

UNIQUENESS OF L-FUNCTIONS IN THE EXTENDED SELBERG CLASS CONCERNING ONE SHARED SET

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Abstract. In this paper, we investigate the value distribution of L-functions in the extended Selberg class. We show how two L-functions L_1 and L_2 satisfying certain condition are uniquely determined by the zero sharing between $P(L_1)$ and $P(L_2)$ for some polynomial P , or by a set sharing between L_1 and L_2 . Considering the most general form of a polynomial in the set sharing we obtain some results which completely generalize and extend some recent results of [X. M. Li, X. R. Du, H. X. Yi, *Dirichlet series satisfying a Riemann type functional equation and sharing one set*, Complex Var. Elliptic Equ., **68**(10) (2023), 1653–1677], which were actually proved as an answer of an analogous question of Gross [F. Gross, *Factorization of meromorphic functions and some open problems*, Complex Analysis (Proc. Conf. Univ. Kentucky, Lexington, Ky., 1976), pp. 51–69, Lect. Notes Math., Vol 599, Springer, Berlin, 1977] for L-functions. We also obtain uniqueness relation between two nonconstant L-functions (belonging to the extended Selberg class) by proving other two results, one concerning a prior result due to Yuan-Li-Yi [Q. Q. Yuan, X. M. Li, H. X. Yi, *Value distribution of L-functions and uniqueness questions of F. Gross*, Lithuanian Math. J., **58** (2018), 249–262] and another related to a result of Hao-Chen [W. J. Hao, J. F. Chen, *Uniqueness theorems for L-functions in the extended Selberg class*, Open Math., **16** (2018), 1291–1299].

1. Introduction

The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is a well known and valuable object in mathematics due to the celebrated Riemann hypothesis conjectured by B. Riemann [14]. The function has various structural properties including the existence of a single pole in $\mathbb{C}$, analytic continuation as meromorphic function and Euler product over the primes. Here and throughout the paper, $\mathbb{C}$ denotes the usual complex plane and s denotes a complex variable of the form $s = \sigma + it$, where σ and t are real numbers, and i is the imaginary unit. L-functions which are defined by taking the

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Riemann zeta function as a prototype but in a wider setting are naturally important, especially in analytic number theory.

In 1989, Selberg introduced a class of L-functions by considering $\zeta(s)$ as a model. Recently, value distribution of L-functions belonging to this class has gained increasing interest from the researchers (see [8, 9, 13, 16, 20]). Before going into the detailed discussion, we must introduce the definition of this class due to Selberg, often referred to as the Selberg Class $\mathcal{S}$. The Selberg class $\mathcal{S}$ of L-functions is defined as the set of all those Dirichlet series $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ of a complex variable s , which is absolutely convergent for $\operatorname{Re}(s) > 1$, and satisfy the following axioms (see [15]):

- (i) *Ramanujan hypothesis*: $a(n) \ll n^\varepsilon$ for each $\varepsilon > 0$;
- (ii) *Analytic continuation*: There is a nonnegative integer m such that $(s-1)^m L(s)$ is an entire function of finite order;
- (iii) *Functional equation*: L satisfies a functional equation of the type

$$\Lambda_L(s) = \omega \overline{\Lambda_L(1 - \bar{s})},$$

where

$$\Lambda_L(s) = L(s) Q^s \prod_{j=1}^K \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q , λ_j , a positive integer K and complex numbers ν_j , ω with $\operatorname{Re}(\nu_j) \geq 0$ and $|\omega| = 1$;

- (iv) *Euler product hypothesis*: L can be written over primes in the following manner:

$$L(s) = \prod_p \exp \left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}} \right)$$

with suitable coefficients $b(p^k)$ such that $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$, where the product is taken over all prime numbers p .

An L-function $L(s)$ is said to be in the extended Selberg class $\mathcal{S}^\sharp$ if it satisfies axioms (i)-(iii) and is not vanishing identically (see [7]). It is worth mentioning that there do exist some $L(s)$ which do not have an Euler product. Moreover, as the class $\mathcal{S}^\sharp$ contains the class $\mathcal{S}$, it will be more reasonable to consider the functions in $\mathcal{S}^\sharp$. In this paper, by an L-function we will always mean an L-function in the extended Selberg class $\mathcal{S}^\sharp$, with the normalizing condition $a(1) = 1$. Naturally, the results found for functions in $\mathcal{S}^\sharp$ are also valid for the functions in $\mathcal{S}$. Let us now mention that if d_L denotes the degree of an L-function $L(s)$, then $d_L := 2 \sum_{j=1}^K \lambda_j$, where λ_j and K are respectively the positive real number and the positive integer defined as in axiom (iii). Unless stated otherwise, the degree of $L(s)$ in the following will always be taken as nonzero (positive). The value distribution of $L(s)$ concerns the distribution of zeros of $L(s) - c$ for any $c \in \mathbb{C} \cup \{\infty\}$. In this paper, we will mainly focus on the problem of determining L-functions by the preimage of subsets of $\mathbb{C}$. For the sake of convenience, the notion of sharing of sets (or values) is now illustrated below, which is actually same as that of the meromorphic functions.

By the term “meromorphic” we understand a function that is meromorphic in $\mathbb{C}$. Let $\mathcal{M}(\mathbb{C})$ denote the class of functions meromorphic in $\mathbb{C}$. Suppose two nonconstant

functions f and g belong to $\mathcal{M}(\mathbb{C})$. For a subset S of $\mathbb{C} \cup \{\infty\}$, define

$$E(S, f) := \cup_{v \in S} \{s \in \mathbb{C} : f(s) = v\},$$

where a zero of $f - v$ with multiplicity p is counted p times in $E(S, f)$, i.e., $E(S, f)$ is a multiset. When we ignore the multiplicities, we replace the symbol $E(S, f)$ by $\overline{E}(S, f)$. If $E(S, f) = E(S, g)$ for some $f, g \in \mathcal{M}(\mathbb{C})$, then f and g are said to share S CM (Counting Multiplicity). In case $\overline{E}(S, f) = \overline{E}(S, g)$, then f, g share S IM (Ignoring Multiplicity). If the set S contains exactly one element, say v , then we write $E(S, f)$ (resp. $\overline{E}(S, f)$) simply as $E(v, f)$ (resp. $\overline{E}(v, f)$), and the definitions of value sharing CM or IM follows in the similar way.

In terms of shared values, Nevanlinna's five-value theorem states that two non-constant meromorphic functions f and g are identically equal if $\overline{E}(v_j, f) = \overline{E}(v_j, g)$ for five distinct values v_1, v_2, v_3, v_4 and v_5 in $\mathbb{C} \cup \{\infty\}$. It was also shown by Nevanlinna (see [3, 19]) that the number "five" is the best possible in the above theorem. In 2007, without putting any restriction on the degree of L-function, Steuding [16, p. 152] first proved a uniqueness result with one shared value. From this result and [5] it is known that two L-functions L_1 and L_2 satisfying axioms (i)-(iii) with $a(1) = 1$ are identical when they share a complex value c ($\neq \infty, 1$) CM. Since L-functions can be analytically continued as meromorphic functions, it becomes quite interesting to investigate to which extent an L-function $L(s)$ can share values or sets with an arbitrary meromorphic function $f(s)$. Being motivated by the well-known Gross' question for meromorphic (or entire) functions (see [2]) Yuan-Li-Yi [20] investigated on the possible relationship between an L-function L and a function f in $\mathcal{M}(\mathbb{C})$ for the situation when L and f share one or two finite sets. In [20], without requiring positive degree of $L(s)$, the authors proved the following result regarding one set sharing between L and f .

THEOREM 1.1 ([20]). *Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles, and let L be a nonconstant L-function. Suppose that $S = \{w : w^n + aw^m + b = 0\}$, where n and m are relatively prime positive integers with $n > 2m + 4$, and a, b are nonzero constants. If f and L share S CM, then $L = f$.*

Several research articles have been devoted in the direction of uniqueness of L-functions under the sharing of one or more sets of values with some meromorphic function. Among these, we take into account only those in this paper which involve preferably one shared set. In the same direction, Hao-Chen [4] obtained the following result.

THEOREM 1.2 ([4]). *Let f be an entire function with $\lim_{Re(s) \rightarrow \infty} f(s) = k$ ($k \neq \infty$), and let $R(w) = 0$ be an algebraic equation with $n \geq 2$ distinct roots, where $R(1), R(k), R(b) \neq 0$. Suppose that $f(s_0) = L(s_0) = b$ for some $s_0 \in \mathbb{C}$. If for a nonconstant L-function L , $E(S, L) = E(S, f)$, where $S = \{w : R(w) = 0\}$, then $R(L) = R(f)$.*

In [1], similar polynomials as of Theorem 1.1 were considered in case of set sharing. In 2011, Li [9] obtained that if two L-functions L_1 and L_2 in $\mathcal{S}^\#$ satisfy the same functional equation, and if $\overline{E}(c_j, L_1) = \overline{E}(c_j, L_2)$ for two distinct finite complex values

c_1 and c_2 , then $L_1 = L_2$. This clearly shows that the uniqueness relation $L_1 = L_2$ holds for a relaxed nature of sharing as compared to Steuding's theorem (see [9, 16]). In connection with the above result of Li [9] the following question was raised by Li-Du-Yi [12] for L-functions.

QUESTION 1.3. Is there a finite set S such that any two nonconstant L-functions L_1 and L_2 are identical if $E(S, L_1) = E(S, L_2)$?

In the same paper, the authors [12] got affirmative answers to the above question by means of the following two theorems.

THEOREM 1.4 ([12]). *Let L_1 and L_2 be two L-functions in the extended Selberg class $\mathcal{S}^\sharp$, which satisfy the same functional equation. Suppose that $S = \{c_1, c_2, c_3\}$, where c_1, c_2, c_3 are any three distinct finite complex numbers. If $E(S, L_1) = E(S, L_2)$, then $L_1 = L_2$.*

THEOREM 1.5 ([12]). *Let L_1 and L_2 be two L-functions in the extended Selberg class $\mathcal{S}^\sharp$, which satisfy the same functional equation. Suppose that $S = \{c_1, c_2\}$, where c_1, c_2 are two distinct finite complex numbers. If $E(S, L_1) = E(S, L_2)$, then $L_1 = L_2$.*

In Theorems 1.1 and 1.2, the set S is given by the roots of some polynomials. The sets involved in the sharing in Theorems 1.4 and 1.5 are actually $\{w \in \mathbb{C} : P_1(w) = 0\}$ and $\{w \in \mathbb{C} : P_2(w) = 0\}$, where $P_1(w)$ and $P_2(w)$ are polynomials of the form $(w - c_1)(w - c_2)(w - c_3)$ and $(w - c_1)(w - c_2)$, respectively. In fact, Theorem 1.4 (resp. Theorem 1.5) implies that if $P_1(L_1)$ and $P_1(L_2)$ (resp. $P_2(L_1)$ and $P_2(L_2)$) share 0 CM, then L_1 and L_2 are identical, provided they satisfy the same functional equation. On the other hand, it is still unknown whether Theorem 1.1 holds for two nonconstant L-functions L_1 and L_2 in $\mathcal{S}^\sharp$, or for some other values of n . However, in [17], Wu-Hu proved a uniqueness result with L-functions for a set $S \subset \mathbb{C} \setminus \{1\}$ with n elements of very special type. Regarding these observations, it is quite natural to ask the following question.

QUESTION 1.6. (i) What will be the conclusions in Theorems 1.4 and 1.5 if $P(L_1)$ and $P(L_2)$ share 0 CM, where $P(w)$ is an arbitrary polynomial of some degree q ?

(ii) Is it possible to replace L and f by L_1 and L_2 of $\mathcal{S}^\sharp$ as well as reduce the lower bound of n in Theorem 1.1?

In this paper, we will find the best possible answer of (i) of Question 1.6 by considering the most generalized form of a polynomial. In this direction, we prove the following results.

THEOREM 1.7. *Let L_1 and L_2 be two nonconstant L-functions in the extended Selberg class $\mathcal{S}^\sharp$, and let L_1, L_2 satisfy the same functional equation. Suppose that $P(w) = 0$ is a polynomial equation with $l (\geq 2)$ distinct roots. If $E(0, P(L_1)) = E(0, P(L_2))$, then $L_1 = L_2$.*

COROLLARY 1.8. *Let L_1 and L_2 be two nonconstant L-functions in the extended Selberg class $\mathcal{S}^\sharp$, and let L_1, L_2 satisfy the same functional equation. If $E(S, L_1) = E(S, L_2)$, where $S = \{\gamma_1, \gamma_2, \dots, \gamma_l\} \subset \mathbb{C}$, $l \geq 2$, then $L_1 = L_2$.*

REMARK 1.9. It is obvious that Theorems 1.4 and 1.5 can be obtained as a special case of Corollary 1.8. Thus Corollary 1.8 is a generalization of both Theorems 1.4 and 1.5.

Taking into account (ii) of Question 1.6 we have the next theorem.

THEOREM 1.10. *Let L_1 and L_2 be two nonconstant L-functions in the extended Selberg class $\mathcal{S}^\sharp$, and let L_1, L_2 satisfy the same functional equation. Suppose $P(w) = w^n + aw^m + b$, where n and m are positive integers with $n > m$, $\frac{b^{n-m}}{a^n} \neq \frac{(-1)^n m^m (n-m)^{n-m}}{n^n}$, and a, b are two nonzero constants. If $E(S, L_1) = E(S, L_2)$, where $S = \{w : P(w) = 0\}$, then $L_1 = L_2$.*

REMARK 1.11. Theorem 1.10 suggests that the number of values of n for which the conclusion holds is substantially increased with respect to Theorem 1.1. Also, while replacing L, f by L_1, L_2 one does not necessarily need $\gcd(m, n) = 1$.

As a variant of Theorem 1.1, Hao-Chen [4] obtained the following result.

THEOREM 1.12 ([4]). *Let f be an entire function with $\lim_{\operatorname{Re}(s) \rightarrow \infty} f(s) = k$ ($k \neq \infty$). Let $S = \{\gamma_1, \gamma_2, \dots, \gamma_l\} \subset \mathbb{C} \setminus \{1, k, \beta\}$, where $\gamma_1, \gamma_2, \dots, \gamma_l$ are all distinct roots of the algebraic equation $w^{n+m} + aw^n + b = 0$, $1 \leq l \leq n+m$, n and m are two positive integers with $n > m+2$, and a, b are finite nonzero constants. If f and a nonconstant L-function L share S CM, and $f(s_0) = L(s_0) = \beta$ for some $s_0 \in \mathbb{C}$ then $f = tL$, where t is a constant such that $t^d = 1$, $d = \gcd(n, m)$.*

Regarding Theorem 1.12, considering L_1 and L_2 we deduce an application of Theorem 1.7 by means of the following result.

THEOREM 1.13. *Let L_1 and L_2 be two nonconstant L-functions in $\mathcal{S}^\sharp$, which satisfy the same functional equation and let S be a subset of $\mathbb{C}$ consisting of all distinct roots of the algebraic equation as given in Theorem 1.12 with $n > m$ and $\frac{b^m}{a^{n+m}} \neq \frac{(-1)^{n+m} n^n m^m}{(n+m)^{n+m}}$. If L_1 and L_2 share S CM, then $L_1 = L_2$.*

REMARK 1.14. In Theorem 1.13, the uniqueness relation is obtained without the requirement of the condition that $f(s_0) = L(s_0) = \beta$ for some $s_0 \in \mathbb{C}$.

Let us denote by $\mathcal{S}_0^\sharp$ the class of those L-functions belonging to $\mathcal{S}^\sharp$ with $a(1) = 1$, which have nonzero degree and satisfy the same functional equation. To state the next result we need the following definition.

DEFINITION 1.15 ([10]). If for any two functions f, g in $\mathcal{M}(\mathbb{C})$, there exists some set $S \subset \mathbb{C}$ such that $E(S, f) = E(S, g)$ implies $f = g$, then S is called unique range set (URS) of meromorphic functions.

Replacing f, g by two L-functions in $\mathcal{S}^\sharp$ we can similarly define URS for L-functions in $\mathcal{S}^\sharp$. Choosing the class $\mathcal{S}_0^\sharp$ from Theorem 1.7 we immediately have the following result.

COROLLARY 1.16. *Let $S = \{\gamma_1, \gamma_2, \dots, \gamma_l\} \subset \mathbb{C}$, $l \geq 2$. Then S is URS of L-functions in $\mathcal{S}_0^\sharp$.*

To prove the main results, we will be greatly dependent on Nevanlinna theory of value distribution. For this, we assume that the readers are well familiar with the standard notations and fundamental results of the theory (for details, see [3, 18, 19]). For convenience, we must mention that the symbols $T(r, f)$, $m(r, f)$ and $N(r, f)$ (resp. $\bar{N}(r, f)$) are used to denote the characteristic function, the proximity function, the integrated counting function (resp. reduced counting function) of poles of some meromorphic f , respectively. For some $a \in \mathbb{C}$, we define $N(r, a; f) = N(r, \frac{1}{f-a})$ and $\bar{N}(r, a; f) = \bar{N}(r, \frac{1}{f-a})$. Moreover, $S(r, h)$ will mean any quantity satisfying $S(r, h) = O(\log(rT(r, h)))$ ($r \rightarrow \infty$) possibly outside a set of finite Lebesgue measure. In particular, if h is of finite order, then $S(r, h) = O(\log r)$ ($r \rightarrow \infty$) for all r . The order $\rho(f)$ of some $f \in \mathcal{M}(\mathbb{C})$ is defined as $\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r}$.

2. Lemmas

We present some lemmas that will be useful in proving the main results of this paper.

LEMMA 2.1 ([19, Theorem 1.14]). *Let nonconstant $f(s), g(s) \in \mathcal{M}(\mathbb{C})$. If $\rho(f)$ and $\rho(g)$ are the orders of f and g respectively, then*

$$\begin{aligned} \rho(fg) &\leq \max\{\rho(f), \rho(g)\} \\ \rho(f+g) &\leq \max\{\rho(f), \rho(g)\}. \end{aligned}$$

LEMMA 2.2 ([19, Theorem 1.42]). *Let a nonconstant function $f(s) \in \mathcal{M}(\mathbb{C})$. If 0 and ∞ are two Picard exceptional values of f , then $f(s) = e^{h(s)}$ for some entire function $h(s)$.*

LEMMA 2.3. *Let L_1 and L_2 be two nonconstant L-functions in $\mathcal{S}^\sharp$, and L_1, L_2 satisfy the same functional equation. If $L_1 + cL_2 \not\equiv 0$ for some nonzero constant c , then $L_1(s), L_2(s), (L_1 + cL_2)(s)$ and $\prod_{j=1}^K (\Gamma(\lambda_j s + \nu_j))^{-1}$ have the same zeros in $\{s \in \mathbb{C} : \operatorname{Re}(s) < -\kappa_0 \text{ and } |\operatorname{Im}(s)| < \kappa_1\}$, where κ_0 and κ_1 are two sufficiently large positive numbers such that none of the functions L_1, L_2 and $L_1 + cL_2$ has a zero in $\{s \in \mathbb{C} : \operatorname{Re}(s) > \kappa_0\}$. Furthermore, in the region $\{s \in \mathbb{C} : \operatorname{Re}(s) < -\kappa_0 \text{ and } |\operatorname{Im}(s)| < \kappa_1\}$, each common zero of L_1, L_2 and $L_1 + cL_2$ have the same multiplicity 1 for sufficiently large $\kappa_0, \kappa_1 > 0$.*

Proof. Noting that L_1 and L_2 satisfy the same functional equation, by axiom (iii) of the definition of L-function, we readily have

$$L_1(s) = \omega \chi(s) \overline{L_1(1 - \bar{s})}, \quad L_2(s) = \omega \chi(s) \overline{L_2(1 - \bar{s})}, \quad (1)$$

$$\text{where} \quad \chi(s) = \frac{Q^{1-2s} \prod_{j=1}^K \Gamma(\lambda_j(1-s) + \bar{\nu}_j)}{\prod_{j=1}^K \Gamma(\lambda_j s + \nu_j)}. \quad (2)$$

Here Q , λ_j , ν_j , ω are same as in axiom (iii). From (1) we obtain

$$L_1(s) + cL_2(s) = \omega\chi(s)\{\overline{L_1(1-\bar{s})} + c\overline{L_2(1-\bar{s})}\}. \quad (3)$$

It is easy to see that as $L_1(s)$ and $L_2(s)$ are convergent Dirichlet series, under the assumption $L_1 + cL_2 \neq 0$, we can get a right half-plane $\{s \in \mathbb{C} : \operatorname{Re}(s) \geq \kappa_0\}$ determined by a sufficiently large positive constant κ_0 such that none of $L_1(s)$, $L_2(s)$ and $L_1(s) + cL_2(s)$ has a zero in $\{s \in \mathbb{C} : \operatorname{Re}(s) \geq \kappa_0\}$. In view of the fact that the Gamma function Γ is analytic except for its simple poles at $s = 0, -1, -2, \dots$ in $\mathbb{C}$ such that $\Gamma(s) \neq 0$ for each $s \in \mathbb{C}$, and that χ has neither a pole nor a zero in $\{s \in \mathbb{C} : |\operatorname{Im}(s)| > \kappa_1\}$ for some sufficiently large positive number κ_1 , we conclude from (1)–(3) that zeros of $L_1(s)$, $L_2(s)$, $(L_1 + cL_2)(s)$ coincides with the zeros of $\prod_{j=1}^K (\Gamma(\lambda_j s + \nu_j))^{-1}$ in $\{s \in \mathbb{C} : \operatorname{Re}(s) < -\kappa_0 \text{ and } |\operatorname{Im}(s)| < \kappa_1\}$ for large $\kappa_0, \kappa_1 > 0$. Moreover, in this region, all of L_1 , L_2 and $L_1 + cL_2$ have the zeros of the same multiplicity (one) as that of the zeros of $\prod_{j=1}^K (\Gamma(\lambda_j s + \nu_j))^{-1}$. $\square$

LEMMA 2.4. *Let L_1 and L_2 be two nonconstant L -functions in $\mathcal{S}^\sharp$, and let L_1, L_2 satisfy the same functional equation. Suppose that $P(w) = 0$ is an algebraic equation, where $P(w) = A(w - \gamma_1)^{m_1} \dots (w - \gamma_{l-1})^{m_{l-1}} (w - 1)^{m_l}$ with $A(\neq 0)$, γ_j ($\gamma_i \neq \gamma_j \neq 1$, $i, j = 1, 2, \dots, l-1$) being constants; $l (\geq 2)$, $m_1, m_2, \dots, m_l$ are positive integers such that $\sum_{i=1}^l m_i = q$. If $\frac{P(L_1(s))}{P(L_2(s))} = (s-1)^p e^{d_1 s + d_2}$, where $d_1, d_2 \in \mathbb{C}$ and p is an integer, then $\frac{P(L_1(s))}{P(L_2(s))} = D$ for some nonzero constant D .*

Proof. From the assumption we can write

$$\frac{(L_1(s) - \gamma_1)^{m_1} \dots (L_1(s) - \gamma_{l-1})^{m_{l-1}} (L_1(s) - 1)^{m_l}}{(L_2(s) - \gamma_1)^{m_1} \dots (L_2(s) - \gamma_{l-1})^{m_{l-1}} (L_2(s) - 1)^{m_l}} = (s-1)^p e^{d_1 s + d_2}. \quad (4)$$

By the same reasoning as adopted in [9, p. 4207], we obtain

$$\frac{D_1}{n_1^\sigma} \leq |L_1(s) - 1| \leq \frac{D_2}{n_1^\sigma} \quad \text{and} \quad L_1'(s) = O\left(\frac{1}{n_1^\sigma}\right), \quad (5)$$

$$\text{and} \quad \frac{D_3}{n_2^\sigma} \leq |L_2(s) - 1| \leq \frac{D_4}{n_2^\sigma} \quad \text{and} \quad L_2'(s) = O\left(\frac{1}{n_2^\sigma}\right), \quad (6)$$

as $\sigma = \operatorname{Re}(s) \rightarrow +\infty$. Here $D_j (> 0)$ ($j = 1, 2, 3, 4$) is a constant, and $n_1 (\geq 2)$ $n_2 (\geq 2)$ are integers. Clearly, from (5) and (6) it follows that

$$\frac{D_1'}{D_4'} \left(\frac{n_2}{n_1}\right)^{\sigma m_l} \leq \left|\frac{L_1(s) - 1}{L_2(s) - 1}\right|^{m_l} \leq \frac{D_2'}{D_3'} \left(\frac{n_2}{n_1}\right)^{\sigma m_l} \quad \text{as } \sigma \rightarrow \infty, \quad (7)$$

where $D_1' = D_1^{m_l}$, $D_2' = D_2^{m_l}$ etc. From (4) and (7) we have

$$\begin{aligned} & \left| \frac{D_1'}{D_4'} \left(\frac{n_2}{n_1}\right)^{\sigma m_l} \left| \left(\frac{L_1(s) - \gamma_1}{L_2(s) - \gamma_1} \right)^{m_1} \left(\frac{L_1(s) - \gamma_2}{L_2(s) - \gamma_2} \right)^{m_2} \dots \left(\frac{L_1(s) - \gamma_{l-1}}{L_2(s) - \gamma_{l-1}} \right)^{m_{l-1}} \right| \right| \\ & \leq |(s-1)^p e^{d_1 s + d_2}| \\ & \leq \frac{D_2'}{D_3'} \left(\frac{n_2}{n_1}\right)^{\sigma m_l} \left| \left(\frac{L_1(s) - \gamma_1}{L_2(s) - \gamma_1} \right)^{m_1} \left(\frac{L_1(s) - \gamma_2}{L_2(s) - \gamma_2} \right)^{m_2} \dots \left(\frac{L_1(s) - \gamma_{l-1}}{L_2(s) - \gamma_{l-1}} \right)^{m_{l-1}} \right|, \quad (8) \end{aligned}$$

as $\sigma \rightarrow +\infty$. Since $L_i(s) \rightarrow 1$ for $\sigma \rightarrow +\infty$ ($i = 1, 2$), by (8) we obtain

$$\frac{D'_1}{2D'_4} \left(\frac{n_2}{n_1} \right)^{\sigma \cdot m_l} \leq |(s-1)^p e^{d_1 s + d_2}| \leq \frac{2D'_2}{D'_3} \left(\frac{n_2}{n_1} \right)^{\sigma \cdot m_l} \quad (9)$$

as $\sigma \rightarrow +\infty$. We now distinguish the following two cases.

Case 1. Let $d_1 \neq 0$. Then proceeding in the line of [12, Subcase 2.1 of proof of Theorem 1.11, pp. 1672-1674] and using (9) we obtain that this case is impossible. Here we omit the details.

Case 2. Let $d_1 = 0$. Then (4) can be rewritten as

$$\frac{(L_1(s) - \gamma_1)^{m_1} \dots (L_1(s) - \gamma_{l-1})^{m_{l-1}} (L_1(s) - 1)^{m_l}}{(L_2(s) - \gamma_1)^{m_1} \dots (L_2(s) - \gamma_{l-1})^{m_{l-1}} (L_2(s) - 1)^{m_l}} = (s-1)^p e^{d_2}. \quad (10)$$

In this case, (9) reduces to

$$\frac{D'_1}{2D'_4} \left(\frac{n_2}{n_1} \right)^{\sigma \cdot m_l} \leq |(s-1)^p e^{d_2}| \leq \frac{2D'_2}{D'_3} \left(\frac{n_2}{n_1} \right)^{\sigma \cdot m_l} \quad (11)$$

as $\sigma \rightarrow +\infty$. From (11) it follows that $p = 0$ and $n_1 = n_2$ (≥ 2). Consequently, we have from (10) that

$$\frac{(L_1(s) - \gamma_1)^{m_1} \dots (L_1(s) - \gamma_{l-1})^{m_{l-1}} (L_1(s) - 1)^{m_l}}{(L_2(s) - \gamma_1)^{m_1} \dots (L_2(s) - \gamma_{l-1})^{m_{l-1}} (L_2(s) - 1)^{m_l}} = D,$$

where $D = e^{d_2} \neq 0$. Thus the proof of the lemma is completed. $\square$

LEMMA 2.5 ([11, Lemma 2.7]). Let $P(w) = w^n + aw^m + b$, where m and n are positive integers such that $n > m$, a and b are finite nonzero complex values. Then the following hold:

(i) The algebraic equation $P(w) = 0$ has no root of multiplicity ≥ 3 ;

(ii) If

$$\frac{b^{n-m}}{a^n} \neq \frac{(-1)^n m^m (n-m)^{n-m}}{n^n}, \quad (12)$$

then the algebraic equation $P(w) = 0$ has n distinct simple roots and no multiple root;

(iii) If n and m are relatively prime and

$$\frac{b^{n-m}}{a^n} = \frac{(-1)^n m^m (n-m)^{n-m}}{n^n}, \quad (13)$$

then the algebraic equation $P(w) = 0$ has exactly $n-1$ distinct roots, among which $n-2$ roots are simple and the remaining one is double.

3. Proofs of the theorems

Proof (Proof of Theorem 1.7). Since L_1 and L_2 are two L-functions in $\mathcal{S}^\sharp$, each L_i ($i = 1, 2$) has at most one pole in $\mathbb{C}$, which may appear only at $s = 1$. By the assumption $P(w)$ is any polynomial with $l(\geq 2)$ distinct roots. Therefore, we can

write

$$P(w) = A(w - \gamma_1)^{m_1}(w - \gamma_2)^{m_2} \dots (w - \gamma_l)^{m_l}, \quad (14)$$

where $l (\geq 2)$, $m_i (\geq 1)$ are integers and $A (\neq 0)$, $\gamma_1, \gamma_2, \dots, \gamma_l$ are constants. To proceed further, let us consider the following two cases.

Case 1. Suppose that $P(1) \neq 0$. Clearly, $\gamma_j \neq 1$ for each $j = 1, 2, \dots, l$. Since $P(L_1)$ and $P(L_2)$ share 0 CM, we clearly obtain a function $\mathcal{L}(s)$ given by

$$\mathcal{L}(s) = \frac{P(L_1(s))(s-1)^p}{P(L_2(s))}, \quad (15)$$

such that $\mathcal{L}(s)$ has neither zero nor pole in $\mathbb{C}$. Here p is an integer, in particular, if L_1 and L_2 are both entire functions, then $p = 0$ and $\mathcal{L}(s)$ takes the form $\mathcal{L}(s) = \frac{P(L_1(s))}{P(L_2(s))}$. If d_{L_i} denotes the degree of L_i , then by Steuding [16, p. 150], we obtain that

$$T(r, L_i) = N\left(r, \frac{1}{L_i}\right) + O(r) = \frac{d_{L_i}}{\pi} r \log r + O(r), \quad i = 1, 2. \quad (16)$$

Using the definition of order of meromorphic function we deduce that

$$\rho(L_1) = 1, \quad \rho(L_2) = 1. \quad (17)$$

From Nevanlinna's first fundamental theorem and (16) we also obtain (see [16, pp. 152-153]):

$$\rho\left(\frac{1}{L_i - \gamma_j}\right) = \rho(L_i - \gamma_j) = \rho(L_i) = 1, \quad i = 1, 2; j = 1, 2, \dots, l. \quad (18)$$

By (17), (18) and Lemma 2.1 we have $\rho(\mathcal{L}(s)) \leq \max\{\rho(P(L_1)), \rho(P(L_2))\} \leq 1$. This together with Lemma 2.2 implies that $\mathcal{L}(s)$ is of the form

$$\mathcal{L}(s) = e^{\phi(s)}, \quad (19)$$

where $\phi(s)$ is a polynomial of degree at most 1. Let $\phi(s) = c_1 s + c_2$, where c_1 and c_2 are constants. Then from (15) and (19) we have

$$\frac{P(L_1(s))}{P(L_2(s))} = (s-1)^{p_1} e^{c_1 s + c_2}, \quad (20)$$

where $p_1 = -p$ is an integer. Noting that $L_i(s) = \sum_{n=1}^{\infty} \frac{a_i(n)}{n^s}$ with $a_i(1) = 1$, we deduce

$$\lim_{\sigma \rightarrow +\infty} L_i(s) = 1, \quad i = 1, 2. \quad (21)$$

As $P(1) \neq 0$, using (21) we clearly obtain

$$\lim_{\sigma \rightarrow +\infty} \frac{P(L_1(s))}{P(L_2(s))} = \prod_{j=1}^l \lim_{\sigma \rightarrow +\infty} \left(\frac{L_1(s) - \gamma_j}{L_2(s) - \gamma_j} \right)^{m_j} = 1. \quad (22)$$

Let us suppose that $c_1 = c_{11} + ic_{12}$ and $c_2 = c_{21} + ic_{22}$, where $c_{11}, c_{12}, c_{21}, c_{22}$ are real numbers. With these values and that $s = \sigma + it$, from (20) and (22) it now follows that

$$\frac{P(L_1(s))}{P(L_2(s))} = (\sigma + it - 1)^{p_1} e^{c_{11}\sigma - c_{12}t + c_{21} + i(c_{12}\sigma + c_{11}t + c_{22})} \quad (23)$$

and
$$\lim_{\sigma \rightarrow +\infty} (\sigma + it - 1)^{p_1} e^{c_{11}\sigma - c_{12}t + c_{21} + i(c_{12}\sigma + c_{11}t + c_{22})} = 1. \quad (24)$$

If possible, let $c_{11} \neq 0$. Then for $c_{11} > 0$, in view of (23), we have $\lim_{\sigma \rightarrow +\infty} \left| \frac{P(L_1(s))}{P(L_2(s))} \right| = +\infty$, which is a contradiction to (24). Similarly, for $c_{11} < 0$, $\lim_{\sigma \rightarrow +\infty} \left| \frac{P(L_1(s))}{P(L_2(s))} \right| = 0$, which is also a contradiction. Therefore $c_{11} = 0$. Then by the same reasoning we must have $p_1 = 0$. Therefore, from (23) we get $\left| \frac{P(L_1(s))}{P(L_2(s))} \right| = e^{-c_{12}t + c_{21}}$. Also, by (24) we have $\lim_{\sigma \rightarrow +\infty} e^{-c_{12}t + c_{21}} = e^{-c_{12}t + c_{21}} = 1$. Thus $\left| \frac{P(L_1(s))}{P(L_2(s))} \right| = 1$. This together with (22) gives

$$\frac{P(L_1(s))}{P(L_2(s))} \equiv 1. \quad (25)$$

If $P(w)$ is a polynomial of degree q , then (14) can be rewritten as

$$P(w) = a_q w^q + a_{q-1} w^{q-1} + \dots + a_1 w + a_0,$$

where $a_q (\neq 0), a_{q-1}, \dots, a_0 \in \mathbb{C}$, $q = \sum_{i=1}^l m_i$. Therefore, in view of (25) it follows that

$$a_q (L_1^q - L_2^q) + a_{q-1} (L_1^{q-1} - L_2^{q-1}) + \dots + a_2 (L_1^2 - L_2^2) + a_1 (L_1 - L_2) = 0$$

i.e.,

$$(L_1 - L_2) \left[a_q \sum_{k=0}^{q-1} L_1^{q-1-k} L_2^k + a_{q-1} \sum_{k=0}^{q-2} L_1^{q-2-k} L_2^k + \dots + a_2 \sum_{k=0}^1 L_1^{1-k} L_2^k + a_1 \right] = 0. \quad (26)$$

If $L_1 = L_2$, then (26) holds and we have nothing to prove. Therefore, in what follows, let us assume that $L_1 \neq L_2$. Then from (26) we get

$$a_q \sum_{k=0}^{q-1} L_1^{q-1-k} L_2^k + a_{q-1} \sum_{k=0}^{q-2} L_1^{q-2-k} L_2^k + \dots + a_2 \sum_{k=0}^1 L_1^{1-k} L_2^k + a_1 = 0. \quad (27)$$

Since $L_1, L_2 \in \mathcal{S}^\#$ satisfy the same functional equation, by Lemma 2.3 there exists a common zero of L_1, L_2 which is also a zero of $L_1 - L_2$ in some region $\{s \in \mathbb{C} : \operatorname{Re}(s) < -\kappa_0 \text{ and } |\operatorname{Im}(s)| < \kappa_1\}$ for some sufficiently large positive constants κ_0 and κ_1 such that each of L_1, L_2 and $L_1 - L_2$ is zero-free in $\{s \in \mathbb{C} : \operatorname{Re}(s) \geq \kappa_0\}$. Considering such a common zero, say s_0 , satisfying $L_1(s_0) = L_2(s_0) = 0$, we obtain from (27) that $a_1 = 0$. Suppose that $a_2 \neq 0$. If $q = 2$, then we arrive at a contradiction as $L_1 \neq L_2$ and $a_1 = 0$. For the case $q > 2$, (27) reduces to

$$a_q \sum_{k=0}^{q-1} L_1^{q-1-k} L_2^k + a_{q-1} \sum_{k=0}^{q-2} L_1^{q-2-k} L_2^k + \dots + a_3 \sum_{k=0}^2 L_1^{2-k} L_2^k = -a_2 (L_1 + L_2). \quad (28)$$

If $L_1 + L_2 \neq 0$, then likewise in the above paragraph, using Lemma 2.3 we can obtain a region $\{s \in \mathbb{C} : \operatorname{Re}(s) < -\kappa'_0 \text{ and } |\operatorname{Im}(s)| < \kappa'_1\}$ in which L_1, L_2 and $L_1 + L_2$ have common zeros with the same multiplicities (one) for some sufficiently large positive constants κ'_0 and κ'_1 . Let s_1 be a simple zero of both L_1 and L_2 in this region. Then s_1 is a zero of the left-hand side of (28) with multiplicity at least 2, while it is a zero of the right-hand side of (28) with multiplicity 1. Thus we arrive

at a contradiction. Therefore, we have $L_1 + L_2 = 0$, which is again a contradiction in view of (21). Thus, assuming $L_1 \neq L_2$ we obtain an impossibility for $a_2 \neq 0$.

Next, suppose that $a_2 = 0$. Let $k_0 (\geq 3)$ be the smallest positive integer such that $a_{k_0} \neq 0$. Obviously, $q \neq k_0$, otherwise we arrive at a contradiction as $L_1 \neq L_2$ and $a_j = 0$ for $1 \leq j \leq k_0 - 1$. Then (26) can be written as

$$a_q(L_1^q - L_2^q) + a_{q-1}(L_1^{q-1} - L_2^{q-1}) + \dots + a_{k_0+1}(L_1^{k_0+1} - L_2^{k_0+1}) = -a_{k_0}(L_1^{k_0} - L_2^{k_0}). \quad (29)$$

Assume that $L_1^{k_0} \neq L_2^{k_0}$. As $L_1, L_2 \in \mathcal{S}^\#$ satisfy the same functional equation, from Lemma 2.3 we see that there exists some region $\{s \in \mathbb{C} : \operatorname{Re}(s) < -\kappa''_0 \text{ and } |\operatorname{Im}(s)| < \kappa''_1\}$ in which $L_1(s), L_2(s), (L_1 - \alpha_1 L_2)(s), (L_1 - \alpha_2 L_2)(s), \dots, (L_1 - \alpha_{k_0} L_2)(s)$ have common simple zeros for some sufficiently large positive constants κ''_0 and κ''_1 such that these functions have no zero in $\{s \in \mathbb{C} : \operatorname{Re}(s) \geq \kappa''_0\}$. Here $\alpha_1, \alpha_2, \dots, \alpha_{k_0}$ are the distinct roots of $s^{k_0} = 1$. Considering s_2 as a common simple zero of L_1 and L_2 in the above region of common zeros, it is clear that s_2 is a zero of the left-hand side of (29) with multiplicity at least $k_0 + 1$, and a zero of the right-hand side of (29) with multiplicity k_0 . Hence we arrive at a contradiction. Therefore, we have $L_1^{k_0} = L_2^{k_0}$, which implies $L_1 = \alpha_j L_2$, where $\alpha_j^{k_0} = 1$, and hence $L_1 = L_2$ in view of (21). This contradicts the assumption $L_1 \neq L_2$.

Case 2. Suppose that $P(1) = 0$. Then exactly one of $\gamma_1, \gamma_2, \dots, \gamma_l$ is 1. Without loss of generality, suppose that $\gamma_l = 1$. Clearly, $P(w)$ can be rewritten as

$$P(w) = A(w - \gamma_1)^{m_1} \dots (w - \gamma_{l-1})^{m_{l-1}} (w - 1)^{m_l}. \quad (30)$$

Also, by the assumption we have, $P(L_1)$ and $P(L_2)$ share 0 CM. Then it is easy to see that the relations (15)-(20) hold. By (30) we have

$$\frac{(L_1(s) - \gamma_1)^{m_1} \dots (L_1(s) - \gamma_{l-1})^{m_{l-1}} (L_1(s) - 1)^{m_l}}{(L_2(s) - \gamma_1)^{m_1} \dots (L_2(s) - \gamma_{l-1})^{m_{l-1}} (L_2(s) - 1)^{m_l}} = (s - 1)^{p_1} e^{c_1 s + c_2},$$

for some integer p_1 . Obviously, by Lemma 2.4 we obtain

$$\frac{P(L_1(s))}{P(L_2(s))} = D, \quad (31)$$

where $D (\neq 0) \in \mathbb{C}$. Since degree of the polynomial $P(w)$ is q , we have

$$\begin{aligned} & a_q L_1^q + a_{q-1} L_1^{q-1} + \dots + a_1 L_1 + a_0 \\ &= D a_q L_2^q + D a_{q-1} L_2^{q-1} + \dots + D a_1 L_2 + D a_0. \end{aligned} \quad (32)$$

Assume that $L_1 \neq L_2$. Since L_1, L_2 satisfy the same functional equation, there exists a region of common zeros of L_1, L_2 and $L_1 - L_2$ given by $\{s \in \mathbb{C} : \operatorname{Re}(s) < -\kappa_0 \text{ and } |\operatorname{Im}(s)| < \kappa_1\}$ for some sufficiently large positive constants κ_0 and κ_1 . With such a common zero of L_1, L_2 we easily obtain from (32) that $a_0(1 - D) = 0$. If $D = 1$, then (32) reduces to (26), and so by the arguments (26) onwards in **Case 1.**, we obtain that $L_1 \neq L_2$ is not possible. Suppose that $D \neq 1$. Then using $a_0 = 0$, (32) can be rewritten as

$$a_q(L_1^q - D L_2^q) + a_{q-1}(L_1^{q-1} - D L_2^{q-1}) + \dots + a_1(L_1 - D L_2) = 0. \quad (33)$$

Let $j_0 (\geq 1)$ be the smallest positive integer such that $a_{j_0} \neq 0$. Clearly, $j_0 \neq q$,

otherwise by (21) we have a contradiction. Then (33) reduces to

$$\begin{aligned} & a_q(L_1^q - DL_2^q) + a_{q-1}(L_1^{q-1} - DL_2^{q-1}) + \dots + a_{j_0+1}(L_1^{j_0+1} - DL_2^{j_0+1}) \\ &= -a_{j_0}(L_1^{j_0} - DL_2^{j_0}). \end{aligned} \quad (34)$$

If $L_1^{j_0} \neq DL_2^{j_0}$, then from Lemma 2.3 we see that $L_1(s)$, $L_2(s)$, $(L_1 - \beta_1 L_2)(s)$, $(L_1 - \beta_2 L_2)(s)$, $\dots$, $(L_1 - \beta_{j_0} L_2)(s)$ have common simple zeros in some region $\text{Re}(s) < -\kappa'''_0$ for a sufficiently large $\kappa'''_0 > 0$, where $\beta_1, \beta_2, \dots, \beta_{j_0}$ are the distinct roots of $s^{j_0} = D$. Considering a common simple zero of L_1 and L_2 in this region, it is clear that multiplicity of the zero is at least $j_0 + 1$ and exactly j_0 in the left-hand and right-hand side of (34), respectively. This is clearly a contradiction. Therefore, we have $L_1^{j_0} = DL_2^{j_0}$, and hence $L_1 = L_2$ in view of (21). This clearly contradicts the assumption $L_1 \neq L_2$.

Therefore, combining **Case 1.** and **Case 2.** it follows that $L_1 = L_2$. This completes the proof of Theorem 1.7. $\square$

Proof (Proof of Corollary 1.8). By the assumption L_1, L_2 share $S = \{\gamma_1, \gamma_2, \dots, \gamma_l\}$ CM, where $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \dots, l$. Then from the definition of set sharing $(L_1 - \gamma_1)(L_1 - \gamma_2) \dots (L_1 - \gamma_l)$ and $(L_2 - \gamma_1)(L_2 - \gamma_2) \dots (L_2 - \gamma_l)$ have the same zeros counting multiplicity. In other words, $P_1(L_1)$ and $P_1(L_2)$ share 0 CM, where $P_1(w) = (w - \gamma_1)(w - \gamma_2) \dots (w - \gamma_l)$. Since $l \geq 2$, the conclusion follows from Theorem 1.7. $\square$

Proof (Proof of Theorem 1.10). Since n and m are positive integers with $n > m \geq 1$, $P(w)$ is certainly a polynomial of degree $n (\geq 2)$. On the other hand, by Lemma 2.5 we see that $P(w) = 0$ has exactly $n (\geq 2)$ distinct roots as $\frac{b^{n-m}}{a^n} \neq \frac{(-1)^n m^m (n-m)^{n-m}}{n^n}$. Since L_1 and L_2 share S CM, $P(L_1)$ and $P(L_2)$ share 0 CM. Thus by the lines of proof of Theorem 1.7 the conclusion can be obtained. Hence we omit the details. $\square$

Proof (Proof of Theorem 1.13). Since $n > m$, the equation $P(w) = 0$ has $n + m (\geq 2)$ distinct roots, in view of Lemma 2.5. Then using the sharing condition between $P(L_1)$ and $P(L_2)$, and proceeding in the lines of proof of Theorem 1.7 we obtain the conclusion. $\square$

4. Further remarks

In Theorems 1.4, 1.5 and also in Theorems 1.7-1.13, L_1 and L_2 are L-functions which have nonzero degree and satisfy the same functional equation. The case where L-functions have degree zero remains unresolved in relation to these theorems. Therefore, it is inevitable to raise the following open question.

QUESTION 4.1. How does Theorem 1.7 hold if the two L-functions L_1 and L_2 are of degree zero?

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