \geq 7$, we get $\left|e^{\Lambda_1} - 1\right| < \dfrac{13.7}{\lambda^n} < \dfrac{1}{2}$, which also implies that $\dfrac{1}{2} < e^{\Lambda_1} < \dfrac{3}{2}$.

If $\Lambda_1 > 0$, then $\quad 0 < \Lambda_1 < e^{\Lambda_1} - 1 = |e^{\Lambda_1} - 1| < \dfrac{13.7}{\lambda^n}$.

If $\Lambda_1 < 0$, then $\quad 0 < |\Lambda_1| < e^{|\Lambda_1|} - 1 = e^{-\Lambda_1}(1 - e^{\Lambda_1}) < \dfrac{27.4}{\lambda^n}$.

In any case, it is always holds true $0 < |\Lambda_1| < \dfrac{27.4}{\lambda^n}$, which implies that

$$0 < \left| \ell \frac{\log b}{\log \lambda\alpha} - n + \frac{\log\left(d\sqrt{5}/a(b-1)\right)}{\log \lambda\alpha} \right| < 25.3 \cdot \lambda^{-n}.$$

It is easy to see that $\dfrac{\log b}{\log \lambda\alpha}$ is irrational. In fact, if $\dfrac{\log b}{\log \lambda\alpha} = \dfrac{p}{q}$ ($p, q \in \mathbb{Z}$ and $p > 0, q > 0, \gcd(p, q) = 1$), then $(\lambda\alpha)^p = b^q \in \mathbb{Z}$, which is an absurdity. Now, we will apply Lemma 2.3 with

$$\tau := \frac{\log b}{\log \lambda\alpha}, \quad \mu := \frac{\log\left(d\sqrt{5}/a(b-1)\right)}{\log \lambda\alpha}, \quad A := 25.3, \quad B := \lambda,$$

and $w := n$. Note that $\ell < 10^{20}$, so we can take $M := 10^{20}$. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent until we find the one that satisfies the conditions. We used Mathematica to obtain the results given in following table.

b	2	3	4	5	6	7	8	9
q_t	q_{37}	q_{41}	q_{39}	q_{41}	q_{50}	q_{43}	q_{47}	q_{35}
$n \leq$	108	116	117	110	114	121	120	125
$\ell \leq$	270	290	292	275	285	302	300	312
$\epsilon >$	0.4	0.05	0.06	0.18	0.02	0.01	0.01	0.0009

So, the bounds $n \leq 125$ and $\ell \leq 312$ hold in all cases. Hence, it remains to check equation (6) for $1 \leq n \leq 125$ and $1 \leq \ell \leq 312$. A quick inspection using Maple reveals that the only solutions of Diophantine equation (6) are those mentioned in the statement of Theorem 3.4. $\square$

3.2 On b repdigits as sum of Fibonacci and Tribonacci numbers

In this subsection, we will follow the method in Subsection 3.1. Our result is as follows.

THEOREM 3.6. *Let $b \geq 2$ be an integer. Then, the Diophantine equation*

$$F_n + T_n = d\left(\frac{b^\ell - 1}{b - 1}\right), \tag{17}$$

has only finitely many solutions in integers (n, b, d, ℓ) such that $n, \ell \geq 1$ and $1 \leq d \leq b - 1$. Moreover, we have $n < 7.5 \times 10^{16} \log^3 b$ and $\ell < 1.2 \times 10^{17} \log^3 b$.

For $n = 1$, it is easy to show that all solutions of equation (17) are of the form $(b, n, \ell, d) = (b, 1, 1, 2)$ with $b \geq 3$. Now, we assume that $n \geq 2$. The next lemma relates the sizes of n, b, and ℓ.

LEMMA 3.7. *All solutions of the Diophantine equation* (17) *satisfy*

$$(\ell - 1)\frac{\log b}{\log \alpha} + \frac{\log(\alpha/2)}{\log \alpha} < n < \ell \frac{\log b}{\log \lambda} + 2.$$

Proof. Using inequalities (2) and (5), one can see that

$$\lambda^{n-2} < \lambda^{n-2} + \alpha^{n-2} \leq F_n + T_n = d\left(\frac{b^\ell - 1}{b - 1}\right) < b^\ell. \tag{18}$$

Taking the logarithm of the extreme sides of (18), we get

$$n \log \lambda < \ell \log b + 2 \log \lambda. \tag{19}$$

For the lower bound, we have from (2) and (5) that

$$b^{\ell-1} < d\left(\frac{b^\ell - 1}{b - 1}\right) = F_n + T_n \leq \lambda^{n-1} + \alpha^{n-1} < 2\alpha^{n-1}.$$

Taking the logarithm on both sides, we get that

$$(\ell - 1) \log b < \log 2 + (n - 1) \log \alpha,$$

which leads to

$$(\ell - 1) \log b + \log(\alpha/2) < n \log \alpha. \tag{20}$$

Combining (19) and (20) we obtain the desired inequalities. $\square$

Now, let us finish the proof of Theorem 3.6 by substituting (3) and (4) in (17) to have

$$\frac{db^{\ell}}{b-1} - \frac{d}{b-1} = \frac{\lambda^n}{\sqrt{5}} + \nu + a\alpha^n + e_n,$$

which leads to

$$a\alpha^n - \frac{db^{\ell}}{b-1} = -\frac{d}{b-1} - \frac{\lambda^n}{\sqrt{5}} - \nu - e_n. \tag{21}$$

Taking the absolute value of both sides of (21), we get for $n \geq 1$ that

$$\begin{aligned}\left|a\alpha^n - \frac{db^{\ell}}{b-1}\right| &\leq \frac{d}{b-1} + \frac{\lambda^n}{\sqrt{5}} + |e_n| + |\nu| \\ &< 1 + \frac{\lambda^n}{\sqrt{5}} + \frac{1}{\sqrt{5}} + \left(\frac{1}{\sqrt{\alpha}}\right)^n \\ &< \lambda^n \left[\frac{1}{\lambda^n} + \frac{1}{\sqrt{5}} + \frac{1}{\lambda^n\sqrt{5}} + \left(\frac{1}{\lambda\sqrt{\alpha}}\right)^n\right] < 1.8 \cdot \lambda^n.\end{aligned}$$

Hence, we have

$$\left|a\alpha^n - \frac{db^{\ell}}{b-1}\right| < 1.8 \cdot \lambda^n. \tag{22}$$

Dividing now both sides of (22) by $a\alpha^n$, we get

$$\left|1 - b^{\ell} \cdot \alpha^{-n} \cdot \frac{d}{a(b-1)}\right| < \frac{10}{(\alpha/\lambda)^n}. \tag{23}$$

Let

$$\Gamma_2 := b^{\ell} \cdot \alpha^{-n} \cdot \frac{d}{a(b-1)} - 1. \tag{24}$$

Next, we apply Theorem 2.1 to Γ_2. First, we need to check that $\Gamma_2 \neq 0$. If it wasn't, then we would get that $db^{\ell} = \alpha^n a(b-1)$.

Now, we apply the automorphism σ of the Galois group $\mathcal{G}$ on both sides and take absolute values as follows: $b^{\ell} \leq |d|b^{\ell} = (b-1)|b'||\beta|^n < b-1$, which contradicts the fact that $\ell \geq 2$. We conclude that $\Gamma_2 \neq 0$. So, we apply Theorem 2.1 to (24) with $s := 3$ and $(\gamma_1, b_1) := (b, \ell)$, $(\gamma_2, b_2) := (\alpha, -n)$, $(\gamma_3, b_3) := \left(\frac{d}{a(b-1)}, 1\right)$. Thus, we have $\mathbb{L} = \mathbb{Q}(\gamma_1, \gamma_2, \gamma_3) = \mathbb{Q}(\alpha)$ since $a = \alpha/(\alpha^2 + 2\alpha + 3)$, so $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 3$. Note that $h(\gamma_1) = \log b$, $h(\gamma_2) = (\log \alpha)/3$ and

$$h(\gamma_3) \leq h\left(\frac{d}{b-1}\right) + h(a) = \log\left(\max\{b-1, d\}\right) + \frac{1}{3}\log 44 = \log(b-1) + \frac{1}{3}\log 44.$$

Therefore, we take $A_1 = 3\log b$, $A_2 = \log \alpha$ and $A_3 := 3\log(b-1) + \log 44$. Since $n \geq 2$ and $B \geq \max\{|b_1|, |b_2|, |b_3|\}$, we take $B = \max\{n, \ell\}$. Hence, we get

$$\log|\Gamma_2| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2(1 + \log 3) \cdot (1 + \log B) \cdot A_1 A_2 A_3 \tag{25}$$

with

$$A_1 A_2 A_3 = 3\log b \cdot \log \alpha \cdot (3\log(b-1) + \log 44) < 18 \cdot \log \alpha \cdot \log^2 b. \tag{26}$$

In above inequality, we use the fact that $3\log(b-1)+\log 44 < 6\log b$, for $b \geq 2$. Thus, from (23), (25), and (26), we get

$$n\log\left(\frac{\alpha}{\lambda}\right) - \log 10 < 3\times 10^{13}\cdot(1+\log B)\cdot \log^2 b,$$

which leads to $n < 2.5\times 10^{14}\cdot(1+\log B)\cdot\log^2 b$. Now, we study the following two cases according to the values of B.

Case a: $\boldsymbol{B = n}$. Then, for $n \geq 2$ we obtain $n < 6.2\times 10^{14}\cdot\log^2 b\cdot\log n$.

Case b: $\boldsymbol{B = \ell}$. We have

$$n < 2.5\times 10^{14}\cdot(1+\log\ell)\cdot\log^2 b. \tag{27}$$

By Lemma 3.7 one can easily see that $\ell < 1.5n$, and thus by inequality (27) we get $n < 7.6\times 10^{14}\cdot\log^2 b\cdot\log n$.

So, in all cases we conclude that $n < 7.6\times 10^{14}\cdot\log^2 b\cdot\log n$ holds for $n \geq 2$. To obtain an upper bound of n in term of b we will apply Lemma 2.2 with $l = 1$, $L = n$ and $H = 7.6\times 10^{14}\cdot\log^2 b$. Thus, we obtain $n < 1.52\times 10^{15}\cdot\log^2 b\cdot(34.3+2\log(\log b))$.

Since $34.3 + 2\log(\log b) < 49\log b$ for $b \geq 2$, we deduce that $n < 7.5\times 10^{16}\cdot\log^3 b$ and $\ell < 1.2\times 10^{17}\cdot\log^3 b$. This completes the proof of Theorem 3.6.

Remark 3.8. One can use the inequalities in Theorem 3.6 to compute all the solutions to equation (17), for every fixed b.

Now, as an illustration, we solve equation (17), for $2 \leq b \leq 9$. The result in these cases in the following result.

Theorem 3.9. *The only solutions* (b, n, ℓ, d) *of the Diophantine equation* (17) *are in*

$$\left\{\begin{array}{c} (3,1,1,2),\ (3,2,1,2),\ (4,1,1,2),\ (4,2,1,2),\ (5,1,1,2),\ (5,2,1,2),\\ (5,3,1,4),\ (6,1,1,2),\ (6,2,1,2),\ (6,3,1,4),\ (7,1,1,2),\ (7,2,1,2),\\ (7,3,1,4),\ (8,1,1,2),\ (8,2,1,2),\ (8,3,1,4),\ (8,4,1,7),(9,1,1,2),\\ (9,2,1,2),\ (9,3,1,4),\ (9,4,1,7),\ (3,3,2,1),\ (5,5,2,2),(6,4,2,1),\\ (6,6,2,3),\ (2,4,3,1),\ (4,6,3,1) \end{array}\right\}.$$

Considering $b = 2$ in equation (17), we will solve the following equation $F_n + T_n = 2^\ell - 1$, which allows to find all Mersenne numbers that are sum of Fibonacci and Tribonacci numbers. Thus, we have the following consequence.

Corollary 3.10. *The number 7 is the only Mersenne number which is a sum of Fibonacci and Tribonacci numbers. Namely, we have* $F_4 + T_4 = 2^3 - 1$.

Proof (of Theorem 3.9). For $2 \leq b \leq 9$, we get from Theorem 3.6 that $n < 8\times 10^{17}$ and $\ell < 1.3\times 10^{18}$. Now, we need to reduce the upper bound for n and ℓ by applying Lemma 2.3. Put

$$\Lambda_2 := \log(\Gamma_2 + 1) = \ell\log b - n\log\alpha + \log\left(\frac{d}{a(b-1)}\right).$$

From (23), we conclude that $\left|e^{\Lambda_2}-1\right| < \dfrac{10}{(\alpha/\lambda)^n}$. Note that if $n \geq 24$, then $|e^{\Lambda_2}-1| < \dfrac{10}{(\alpha/\lambda)^n} < \dfrac{1}{2}$, which implies that $\dfrac{1}{2} < e^{\Lambda_2} < \dfrac{3}{2}$.

If $\Lambda_2 > 0$, then $\quad 0 < \Lambda_2 < e^{\Lambda_2} - 1 = |e^{\Lambda_2} - 1| < \dfrac{10}{(\alpha/\lambda)^n}$.

If $\Lambda_2 < 0$, then $\quad 0 < |\Lambda_2| < e^{|\Lambda_2|} - 1 = e^{|\Lambda_2|}(1 - e^{-|\Lambda_2|}) < \dfrac{20}{(\alpha/\lambda)^n}$.

In all cases, we have $0 < |\Lambda_2| < \dfrac{20}{(\alpha/\lambda)^n}$, which implies

$$0 < \left|\ell\frac{\log b}{\log\alpha} - n + \frac{\log(d/a(b-1))}{\log\alpha}\right| < 33.1 \cdot (\alpha/\lambda)^{-n}. \tag{28}$$

According to (28) and Lemma 2.3, we take $M := 1.3 \times 10^{18}$. Next, it is easy to see that $\dfrac{\log b}{\log\alpha}$ is irrational. To apply Lemma 2.3, we define the following quantities $\tau := \dfrac{\log b}{\log\alpha}$, $\mu := \dfrac{\log(d/a(b-1))}{\log\alpha}$, $A := 33.1$, $B := \alpha/\lambda$, and $w := n$. With the help of Mathematica, we found the results mentioned in the following table.

b	2	3	4	5	6	7	8	9
q_t	q_{39}	q_{35}	q_{43}	q_{42}	q_{38}	q_{41}	q_{37}	q_{37}
$n \leq$	194	191	197	191	194	201	210	198
$\ell \leq$	291	286	295	286	291	301	315	297
$\epsilon >$	0.28	0.15	0.13	0.21	0.05	0.05	0.001	0.02

It follows that the bounds $n \leq 210$ and $\ell \leq 315$ hold in all cases according to the values of b. To finish the proof, we use a simple routine written in Maple which reveals that the only solutions of the Diophantine equation (17) are those listed in the statement of Theorem 3.9. □

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