

HANDLE DECOMPOSITION FOR A CLASS OF COMPACT ORIENTABLE PL 4-MANIFOLDS

Anshu Agarwal, Biplab Basak and Manisha Binjola

Abstract. For a compact orientable PL 4-manifold M with boundary, let $\hat{M}$ be the singular manifold obtained by capping of ∂M . In this article, we explore the class of compact orientable PL 4-manifolds with empty or connected boundary, whose fundamental groups have rank 1, and their corresponding singular manifolds admit weak semi-simple crystallizations. First, we show that if M is a closed orientable PL 4-manifold belonging to this class, then there exist complementary submanifolds V and V' with a shared boundary such that $\mathcal{G}(V') \geq \mathcal{G}(M) \geq \mathcal{G}(V)$, where $\mathcal{G}(M)$, $\mathcal{G}(V)$, and $\mathcal{G}(V')$ denote the regular genera of M , V , and V' , respectively.

Next, we provide a handle decomposition for a compact orientable PL 4-manifold M with connected non-spherical boundary from this class. If the rank of the fundamental group of $\hat{M}$, m' , is 1, then M admits a handle decomposition that takes one of the following forms:

1) one 0-handle, one 1-handle, k 2-handles and one 3-handle, where $k = 2 + \beta_2(M) - \beta_1(M) - \beta_1(\hat{M})$, or

2) one 0-handle, two 1-handles, k 2-handles and one 3-handle, where $k = 3 + \beta_2(M) - \beta_1(M) - \beta_1(\hat{M})$.

Further, if $m' = 0$, then M admits a handle decomposition which consists of one 0-handle, one 1-handle and $\beta_2(M)$ 2-handles.

We further demonstrate that no manifold from this class with empty or connected spherical boundary can have a fundamental group isomorphic to a finite cyclic group. Finally, we provide a handle decomposition for such manifolds with empty or connected spherical boundary.

1. Introduction

A crystallization (Γ, γ) of a compact PL d -manifold is a specialized edge-colored graph that encodes the manifold's structure (see Subsection 2.1). Pezzana proved that every closed PL d -manifold has a crystallization (see [17]). This result was later broadened to include singular d -manifolds (cf. [7–12]). For a compact PL d -manifold M with

2020 *Mathematics Subject Classification*: 54B15, 54C25, 57Q15, 57M15, 05C15

Keywords and phrases: PL-manifolds; crystallizations; regular genus; handle decomposition.

boundary, a singular d -manifold $\hat{M}$ can be formed by capping off each boundary component of ∂M . There is a one-one correspondence between the class of compact PL 4-manifolds with empty or non-spherical boundary components and the class of singular manifolds (see Remark 2.2). If M is a compact PL 4-manifold with spherical boundary components, then $\hat{M}$ is a closed PL 4-manifold. If M is a closed PL 4-manifold, then $\hat{M} = M$.

The concept of “regular genus” for closed PL d -manifolds, introduced in [16], generalizes the classical genus in dimension 2 and the Heegaard genus in dimension 3 (see Subsection 2.2 for details). This idea was later extended to compact PL d -manifolds with non-spherical boundary components for $d \geq 2$ through the framework of singular d -manifolds (cf. [7–12]).

In [1], the concept of weak semi-simple crystallizations for closed PL 4-manifolds was introduced, while in [7], the notion of weak semi-simple gems for compact PL 4-manifolds with connected boundary was developed. This paper focuses on compact orientable PL 4-manifolds with empty or connected boundary, whose fundamental groups have rank 1 and whose associated singular manifolds admit weak semi-simple crystallizations.

In [10], the authors proved that the weak simple class (weak semi-simple for simply-connected 4-manifolds) of compact simply-connected 4-manifolds with empty or connected boundary admit a special handle decomposition that lacks 1-handles and 3-handles. The original problem for closed PL 4-manifolds was posed by Kirby and can be formulated as follows: “Does every simply-connected closed 4-manifold have a handlebody decomposition with no 1-handles and 3-handles?” A lot of work has been performed over the decades, including studies on manifolds with boundaries, such as Trace’s work in [19, 20].

First, we prove that for a closed orientable 4-manifold M with the non-trivial cyclic fundamental group admitting a weak semi-simple crystallization, the combinatorial invariant regular genus of M lies between the regular genera of its complementary submanifolds V and V' (see Theorem 4.2).

In [3], it has been shown that for the class of closed orientable 4-manifolds with infinite cyclic fundamental group admitting semi-simple crystallizations, there exists a handle decomposition such that the number of 2-handles depends on the second Betti number of the manifold, and the number of other h -handles ($h \leq 4$) is at most 2.

In this article, we extend this work from the class of closed orientable PL 4-manifolds with rank 1 fundamental groups and admitting semi-simple crystallizations to a larger class of compact orientable PL 4-manifolds with empty or connected boundary that admit weak semi-simple crystallizations and have non-trivial cyclic fundamental groups. For a compact orientable PL 4-manifold M with connected non-spherical boundary within this class, we construct a handle decomposition. If the rank of the fundamental group of $\hat{M}$, m' , is 1, then M admits a handle decomposition that takes one of the following forms:

- 1) one 0-handle, one 1-handle, k 2-handles and one 3-handle, where $k = 2 + \beta_2(M) - \beta_1(M) - \beta_1(\hat{M})$, or
- 2) one 0-handle, two 1-handles, k 2-handles and one 3-handle, where $k = 3 + \beta_2(M) -$

$$\beta_1(M) - \beta_1(\hat{M}).$$

Further, if $m' = 0$, then M admits a handle decomposition which consists of one 0-handle, one 1-handle and $\beta_2(M)$ 2-handles (see Theorem 4.4). In proving this theorem, we shall extensively use the notion of regular neighborhoods [18]. The proof of this theorem implies that any manifold with empty or connected spherical boundary of this class can never have a finite cyclic fundamental group, and we also provide a handle decomposition for such manifolds (see Corollary 4.5).

2. Preliminaries

The crystallization theory provides a tool for representing piecewise-linear (PL) manifolds of any dimension combinatorially using edge-colored graphs.

2.1 Crystallization

In this article, we study multigraphs $\Gamma = (V(\Gamma), E(\Gamma))$ without loops, where the edges are assigned colors (or labels) from the set $\Delta_d := \{0, 1, \dots, d\}$. The elements of Δ_d are called the *colors* of Γ . A $(d+1)$ -regular colored graph is defined as a pair (Γ, γ) , where each vertex of Γ has degree $d+1$, and there exists a surjective map $\gamma : E(\Gamma) \rightarrow \Delta_d$ such that any two adjacent edges e_1 and e_2 satisfy $\gamma(e_1) \neq \gamma(e_2)$. When the edge coloring is clear from the context, we may simply refer to the $(d+1)$ -regular colored graph as Γ . For $\mathcal{C} \subseteq \Delta_d$ with cardinality k , the graph $\Gamma_{\mathcal{C}} = (V(\Gamma), \gamma^{-1}(\mathcal{C}))$ is a k -regular colored graph with edge-coloring $\gamma|_{\gamma^{-1}(\mathcal{C})}$. For a color set $\{j_1, j_2, \dots, j_k\} \subset \Delta_d$, $g(\Gamma_{\{j_1, j_2, \dots, j_k\}})$ or $g_{\{j_1, j_2, \dots, j_k\}}$ denotes the number of connected components of the graph $\Gamma_{\{j_1, j_2, \dots, j_k\}}$. A graph (Γ, γ) is called *contracted* if the subgraph $\Gamma_{\hat{j}} = \Gamma_{\Delta_d \setminus \{j\}}$ is connected for all $j \in \Delta_d$. We refer to [6] for standard terminologies on graphs. All spaces and maps will be considered in PL-category [18].

For a $(d+1)$ -regular colored graph (Γ, γ) , a corresponding colored d dimensional simplicial cell complex $\mathcal{K}(\Gamma)$ is constructed as follows:

- for each vertex $v \in V(\Gamma)$, take a d -simplex $\sigma(v)$ with vertices labeled by Δ_d ,
- corresponding to each edge of color j between $v_1, v_2 \in V(\Gamma)$, identify the $(d-1)$ -faces of $\sigma(v_1)$ and $\sigma(v_2)$ opposite to j -labeled vertices such that the same labeled vertices coincide.

The geometric carrier $|\mathcal{K}(\Gamma)|$ is a d -pseudomanifold, and (Γ, γ) is referred to as a *gem* (graph encoded manifold) of any d -pseudomanifold homeomorphic to $|\mathcal{K}(\Gamma)|$, or equivalently, it is said to represent the d -pseudomanifold. Since the vertex set of $\mathcal{K}(\Gamma)$ can be $(d+1)$ -colored, we refer to $\mathcal{K}(\Gamma)$ as a *colored triangulation* of the d -pseudomanifold $|\mathcal{K}(\Gamma)|$. For further details on CW complexes and related notions, see [5].

From this construction, it is straightforward to observe that for any subset $\mathcal{C} \subset \Delta_d$ of cardinality $k+1$, $\mathcal{K}(\Gamma)$ has as many k -simplices with vertices labeled by $\mathcal{C}$ as there are connected components in $\Gamma_{\Delta_d \setminus \mathcal{C}}$ [14]. The *disjoint star* of a simplex $\sigma \in \mathcal{K}(\Gamma)$ is

a simplicial cell complex comprising all d -dimensional simplices of $\mathcal{K}(\Gamma)$ that contain σ as a face, with the identification of their $(d-1)$ -faces containing σ as in $\mathcal{K}(\Gamma)$. The *disjoint link* of $\sigma \in \mathcal{K}(\Gamma)$ is the subcomplex of its disjoint star formed by all simplices that do not intersect σ .

DEFINITION 2.1. A *singular PL d -manifold* is a closed d dimensional polyhedron with a colored triangulation such that the disjoint links of its vertices are closed $(d-1)$ -manifolds. A vertex whose link is not a sphere is called a singular vertex. It follows that a closed (PL) d -manifold is simply a singular (PL) d -manifold with no singular vertices.

From the correspondence between $(d+1)$ -regular colored graphs (Γ, γ) and d dimensional pseudomanifolds, it can be easily visualised that:

- 1) $|\mathcal{K}(\Gamma)|$ is a closed PL d -manifold if and only if for each $c \in \Delta_d$, Γ_c represents $\mathbb{S}^{d-1}$.
- 2) $|\mathcal{K}(\Gamma)|$ is a singular PL d -manifold if and only if for each $c \in \Delta_d$, Γ_c represents a closed PL $(d-1)$ -manifold.

If Γ_j does not represent the $(d-1)$ -sphere, then the color j is called a singular color.

REMARK 2.2 ([10, 12]). There is a one-to-one correspondence between the class of singular (PL) d -manifolds and the class of all compact (PL) d -manifolds with empty or non-spherical boundary components. If M is a singular d -manifold, then removing a small open neighborhood around each of its singular vertices yields a compact d -manifold $\check{M}$ with non-spherical boundary components. Moreover, $M = \check{M}$ if and only if M is a closed d -manifold. Conversely, if M is a compact d -manifold with non-spherical boundary components, a corresponding singular d -manifold $\hat{M}$ can be obtained by capping off each boundary component of ∂M . If M is closed, then $M = \hat{M}$.

From this point forward, when we refer to a graph representing a compact PL 4-manifold M with non-spherical boundary components, we mean the graph that represents the corresponding singular manifold $\hat{M}$, obtained by capping off each boundary component of M with a cone. In this article, we focus on compact PL 4-manifolds M with empty or connected non-spherical boundary. As a result, we will consider only 5-regular colored graphs. Without loss of generality, we assume that color 4 is the only possible singular color of Γ , the gem of $\hat{M}$.

DEFINITION 2.3. A $(d+1)$ -regular colored gem of a singular d -manifold with at most one singular vertex is called a *crystallization* of M if it is contracted.

In the colored triangulation of M corresponding to a crystallization, there are precisely $d+1$ vertices. The foundation of crystallization theory lies in Pezzana's existence theorem [17], which proves that every closed PL d -manifold admits a crystallization. Additionally, the existence of crystallizations was established for singular (PL) d -manifolds with a single singularity in [12]. This concept has since been further developed in various articles [7–11].

2.2 Regular Genus

It is well known that a $(d + 1)$ -regular colored bipartite (resp. non-bipartite) graph (Γ, γ) admits regular embedding into the orientable (resp. non-orientable) surface F_ε (corresponding to cyclic permutation ε of Δ_d) of Euler characteristic

$$\chi_\varepsilon(\Gamma) = \sum_{i \in \mathbb{Z}_{d+1}} g_{\{\varepsilon_i, \varepsilon_{i+1}\}} + (1 - d) \frac{V(\Gamma)}{2},$$

where subscripts are taken in modulo d . Additionally, (Γ, γ) does not admit any regular embedding into a non-orientable (resp. orientable) surface [15]. Using this, the concept of *regular genus* of closed and singular manifolds was introduced in [16] and [12], respectively, extending the notions of *genus of surfaces* and *Heegard genus of 3-manifolds*. The regular genus is a PL-invariant for PL manifolds of any dimension. It is known that a gem (Γ, γ) of a manifold M is a bipartite graph if and only if M is orientable. For $d \geq 3$, the regular genus $\rho(\Gamma)$ of (Γ, γ) is defined as

$$\rho(\Gamma) = \min\{\rho_\varepsilon(\Gamma) \mid \varepsilon \text{ is a cyclic permutation of } \Delta_d\},$$

where the genus (resp. half of genus) $\rho_\varepsilon(\Gamma)$ of F_ε satisfies

$$\rho_\varepsilon(\Gamma) = 1 - \frac{\chi_\varepsilon(\Gamma)}{2}.$$

The regular genus of M (closed or singular PL manifold) is defined as

$$\mathcal{G}(M) = \min\{\rho(\Gamma) \mid (\Gamma, \gamma) \text{ represents } M\}.$$

For a compact PL 4-manifold M with connected non-spherical boundary, the (generalized) regular genus $\mathcal{G}(M)$ of M is defined as the regular genus $\mathcal{G}(\hat{M})$ of $\hat{M}$.

PROPOSITION 2.4 ([11]). *Let M be a compact PL 4-manifold with connected non-spherical boundary. Then $\mathcal{G}(M) \geq 2\chi(M) + 3m + 2m' - 2$, where m and m' are the ranks of the fundamental groups of M and $\hat{M}$, respectively.*

In [2], a lower bound is provided for the regular genus of compact PL 4-manifolds with boundary. However, the definition of regular genus of compact PL 4-manifolds with boundary used there differs from the definition of regular genus presented in this article.

3. Weak semi-simple crystallizations of compact 4-manifolds

In [1] and [7], the concept of weak semi-simple crystallizations of compact PL 4-manifolds with empty or connected non-spherical boundary was introduced. These crystallizations are minimal with respect to regular genus among the graphs representing the same 4-manifold.

DEFINITION 3.1. Let M be a compact PL 4-manifold with empty or connected non-spherical boundary. A 5-regular colored graph Γ representing M is called *weak semi-simple* with respect to a permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_4 = 4)$ if $g_{\{\varepsilon_j, \varepsilon_{j+2}, \varepsilon_{j+4}\}} = m + 1$

$\forall j \in \{0, 2, 4\}$ and $g_{\{\varepsilon_j, \varepsilon_{j+2}, \varepsilon_{j+4}\}} = m' + 1 \forall j \in \{1, 3\}$ (additions are modulo 5 in subscripts), where m and m' are the ranks of the fundamental groups of M and $\hat{M}$, respectively.

Note that $m' \leq m$ and if Γ is weak semi-simple, then it is contracted as well, and hence Γ is a crystallization.

Let (Γ, γ) be a crystallization of a singular PL 4-manifold M and $\mathcal{K}(\Gamma)$ be the corresponding colored triangulation with the vertex set Δ_4 . If $B \subset \Delta_4$, then $\mathcal{K}(B)$ denotes the subcomplex of $\mathcal{K}(\Gamma)$ generated by the vertices $i \in B$. If $\text{Sd } \mathcal{K}(\Gamma)$ is the first barycentric subdivision of $\mathcal{K}(\Gamma)$, then $F(i, j)$ (resp. $F(i, j, k)$) is the largest subcomplex of $\text{Sd } \mathcal{K}(\Gamma)$, disjoint from $\text{Sd } \mathcal{K}(i, j) \cup \text{Sd } \mathcal{K}(\Delta_4 \setminus \{i, j\})$ (resp. $\text{Sd } \mathcal{K}(i, j, k) \cup \text{Sd } \mathcal{K}(\Delta_4 \setminus \{i, j, k\})$). Then the polyhedron $|F(i, j)|$ (resp. $|F(i, j, k)|$) is a closed PL 3-manifold which partitions M into two 4-manifolds $N(i, j)$ and $N(\Delta_4 \setminus \{i, j\})$ (resp. $N(i, j, k)$ and $N(\Delta_4 \setminus \{i, j, k\})$) with $|F(i, j)|$ (resp. $|F(i, j, k)|$) as common boundary. Further, $N(i, j)$ (resp. $N(i, j, k)$) is a regular neighbourhood of $|\mathcal{K}(i, j)|$ (resp. $|\mathcal{K}(i, j, k)|$) in $|\mathcal{K}(\Gamma)|$ (see [3, 10, 13, 16] for more details). Thus, M has a decomposition as $M = N(i, j) \cup_\phi N(\Delta_4 \setminus \{i, j\})$, where ϕ is a boundary identification.

DEFINITION 3.2 (Complementary Submanifolds). Let M be a compact orientable PL 4-manifold with empty or connected boundary, and $\hat{M}$ be its corresponding singular manifold. Let $\hat{M}$ admit weak semi-simple crystallization (Γ, γ) . Without loss of generality, we write $\hat{M} = N(1, 4) \cup N(0, 2, 3)$. We denote $N(1, 4)$ and $N(0, 2, 3)$ by V and V' respectively. We call V and V' the *complementary submanifolds* of $\hat{M}$.

LEMMA 3.3. *Let M be a compact orientable PL 4-manifold with empty or connected non-spherical boundary, and $\hat{M}$ be its corresponding singular manifold. Let M admit weak semi-simple crystallization (Γ, γ) and V, V' be the complementary submanifolds as in Definition 3.2. Then $V \cap V'$ is orientable and V is PL homeomorphic to the boundary connected sum of $\#_{m'}(\mathbb{S}^1 \times \mathbb{B}^3)$ and $C(\partial M)$, the cone over ∂M , where m' is the rank of $\pi_1(\hat{M})$.*

Proof. Using the Mayer-Vietoris exact sequence for the triple $(\hat{M}, V, V')$, we obtain $0 \rightarrow H_4(\hat{M}) \rightarrow H_3(\partial V) \rightarrow 0$. This shows that $\hat{M}$ is orientable if and only if ∂V is orientable. Since M is orientable, it follows that $\hat{M}$ is orientable, and thus ∂V is also orientable. Consequently, as $\partial V = V \cap V'$, $V \cap V'$ is orientable.

Given that $g_{\{0, 2, 3\}} = m' + 1$, there are exactly $m' + 1$ edges of color $\{1, 4\}$. Therefore, V is PL-homeomorphic to the boundary connected sum $\#_{m'}(\mathbb{S}^1 \times \mathbb{B}^3) \# C(\partial M)$, where $C(\partial M)$ is the cone over ∂M . This further implies that $\partial V = \#_{m'}(\mathbb{S}^1 \times \mathbb{S}^2) \# \partial M$. $\square$

The following result on isomorphism between cohomology and homology groups of a 4-manifold with a connected boundary and its associated singular manifold, respectively, was given in [10, Proposition 5].

PROPOSITION 3.4. *Let M be a compact orientable PL 4-manifold with connected boundary and $\hat{M}$ be the singular manifold by capping off its boundary. Then*

$$H_k(\hat{M}) \cong H^{4-k}(M) \text{ for } k \in \{1, 2, 3, 4\}.$$

LEMMA 3.5. *Let M be a compact orientable PL 4-manifold with empty or connected non-spherical boundary admitting a weak semi-simple crystallization. Let V and V' be the spaces, as in Definition 3.2. Then*

$$\beta_1(M) + \beta_1(\hat{M}) - \beta_2(M) + \beta_2(V') - \beta_1(V') = m', \quad (1)$$

where m' is the rank of the fundamental group of $\hat{M}$.

Proof. Since V' is regular neighbourhood of 2-dimensional polyhedron $|\mathcal{K}(0, 2, 3)|$, it deformation retracts onto $|\mathcal{K}(0, 2, 3)|$ and thus $H_i(V') = 0$ for $i > 2$. The Mayer-Vietoris sequence of the triple $(\hat{M}, V, V')$ gives the following long exact sequence.

$$\begin{aligned} 0 \longrightarrow H_3(\hat{M}) \longrightarrow H_2(\partial V) \longrightarrow H_2(V) \oplus H_2(V') \longrightarrow H_2(\hat{M}) \longrightarrow H_1(\partial V) \\ \downarrow \\ 0 \longleftarrow H_1(\hat{M}) \longleftarrow H_1(V) \oplus H_1(V') \end{aligned}$$

By Proposition 3.4 and Universal Coefficient Theorem, we have $H_3(\hat{M}) \cong H^1(M) \cong FH_1(M)$ and $H_2(\hat{M}) \cong H^2(M) \cong FH_2(M) \oplus TH_1(M)$. Now,

$$\begin{aligned} H_2(\partial V) &= H_2(\#_{m'}(\mathbb{S}^1 \times \mathbb{S}^2) \# \partial M) = H_2(\#_{m'}(\mathbb{S}^1 \times \mathbb{S}^2)) \oplus H_2(\partial M) \\ &\cong \oplus_{m'} \mathbb{Z} \oplus H^1(\partial M) \cong \oplus_{m'} \mathbb{Z} \oplus FH_1(\partial M), \end{aligned}$$

where the third isomorphism is due to Poincaré duality. Similarly, we have $H_1(\partial V) = \oplus_{m'} \mathbb{Z} \oplus H_1(\partial M)$. Also, for $i = 1, 2$; $H_i(V) = H_i(\#_{m'}(\mathbb{S}^1 \times \mathbb{B}^3)) \oplus H_i(C)$, where C is cone over ∂M . This implies, $H_2(V) = 0$ and $H_1(V) = \oplus_{m'} \mathbb{Z}$. Therefore, the above exact sequence simplifies to the following.

$$\begin{aligned} 0 \longrightarrow FH_1(M) \longrightarrow \oplus_{m'} \mathbb{Z} \oplus FH_1(\partial M) \longrightarrow H_2(V') \longrightarrow FH_2(M) \oplus TH_1(M) \\ \downarrow \\ 0 \longleftarrow H_1(\hat{M}) \longleftarrow \oplus_{m'} \mathbb{Z} \oplus H_1(V') \longleftarrow \oplus_{m'} \mathbb{Z} \oplus H_1(\partial M) \end{aligned}$$

Since the alternate sum of the number of free generators of finitely generated abelian groups in an exact sequence is zero, the result follows. $\square$

It is known that every compact PL 4-manifold M with empty or connected boundary admits a handle decomposition, i.e.,

$$\begin{aligned} M &= H^{(0)} \cup (H_1^{(1)} \cup \dots \cup H_{d_1}^{(1)}) \cup (H_1^{(2)} \cup \dots \cup H_{d_2}^{(2)}) \\ &\quad \cup (H_1^{(3)} \cup \dots \cup H_{d_3}^{(3)}) \text{ (if } M \text{ is closed) } \cup H^{(4)}, \end{aligned}$$

where $H^{(0)} = \mathbb{D}^4$ and each k -handle $H_i^{(k)} = \mathbb{D}^k \times \mathbb{D}^{4-k}$ (for $1 \leq k \leq 4$, $1 \leq i \leq d_k$) is attached via a map (embedding) $f_i^{(k)}: \partial \mathbb{D}^k \times \mathbb{D}^{4-k} \rightarrow \partial(H^{(0)} \cup \dots \cup (H_1^{(k-1)} \cup \dots \cup H_{d_{k-1}}^{(k-1)}))$.

LEMMA 3.6. *Let M be a compact orientable PL 4-manifold with empty or connected non-spherical boundary admitting a weak semi-simple crystallization (Γ, γ) and V, V' be as in Definition 3.2. Then, $\text{rank}(\pi_1(\hat{M})) \leq \text{rank}(\pi_1(V'))$.*

Proof. Let m' be the rank of the fundamental group of $\hat{M}$. We have $\hat{M} = V \cup V' = (\#_{m'}(\mathbb{S}^1 \times \mathbb{B}^3) \# C(\partial M)) \cup V'$ (up to homeomorphism), where $C(\partial M)$ is the cone over ∂M . Also, $V \cap V' = \partial V = \partial V' = (\#_{m'}\mathbb{S}^1 \times \mathbb{S}^2) \# \partial M$ from Lemma 3.3. We will use Seifert-Van Kampen Theorem. Let $i_1 : \pi_1(V \cap V') \rightarrow \pi_1(V)$ and $i_2 : \pi_1(V \cap V') \rightarrow \pi_1(V')$ be the maps induced from inclusion maps $j_1 : V \cap V' \rightarrow V$ and $j_2 : V \cap V' \rightarrow V'$, respectively.

Since $V \cap V' = \#_{m'}(\mathbb{S}^1 \times \mathbb{S}^2) \# \partial M$, its fundamental group is generated by at least m' elements, say $\pi_1(V \cap V') = \langle \alpha_1, \dots, \alpha_{m'}, \gamma_1, \dots, \gamma_{n_1} | d_1, d_2, \dots, d_{n_2} \rangle$ where generator α_i comes from $\#_{m'}(\mathbb{S}^1 \times \mathbb{S}^2)$ for all $1 \leq i \leq m'$, and the remaining generators come from ∂M . Also, we have $\pi_1(V) = \langle \alpha_i \mid 1 \leq i \leq m' \rangle$ and hence $i_1(\alpha_i) = \alpha_i$, $i_1(\gamma_j) = \langle e \rangle$. Now, let $\pi_1(V') = \langle \beta_1, \dots, \beta_{n_3} | z_1, \dots, z_{n_4} \rangle$. Applying the theorem, we get

$\pi_1(\hat{M}) = \langle \alpha_1, \dots, \alpha_{m'}, \beta_1, \dots, \beta_{n_3} | z_1, \dots, z_{n_4}, \alpha_i = i_2(\alpha_i), e = i_2(\gamma_j), 1 \leq i \leq m', 1 \leq j \leq n_1 \rangle$, which proves the result. $\square$

The following is a result that gives a characterization of having a weak semi-simple crystallization for a closed PL 4-manifold.

PROPOSITION 3.7 ([1, 4]). *Let M be a closed PL 4-manifold. Then, M admits a weak semi-simple crystallization if and only if $\mathcal{G}(M) = 2\chi(M) + 5m - 4$, where m is the rank of the fundamental group of M .*

4. Proof of main results

In this section, we consider the class of compact orientable PL 4-manifolds with empty or connected boundary that have fundamental groups of rank 1 (i.e., $m = 1$) such that their corresponding singular manifolds admit weak semi-simple crystallizations. This implies that the fundamental group of a manifold from this class is either $\mathbb{Z}$ or $\mathbb{Z}_k$ for some k , $k \geq 2$. We will significantly rely on the following known fact to prove our main results.

PROPOSITION 4.1 ([18]). *Let M be a manifold and $X \subset \text{int } M$ be a polyhedron. If X collapses onto Y , then a regular neighborhood of X in M is PL-homeomorphic to a regular neighborhood of Y in M .*

THEOREM 4.2. *Let M be a closed orientable PL 4-manifold with $\text{rank}(\pi_1(M)) = 1$. Let (Γ, γ) be a weak semi-simple crystallization of M and V, V' be as in Definition 3.2. Then $\mathcal{G}(V') \geq \mathcal{G}(M) \geq \mathcal{G}(V)$.*

Proof. Since $M = V \cup V'$ admits weak semi-simple crystallization, $\mathcal{G}(M) = 2\chi(M) + 1$ by Proposition 3.7. Since V' is a 4-manifold with connected boundary, Proposition 2.4 and Lemma 3.6 imply that $\mathcal{G}(V') \geq 2\chi(V') + 3 \text{rank}(\pi_1(V')) - 2 = 2\chi(V') + 1$. Also, $\chi(V) = \chi(\mathbb{S}^1 \times \mathbb{B}^3) = 0$ and $\chi(V \cap V') = \chi(\mathbb{S}^1 \times \mathbb{S}^2) = 0$, $\chi(M) = \chi(V) + \chi(V') - \chi(V \cap V') = \chi(V')$. Thus, $\mathcal{G}(V') \geq 2\chi(V') + 1 = 2\chi(M) + 1 = \mathcal{G}(M)$ and one inequality follows. On the other hand, $\mathcal{G}(\mathbb{S}^1 \times \mathbb{B}^3) = 1$ (cf. [11]) implies $\mathcal{G}(M) \geq \mathcal{G}(V)$ and the result follows. $\square$

LEMMA 4.3. *Let M be a compact orientable PL 4-manifold with empty or connected non-spherical boundary, whose fundamental group has rank 1. Let $\hat{M}$ admit weak semi-simple crystallization (Γ, γ) and V, V' be as in Definition 3.2. Then V' , a regular neighborhood of $\mathcal{K}(0, 2, 3) = \mathcal{K}$, is PL homeomorphic to a regular neighborhood of one of the following: $\mathbb{S}^1 \vee \mathbb{S}^1 \vee (\mathbb{S}^2)^\alpha$, $\mathbb{S}^1 \vee (\mathbb{S}^2)^\alpha$ or a space with the trivial fundamental group, where $\alpha = \beta_2(\mathcal{K})$ and $(\mathbb{S}^2)^\alpha$ denotes the wedge product of α copies of the 2-sphere $\mathbb{S}^2$.*

Proof. We shall analyze $\mathcal{K}(0, 2, 3)$ to prove this. Let k be the number of $\{2, 3\}$ -labeled edges, and let $d_1, d_2, \dots, d_k$ denote the $\{2, 3\}$ -labeled edges. Also, by the definition of weak semi-simple and the correspondence between the number of components of Γ and the number of simplices in the corresponding simplicial cell complex, there are exactly two $\{0, 2\}$ - and $\{0, 3\}$ -labeled edges each. We denote the two edges with endpoints labeled by color 0 and 2 (resp. 0 and 3) by l_1 and l_2 (resp. by r_1 and r_2). Thus, there can be at most four distinct triangles with vertices 0, 2, 3 passing through the edge d_i for each $i \in \{1, 2, \dots, k\}$, as shown in Figure 1. Let $A_{11}^i, A_{12}^i, A_{21}^i$ and A_{22}^i be these four triangles.

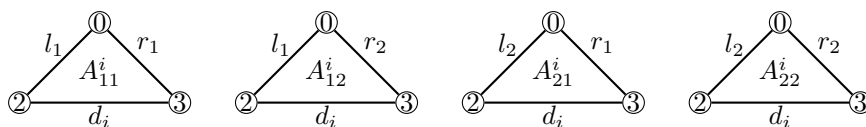


Figure 1: The four possible 2-simplices passing through one $\{2, 3\}$ -labeled edge.

Since the two triangles with all three edges coinciding, make one 2-sphere $\mathbb{S}^2$, the triangles with the same boundary give the copies of 2-sphere. So, from here on in this proof, we assume that no two triangles have the same boundary.

Let p_i be the cardinality of 2-simplices with vertices labeled by $\{0, 2, 3\}$ that pass through the d_i edge.

Case 1 ($p_i + p_j \geq 5 \forall i, j$): We consider the triangles in $\mathcal{K}(0, 2, 3)$ that pass through d_i and d_j , $i \neq j$. The non-trivial loops (if any) should be along the edges $l_1 l_2$, $r_1 r_2$ and $d_i d_j$. It is not difficult to observe that if $p_i + p_j \geq 5$, then all these three possible non-trivial loops $l_1 l_2$, $r_1 r_2$ and $d_i d_j$ are homotopic to identity, i.e., the fundamental group of $\mathcal{K}(0, 2, 3)$ is trivial. Since i and j are arbitrary, we make the general statement on similar lines that for k number of $\{2, 3\}$ -labeled edges, say $d_1, d_2, \dots, d_k$, if $p_i + p_j \geq 5 \forall i, j \in \{1, 2, \dots, k\}$ then the fundamental group of $\mathcal{K}(0, 2, 3)$ is trivial.

Case 2 ($p_i + p_j \geq 4 \forall i, j$ and $p_i + p_j = 4$ for some i, j but $p_i + p_j \neq 4 \forall i, j$): If $p_i = 2 = p_j$, $i \neq j$ and $p_k = 3$ or 4 then the loops $l_1 l_2$, $r_1 r_2$, $d_i d_k$ and $d_j d_k$ are homotopic to identity by using the proof of Case 1 because $p_i + p_k, p_j + p_k \geq 5$. Now, we have to check whether the loop $d_i d_j$ is trivial. Without loss of generality, we assume that one triangle passing through d_i is of the form A_{11}^i in Figure 1. If any of the two triangles passing through d_j is of the form A_{11}^j , then it forms a cone-like shape, and obviously $d_i d_j$ is homotopic to a constant loop. If one triangle is of the

form A_{12}^j (resp., A_{21}^j) in Figure 1 then $d_i d_j$ is homotopic to $r_1 r_2$ (resp., $l_1 l_2$) which is further homotopic to a constant loop. If one triangle is of the form A_{22}^j then another triangle passing through d_j is among A_{11}^j , A_{12}^j or A_{21}^j because $p_j = 2$. Then, A_{11}^j with any of these three makes $d_i d_j$ contractible. Thus, we summarise that $\mathcal{K}(0, 2, 3)$ has the trivial fundamental group.

Case 3 ($p_i = 1$ for some i and $p_j = 3$ or 4 for some j): Let $p_k = 1$ for some k . As the other two edges of the triangle passing through d_k coincide with the edges of other triangles in $\mathcal{K}(0, 2, 3)$ passing through the $\{2, 3\}$ -labeled edge d_j , we can deform thus the triangle passing through d_k from the free edge d_k . On collapsing, we get the space consisting of all the triangles except the one passing through d_k , and this case reduces to **Case 1** or **Case 2**. Therefore, $\mathcal{K}(0, 2, 3)$ has the trivial fundamental group.

Case 4 ($p_i = 2 \forall i$): This case consists of the two triangles passing through each d_i . Let $A_{pq}^i + A_{rs}^i$ be the set of two triangles as in Figure 1 passing through d_i , where $pq, rs \in \{11, 12, 21, 22\}$. We denote the possible combinations of triangles as follows:

Combination	$A_{11}^i + A_{12}^i$	$A_{11}^i + A_{21}^i$	$A_{11}^i + A_{22}^i$	$A_{12}^i + A_{21}^i$	$A_{12}^i + A_{22}^i$	$A_{21}^i + A_{22}^i$
Denoted by	I_i	II_i	III_i	IV_i	V_i	VI_i

Let us first consider that $k = 2$, that is, the number of 23-colored edges are two, say d_i, d_j . We observe the following points:

- 1) The combination of the type $III_i + IV_j$ is not possible; otherwise, $\hat{M}$ will turn out to be non-orientable.
- 2) The combinations $I_i + VI_j$ and $II_i + V_j$ are the same up to the renaming of edges.

Now, as we did in **Case 1**, it is easy to check that all possible combinations are contractible except those shown in Figure 2 up to the renaming of edges. We tabulate all the non-contractible combinations up to the renaming of edges and their corresponding spaces such that they have homeomorphic regular neighborhoods:

$$\begin{array}{ccc} I_i + VI_j & III_i + III_j & III_i + IV_j \\ \mathbb{S}^2 \vee \mathbb{S}^1 & \mathbb{S}^2 \vee \mathbb{S}^1 & \mathbb{RP}^2 \end{array}$$

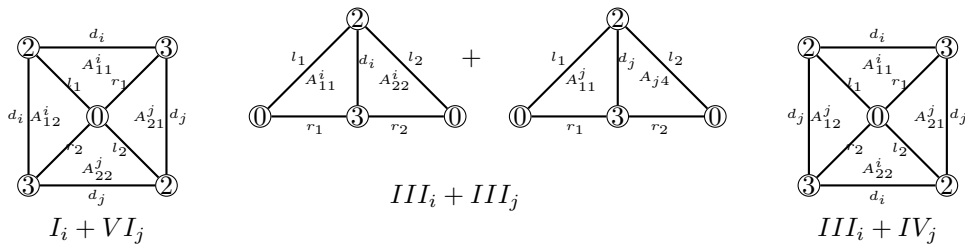


Figure 2: Possible non-contractible combinations up to the renaming of edges for $k = 2$.

Let us consider $k = 3$. We observe that we get one more $\mathbb{S}^2$ in the corresponding space if

- 1) we add I_k or VI_k to $I_i + VI_j$;

2) if we add III_k to $III_i + III_j$;

else the space has the trivial fundamental group. On a similar basis, the above statement is generalized on k as below. Let I^r denote the triangles of type $I_{i_1}, I_{i_2}, \dots, I_{i_r}$ passing through r different edges with points labeled by $\{2, 3\}$.

$$\begin{array}{cc} I^r + VI^s & III^r \\ (\mathbb{S}^2)^{r+s-1} \vee \mathbb{S}^1 & (\mathbb{S}^2)^{r-1} \vee \mathbb{S}^1, \end{array}$$

where $(\mathbb{S}^2)^\alpha$ denoted the wedge product of α copies of $\mathbb{S}^2$.

Besides these combinations for the k number of $\{2, 3\}$ -labeled edges, we get the simply-connected space.

Case 5 ($p_i \in \{1, 2\} \forall i$ but $2 \neq p_i \neq 1 \forall i$): Let $p_j = 1$. From the same observation as in **Case 3**, either we have the space where we can deform the triangle through the free edge d_j and the result is unaffected, or we have the combinations as in Figure 3. The space to which this combination corresponds with respect to the regular neighborhood is $S^1 \vee (S^2)^{i_r-2}$.

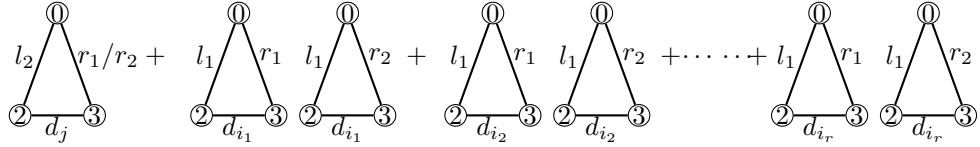


Figure 3

Case 6 ($p_i = 1 \forall i$): Keeping the observation of **Case 3** in consideration, all the combinations in this case can deform as in Figure 4 and its corresponding space is the wedge product of two 1-spheres i.e., $\mathbb{S}^1 \vee \mathbb{S}^1$.

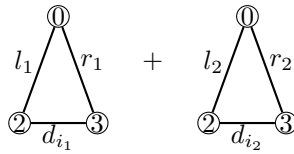


Figure 4

With all these cases, the lemma follows. $\square$

THEOREM 4.4. *Let M be a compact orientable PL 4-manifold with connected non-spherical boundary admitting a weak semi-simple crystallization, such that the rank of its fundamental group is 1. If the rank of the fundamental group of $\hat{M}$, m' , is 1, then M admits a handle decomposition that takes one of the following forms:*

1) one 0-handle, one 1-handle, k 2-handles and one 3-handle, where $k = 2 + \beta_2(M) - \beta_1(M) - \beta_1(\hat{M})$, or

2) one 0-handle, two 1-handles, k 2-handles and one 3-handle, where $k = 3 + \beta_2(M) - \beta_1(M) - \beta_1(\hat{M})$.

Further, if $m' = 0$, then M admits a handle decomposition which consists of one 0-handle, one 1-handle and $\beta_2(M)$ 2-handles.

Proof. Let M admit a weak semi-simple crystallization Γ with respect to the permutation $\epsilon = (0, 1, 2, 3, 4)$. We write $\hat{M} = N(1, 4) \cup N(0, 2, 3) = V \cup V'$, where $V = \#_{m'}(\mathbb{S}^1 \times \mathbb{B}^3) \# C(\partial M)$ by Lemma 3.3. Since $m = 1$, $m' = 0, 1$ and $\pi_1(M) = \mathbb{Z}_k$ (or $\mathbb{Z}$). Lemma 4.3 implies that V' , a regular neighborhood of $\mathcal{K} = \mathcal{K}(0, 2, 3)$, is PL homeomorphic to a regular neighborhood of a CW complex T , which is either $\mathbb{S}^1 \vee \mathbb{S}^1 \vee (\mathbb{S}^2)^{\beta_2(\mathcal{K})}$, $\mathbb{S}^1 \vee (\mathbb{S}^2)^{\beta_2(\mathcal{K})}$ or a simply-connected space, where $(\mathbb{S}^2)^\gamma$ denote the wedge product of γ copies of the 2-sphere $\mathbb{S}^2$.

If $m' = 1$, then due to Lemma 3.6, T cannot be simply-connected, and we are left with two possible choices for T .

Case 1 ($T = \mathbb{S}^1 \vee (\mathbb{S}^2)^{\beta_2(\mathcal{K})}$): In this case, CW complex T consists of one 0-cell, one 1-cell and $\beta_2(\mathcal{K})$ number of 2-cells. For $0 \leq j \leq 2$, small regular neighbourhood of j -cell of T is a j -handle $H^{(j)}$. Since V' deformation retracts onto $\mathcal{K}$, $\beta_2(V') = \beta_2(\mathcal{K})$, we can represent V' as $V' = H^{(0)} \cup H^{(1)} \cup \left(H_1^{(2)} \cup \dots \cup H_{\beta_2(V')}^{(2)} \right)$.

Now, observe that boundary identification between V and V' is an attachment of one 3-handle. Also, $\beta_2(V')$ equals $2 + \beta_2(M) - \beta_1(M) - \beta_1(\hat{M})$ from (1). Therefore, $M = H^{(0)} \cup H^{(1)} \cup \left(H_1^{(2)} \cup \dots \cup H_{\beta_2(V')}^{(2)} \right) \cup H^{(3)}$, where $\beta_2(V')$ is equal to $\beta_2(M)$, $\beta_2(M) + 1$ or $\beta_2(M) + 2$.

Case 2 ($T = \mathbb{S}^1 \vee \mathbb{S}^1 \vee (\mathbb{S}^2)^{\beta_2(\mathcal{K})}$): In this case, the CW complex T consists of one 0-cell, two 1-cells and $\beta_2(\mathcal{K})$ number of 2-cells. Proceeding in the same manner as **Case 1**, we can represent V' as $V' = H^{(0)} \cup (H_1^{(1)} \cup H_2^{(1)}) \cup \left(H_1^{(2)} \cup \dots \cup H_{\beta_2(V')}^{(2)} \right)$.

Now, observe that boundary identification between V and V' is an attachment of one 3-handle. Thus, $M = H^{(0)} \cup (H_1^{(1)} \cup H_2^{(1)}) \cup \left(H_1^{(2)} \cup \dots \cup H_{\beta_2(V')}^{(2)} \right) \cup H^{(3)}$, where $\beta_2(V')$ is equal to $\beta_2(M) + 1$, $\beta_2(M) + 2$ or $\beta_2(M) + 3$ from (1).

Now, if $m' = 0$ then we get $M = V'$ using Lemma 3.3. Since $m = 1$, we get $T = \mathbb{S}^1 \vee (\mathbb{S}^2)^{\beta_2(\mathcal{K})}$ and thus M has a handle decomposition as $M = H^{(0)} \cup H^{(1)} \cup \left(H_1^{(2)} \cup \dots \cup H_{\beta_2(M)}^{(2)} \right)$. This proves the theorem. $\square$

When M is a compact orientable PL 4-manifold with connected spherical boundary, $\hat{M}$ is a closed orientable manifold, and considering (Γ, γ) to be a weak semi-simple crystallization of $\hat{M}$, Lemmas 3.3, 3.5, 3.6 and 4.3 hold true for this case as well.

COROLLARY 4.5. *Let M be a compact orientable PL 4-manifold with empty or connected spherical boundary having the fundamental group of rank 1. Let (Γ, γ) be a weak semi-simple crystallization representing $\hat{M}$ and V, V' be as in Definition 3.2. Then $\pi_1(M) = \mathbb{Z}$ and M admits a handle decomposition as follows:*

$$M = H^{(0)} \cup H^{(1)} \cup \left(H_1^{(2)} \cup \dots \cup H_{\beta_2(M)}^{(2)} \right) \cup H^{(3)} \text{ (if } M \text{ is closed)} \cup H^{(4)}.$$

Proof. By the Seifert-Van Kampen Theorem, we deduce that $\pi_1(\hat{M}) = \pi_1(M)$ if M has a spherical connected boundary. Also, from the proof of Lemma 3.6, we have $\pi_1(\hat{M}) = \pi_1(V')$ if M has empty or connected spherical boundary. Since $m = 1$, $T = \mathbb{S}^1 \vee (\mathbb{S}^2)^{\beta_2(K)}$ by Lemma 4.3. Thus, M with empty or connected spherical boundary cannot have a finite cyclic group as its fundamental group. Due to (1), we get $\beta_2(V') = \beta_2(M)$. Thus,

$$M = H^{(0)} \cup H^{(1)} \cup \left(H_1^{(2)} \cup \dots \cup H_{\beta_2(M)}^{(2)} \right) \cup H^{(3)} \text{ (if } M \text{ is closed)} \cup H^{(4)}.$$

This proves the result. $\square$

REMARK 4.6. In [3], the authors proved that if M is a closed orientable PL 4-manifold having a semi-simple crystallization with the fundamental group isomorphic to $\mathbb{Z}$, then it admits a handle decomposition as one of the following types:

- 1) one 0-handle, two 1-handles, $1 + \beta_2(M)$ 2-handles, one 3-handle and one 4-handle,
 - 2) one 0-handle, one 1-handle, $\beta_2(M)$ 2-handles, one 3-handle and one 4-handle,
- where $\beta_2(M)$ denotes the second Betti number of the manifold M with $\mathbb{Z}$ coefficients. Clearly, Corollary 4.5 refines this result by providing the precise handle decomposition of M .

ACKNOWLEDGEMENT. The authors would like to thank the anonymous referees for many useful comments and suggestions. The second author is supported by the Mathematical Research Impact Centric Support (MATRICS) Research Grant (MTR/2022/000036) by SERB (India).

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(received 15.06.2024; in revised form 29.01.2025; available online 24.09.2025)

Department of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India
E-mail: maz228084@iitd.ac.in
 ORCID iD: <https://orcid.org/0009-0005-4327-054X>

Department of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India
E-mail: biplab@iitd.ac.in
 ORCID iD: <https://orcid.org/0000-0002-4978-7022>

Department of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India
E-mail: binjolamanisha@gmail.com
 ORCID iD: <https://orcid.org/0000-0001-9703-8424>