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A CHARACTERIZATION OF ULAM HYPERSTABILITY

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Abstract. The main result of this work is a simple characterization of the hyperstability of functional equations in a very general framework: for functions with the codomain endowed with a generalized homogeneous premetric. We also give some particularizations on Abelian groups equipped with generalized homogeneous norms, obtaining, among others, improvements of similar known results for Cauchy-type, Jensen-type equations, but also for equations having compound functions as solutions.

1. Introduction

In 2014, Brzdęk [2] observes an interesting property of Cauchy differences: if X and Y are normed linear spaces, r, s are real numbers such that $r+s>0$, $D:=(X\setminus\{0\})\times(X\setminus\{0\})$ and $f:X\to Y$, then

$$\sup_{(x,y)\in D}\|x\|^r\|y\|^s\|f(x+y)-f(x)-f(y)\|\in\{0,\infty\}.$$

In other words, an arbitrary function $f:X\to Y$ is either *Cauchy on* D, i.e.

$$f(x+y)=f(x)+f(y)\qquad\text{for all}\quad(x,y)\in D\tag{1}$$

or

$$\sup_{(x,y)\in D}\|x\|^r\|y\|^s\|f(x+y)-f(x)-f(y)\|=\infty.$$

This property is a consequence of the φ-hyperstability of equation (1), where $\varphi:D\to(0,\infty)$ is a control function defined by $\varphi(x,y):=\|x\|^{-r}\|y\|^{-s}$.

The question arises whether this phenomenon also occurs for other functional equations, other control functions and what is the connection with the hyperstability.

Hyperstability of functional equations is a special case of their stability – a problem posed by S. M. Ulam [9] in metric spaces. Here we will widen the framework: instead of metric spaces we will use some special premetric spaces. We say that (Y,d) is

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a *generalized premetric space* if the function $d : Y \times Y \to \mathbb{R}_+ := [0, \infty]$ – called a *premetric* – satisfies only the condition: for all $x, y \in Y$

$$d(x, y) = 0 \quad \text{if and only if} \quad x = y.$$

In the following lines, (Y, d) is a generalized premetric space, X and D are nonempty sets, G and $H : Y^X \to Y^D$ are two functions and

$$Gf(z) = Hf(z) \quad \text{for all} \quad z \in D \tag{2}$$

is a functional equation, where $f : X \to Y$ is the unknown. If $\varphi : D \to (0, \infty)$ is a function, then equation (2) is *φ-hyperstable* if and only if all the functions $f : X \to Y$ for which

$$d(Gf(z), Hf(z)) \leq \varphi(z) \quad \text{for all} \quad z \in D \tag{3}$$

are solutions of equation (2). In this context we say that φ is a *control function* for equation (2).

By $S_{(i)}$ we denote the set of solutions of equation (inequality) (i); for instance, equation (2) is φ-hyperstable if and only if $S_{(2)} = S_{(3)}$.

We will give very general conditions under which the φ-hyperstability of equation (2) is equivalent with the alternative

$$\text{either} \quad f \in S_{(2)} \quad \text{or} \quad \sup_{z \in D} \frac{1}{\varphi(z)} d(Gf(z), Hf(z)) = \infty \quad \text{for all} \quad f \in Y^X, \tag{4}$$

in terms of generalized homogeneity of the functions d, G and H. We apply these results within normed groups and, in particular, for Cauchy-type equations and Jensen-type equations.

Hyperstability approached in the broader sense of Ulam stability is analyzed in [3]. Information on the status of Ulam stability research can be found in [5]. Hyperstability conditions for Cauchy-type equations on restricted domains (in the meaning of equation (1), when $D = X^2$) are given in [2] and, in a more general framework (when X is part of an abelian semigroup, $D \subseteq X^2$ and the control function φ is not necessarily of Aoki-Rassias-type), in [7]. The same approach, but for Jensen-type equations, is done in [8]. Sufficient conditions under which alternatives of the form (4) hold for Cauchy equations were first given in [2] for some Aoki-Rassias-type control functions, and in [7] in an extended framework; in [8], a similar result for Jensen-type equation was proved. We can consult [6] for the behavior of generalized homogeneous functions. Details on normed groups can be found in [1].

2. Results

In the following $\mathbb{N}^*$ is the set of positive integers, $\tau \circ g$ denotes the composition of the functions $U \overset{g}{\to} Y \overset{\tau}{\to} Y$, $\tau^0 := \mathrm{id}_Y$ and τ^i is the i-th iteration of τ for $i \in \mathbb{N}^*$.

We show first that alternative (4) is a sufficient condition for the hyperstability of equation (2).

Proposition 2.1. *If $\varphi : D \to (0, \infty)$ is a function that satisfies alternative* (4)*, then equation* (2) *is φ-hyperstable.*

Proof. Suppose there is a function $h \in S_{(3)} \backslash S_{(2)}$. Then $\frac{1}{\varphi(z)} d\left(Gh\left(z\right), Hh\left(z\right)\right) \leq 1$ for $z \in D$ and there exists $z_0 \in D$ for which $\frac{1}{\varphi(z_0)} d\left(Gh\left(z_0\right), Hh\left(z_0\right)\right) > 0$. Consequently $\sup_{z \in D} \frac{1}{\varphi(z)} d\left(Gh\left(z\right), Hh\left(z\right)\right) \in (0, 1]$, which is a contradiction. Therefore equation (2) is φ-hyperstable. □

2.1 Hyperstability on (τ, γ)-spaces

As above, X and D are nonempty sets, (Y, d) is a generalized premetric space and $G, H : Y^X \to Y^D$ are two functions.

We introduce the concept of generalized homogeneity.

Definition 2.2. Let $n \in \mathbb{N}^*$ and $\tau : Y \to Y$ be a function.
1. The function $E : Y^X \to Y^D$ is (τ, τ^n)-*homogeneous* if $E\left(\tau \circ f\right) = \tau^n \circ Ef$ for all $f \in Y^X$.

2. The couple (Y, d) is a (τ, γ)-*space* if d is a generalized premetric on Y, $\tau : Y \to Y$ is surjective function, $\gamma \in (0, \infty) \setminus \{1\}$ and the generalized premetric d is (τ, γ)-*homogeneous*, i.e., $d\left(\tau\left(x\right), \tau\left(y\right)\right) = \gamma d\left(x, y\right)$ for all $x, y \in Y$.

We can consult [6] for the behavior of homogeneous functions in a generalized sense.

Remark 2.3. The study of Ulam stability of linear operators between real or complex vector spaces equipped with gauges was initiated in [4]. Recall that, if Y is a vector space over the field $K \in \{\mathbb{R}, \mathbb{C}\}$, then $\rho : Y \to [0, \infty]$ is a *gauge* if $\rho\left(x\right) = 0$ if and only if $x = 0$, and $\rho\left(\lambda x\right) = |\lambda| \rho\left(x\right)$ for all $x \in Y$ and $\lambda \in K \setminus \{0\}$. In this context, defining $d : Y^2 \to [0, \infty]$, $d\left(x, y\right) := \rho\left(x - y\right)$ and $\tau_\lambda : Y \to Y, \tau_\lambda\left(x\right) := \lambda x$, we remark that (Y, d) is a $(\tau_\lambda, |\lambda|)$-space for all $\lambda \in K \setminus \{0\}$ with $|\lambda| \neq 1$.

This is the framework in which we will prove a complement of Proposition 2.1.

Theorem 2.4. *We assume that X, D are nonempty sets, (Y, d) is a (τ, γ)-space and that $G, H : Y^X \to Y^D$ are (τ, τ^n)-homogeneous functions. If $\varphi : D \to (0, \infty)$ is a function, then dichotomy* (4) *holds if and only if equation* (2) *is φ-hyperstable.*

Proof. 1. From Proposition 2.1 it follows that alternative (4) implies the φ-hyperstability of equation (2).

2. For proving the converse, we first note that, if $x, y \in Y$ such that $\tau\left(x\right) = \tau\left(y\right)$, we have $\gamma d\left(x, y\right) = d\left(\tau\left(x\right), \tau\left(y\right)\right) = 0$ and $x = y$. Hence $\tau : Y \to Y$ is a bijective correspondence.

Since d is (τ, γ)-homogeneous, by induction on $j \in \mathbb{N}^*$ we get

$$d\left(\tau^j\left(x\right), \tau^j\left(y\right)\right) = \gamma^j d\left(x, y\right) \text{ for all } x, y \in Y, j \in \mathbb{N}^*.$$

Replacing x, y by $\tau^{-j}(x), \tau^{-j}(y)$, where $\tau^{-j} := \left(\tau^j\right)^{-1}$, we get $d\left(\tau^{-j}(x), \tau^{-j}(y)\right) = \gamma^{-j} d(x, y)$. Therefore,

$$d\left(\tau^j(x), \tau^j(y)\right) = \gamma^j d(x, y) \quad \text{for all} \quad x, y \in Y, j \in \mathbb{Z}. \tag{5}$$

Similarly, since G and H are (τ, τ^n)-homogeneous,

$$G\left(\tau^j \circ f\right) = \tau^{nj} \circ Gf, H\left(\tau^j \circ f\right) = \tau^{nj} \circ Hf \quad \text{for all} \quad f \in Y^X \text{ and } j \in \mathbb{Z}. \tag{6}$$

Suppose that equation (2) is φ-hyperstable and $f \in Y^X$ such that

$$c := \sup_{z \in D} \frac{1}{\varphi(z)} d(Gf(z), Hf(z)) < \infty.$$

Then,

$$d(Gf(z), Hf(z)) \leq c\varphi(z) \quad \text{for all} \quad z \in D. \tag{7}$$

It is enough to prove that $f \in S_{(2)}$, (i.e. $Gf = Hf$) or, equivalently, $c = 0$.

Let $j \in \mathbb{Z} \setminus \{0\}$ be such that

$$c\gamma^{nj} \leq 1. \tag{8}$$

Using consecutively (6), (5), (7) and (8) we get

$$\begin{aligned} d\left(G\left(\tau^j \circ f\right)(z), H\left(\tau^j \circ f\right)(z)\right) &= d\left(\left(\tau^{nj} \circ Gf\right)(z), \left(\tau^{nj} \circ Hf\right)(z)\right) \\ &= \gamma^{nj} d(Gf(z), Hf(z)) \leq c\gamma^{nj}\varphi(z) \leq \varphi(z) \end{aligned}$$

for all $z \in D$. But equation (2) is φ-hyperstable, hence $\tau^j \circ f \in S_{(2)}$ i.e.,

$$G\left(\tau^j \circ f\right) = H\left(\tau^j \circ f\right), \quad \text{or, by (6),} \quad \tau^{nj} \circ Gf = \tau^{nj} \circ Hf.$$

Since τ^{nj} is injective we get $Gf = Hf$ and $c = 0$. □

When the cardinality of Y – denoted $|Y|$ – is 1, we get $S_{(2)} = S_{(3)} = Y^X$ and equation (2) is φ-hyperstable for every $\varphi : D \to (0, \infty)$. We can analyze the hyperstability of equation (2) when Y (or D) is finite, d is finite, but (2) is not trivial ($S_{(2)} \neq Y^X$).

Corollary 2.5. *Assume that (Y, d), G and H satisfy the assumptions in Theorem 2.4, d is a finite function, $|Y| < \infty$ or $|D| < \infty$, and $S_{(2)} \neq Y^X$. Then there are no functions $\varphi : D \to (0, \infty)$ for which equation (2) is φ-hyperstable.*

Proof. Suppose that $\varphi : D \to (0, \infty)$ is a control function for which equation (2) is φ-hyperstable. Let $f \in Y^X \setminus S_{(2)}$. Then there is $z_0 \in D$ such that $d(Gf(z_0), Hf(z_0)) > 0$, hence

$$\sup_{z \in D} \frac{1}{\varphi(z)} d(Gf(z), Hf(z)) \in (0, \infty)$$

which contradicts the conclusions of the previous theorem. □

2.2 Hyperstability on (m, γ)-groups

A special type of (τ, γ)-space is an (m, γ)-group.

Definition 2.6. We say that $(Y, +, \|\|)$ is an *(m, γ)-group* if $m \in \mathbb{N}^*$, $(Y, +)$ is a uniquely m-divisible normed Abelian group, $\gamma \in (0, \infty) \setminus \{1\}$ and $\|\| : Y \to \mathbb{R}_+$ is a *(m, γ)-homogeneous norm* on Y, i.e., $\|my\| = \gamma \|y\|$ for all $y \in Y$.

We recall that:

- the commutative group $(Y,+)$ is uniquely m-divisible if and only if for all $y \in Y$, the equation $mx = y$ has a unique solution $x \in Y$; we denote $x := m^{-1}y$ or $x := \frac{y}{m}$;
- $\|\| : Y \to \mathbb{R}_+$ is a norm on the commutative group $(Y,+)$ if $\|-y\| = \|y\|$, $\|x+y\| \le \|x\| + \|y\|$ for all $x, y \in Y$ and $\|x\| = 0$ iff $x = 0$. Details on normed groups can be found in [1].

Nontrivial examples of (m,γ)-groups (e.g. normed linear spaces, valuated rings or F spaces) can be found in [6].

Proposition 2.7. *If $(Y,+,\|\|)$ is an (m,γ)-group, then (Y,d) is a (τ,γ)-space, where $d(x,y) := \|x-y\|$ and $\tau(x) := mx$.*

Proof. Of course d is a translation invariant metric on Y and a (τ,γ)-homogeneous function. □

Under the terms of Proposition 2.7, we note that

– $\ker \tau = \{0\}$ (since, for $x \in Y$ with $\tau(x) = 0$ we have $0 = \|mx\| = \gamma \|x\|$ and $x = 0$; hence τ is a monomorphism);

– since $(Y,+)$ is uniquely m-divisible we conclude that the function
$$\sigma : Y \to Y, \quad \sigma(y) := m^{-1}y$$
is an onto and a right inverse of the monomorphism τ (since $\tau\circ\sigma(y) = m\left(m^{-1}y\right) = y$ for all $y \in Y$); therefore τ is an automorphism of Y and $\sigma = \tau^{-1}$.

In this framework, as usual we define
$$m^k g := \tau^k \circ g \quad \text{for} \quad k \in \mathbb{Z} \quad \text{and} \quad g \in Y^X \cup Y^D.$$
In these terms we can characterize the hyperstability of the equation
$$Ef(z) = 0 \text{ for all } z \in D, \tag{9}$$
where $f : X \to Y$ is the unknown, $(Y,+,\|\|)$ is an (m,γ)-group and $E : Y^X \to Y^D$ is a fixed function; if E is (τ,τ^n)-homogeneous we also say that E is (m,m^n)-*homogeneous*, i.e., the function E satisfies $E(mf) = m^n Ef$ for all $f \in Y^X$.

From Proposition 2.7 and Theorem 2.4 we immediately obtain the following characterization of hyperstability.

Corollary 2.8. *Let X, D be nonempty sets, $(Y,+,\|\|)$ be an (m,γ)-group, $E : Y^X \to Y^D$ be an (m,m^n)-homogeneous function, where $n \in \mathbb{Z}\setminus\{0\}$ and $\varphi : D \to (0,\infty)$ be a function. Then equation* (9) *is φ-hyperstable if and only if*
$$\textit{either} \quad Ef = 0 \quad \textit{or} \quad \sup_{z\in D} \frac{1}{\varphi(z)} \|Ef(z)\| = \infty,$$
for all $f : X \to Y$.

2.3 Applications to Cauchy-type and Jensen-type equations

Let $(S,+)$ be an Abelian semigroup.

2.3.1. If we assume that the sets

$$X \subseteq S, D \subseteq \left\{(x,y) \in X^2 \mid x+y \in X\right\} \tag{10}$$

are nonempty, $(Y,+,\|\|)$ is an (m,γ)-group and

$$E : Y^X \to Y^D, \quad Ef\,(x,y) := f\,(x+y) - f\,(x) - f\,(y)\,,$$

then we can apply the above results to the Cauchy-type equation $Ea = 0$, i.e.,

$$a\,(x+y) = a\,(x) + a\,(y) \qquad \text{for all} \qquad (x,y) \in D, \tag{11}$$

where $a : X \to Y$ is the unknown; in this case we say that a is *Cauchy* on D. We note that E is (m,m)-homogeneous for $m \in \mathbb{N}^*$. Applying Corollary 2.8 we obtain the following extension of [7, Theorem 2.3].

Corollary 2.9. *We assume that $(Y,+,\|\|)$ is an (m,γ)-group, the sets $X \subseteq S$ and $D \subseteq X^2$ satisfy* (10) *and $\varphi : D \to (0,\infty)$ is a function. Then equation* (11) *is φ-hyperstable if and only if an arbitrary function $f \in Y^X$ is Cauchy on D or*

$$\sup_{(x,y)\in D} \frac{1}{\varphi\,(x,y)} \left\| f\,(x+y) - f\,(x) - f\,(y) \right\| = \infty.$$

2.3.2. Now suppose that the homomorphism

$$S \to S, x \mapsto 2x \text{ is an automorphism;} \tag{12}$$

we denote by $y := \frac{x}{2}$ the unique solution of the equation $2y = x$ for $x \in S$. Also we assume that $(Y,+,\|\|)$ is a $(2,\gamma)$-group, the sets

$$X \subseteq S, \quad D \subseteq \left\{(x,y) \in X^2 | \frac{x+y}{2} \in X\right\} \tag{13}$$

are nonempty and the equation

$$\rho\left(\frac{x+y}{2}\right) = \frac{\rho\,(x) + \rho\,(y)}{2} \text{ for all } (x,y) \in D, \tag{14}$$

have the function $\rho : X \to Y$ as the unknown. Since for the function

$$E : Y^X \to Y^D, \quad E\rho\,(x,y) := \rho\left(\frac{x+y}{2}\right) - \frac{\rho\,(x) + \rho\,(y)}{2},$$

we have $E\,(m\rho) = mE\rho$ for $m \in \mathbb{N}^*$, we can apply Corollary 2.8 and we obtain a generalization of [8, Proposition 2.3].

Corollary 2.10. *We assume that $(S,+)$ is an Abelian semigroup that satisfies* (12)*, $X \subseteq S$ and $D \subseteq X^2$ satisfy* (13)*, $(Y,+,\|\|)$* is a *$(2,\gamma)$-group and $\varphi : D \to (0,\infty)$ is a function. Then equation* (14) *is φ-hyperstable if and only if for all $\rho \in Y^X$*

$$\textit{either} \quad \rho \in S_{(14)} \quad \textit{or} \quad \sup_{(x,y)\in D} \frac{1}{\varphi\,(x,y)} \left\| 2\rho\left(\frac{x+y}{2}\right) - \rho\,(x) - \rho\,(y) \right\| = \infty.$$

Finally we give an concrete example of using the above results. Let $(Y,+,\|\cdot\|)$ be a linear normed space,

$$g:\left(0,\frac{\pi}{2}\right)\to X:=(-\infty,0)\,,\quad g\,(u):=\ln\sin u$$

and $D:=X^2$. In [8] it was proven that the equation

$$f\left(\arcsin\sqrt{\sin u\cdot\sin v}\right)=\frac{f\,(u)+f\,(v)}{2}\text{ for all }u,v\in(0,\frac{\pi}{2}),\tag{15}$$

where $f:(0,\frac{\pi}{2})\to Y$ is the unknown, has the solutions $S_{(15)}=\left\{\rho\circ g\mid\rho\in S_{(14)}\right\}$ (see [8, Proposition 3.11]) and that, for $p,q>0$ and $\varphi:D\to(0,\infty)$, $\varphi\,(u,v):=u^pv^q$, equation (15) is φ-hyperstable (see [8, Corollary 3.13]). Applying Corollary 2.8 we obtain the following dichotomy.

Corollary 2.11. *Suppose that* $(Y,+,\|\cdot\|)$ *is a linear normed space and* $f:(0,\frac{\pi}{2})\to Y$. *Then either*

$$\sup_{u,v\in(0,\frac{\pi}{2})}u^rv^s\left\|2f\left(\arcsin\sqrt{\sin u\cdot\sin v}\right)-f\,(u)-f\,(v)\right\|=\infty\quad\textit{for all } r,s<0$$

or $$\rho\left(\frac{x+y}{2}\right)=\frac{\rho\,(x)+\rho\,(y)}{2}\quad\textit{for all } x,y\in(-\infty,0)\,,$$

where $\rho:(-\infty,0)\to Y,\ \rho\,(x):=f\,(\arcsin e^x)\,.$

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