MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 06.09.2026

research paper
оригинални научни рад
DOI: 10.57016/MV-XWSF5683

SEMILATTICES WITH A CONGRUENCE

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Abstract. A specialization semilattice is a semilattice together with a coarser preorder satisfying a compatibility condition. We show that the category of specialization semilattices is isomorphic to the category of semilattices with a congruence, hence equivalent to the category of semilattice epimorphisms.

Guided by the above example, we recall an "internal" characterization of surjective homomorphisms between general relational systems.

1. Introduction

A *specialization semilattice* is a join semilattice endowed with a further preorder $\sqsubseteq$ which is coarser than the order induced by the semilattice operation and such that

$$a \sqsubseteq b \quad \text{and} \quad a_1 \sqsubseteq b \quad \text{imply} \quad a \vee a_1 \sqsubseteq b, \tag{1}$$

for all the elements a, a_1, b of the semilattice.

If X is a topological space, then $(\mathcal{P}(X), \cup, \sqsubseteq)$ is a specialization semilattice, where $\sqsubseteq$ is defined by $a \sqsubseteq b$ if $a \subseteq Kb$, for a and b subsets of X, and K is closure. In [7, Theorem 5.7] we showed that every specialization semilattice can be embedded into the specialization semilattice associated to some topological space, as described above.

The study of specialization semilattices is motivated by the fact that a function between topological spaces is continuous if and only if the corresponding image function is a homomorphism between the associated specialization semilattices [7, Proposition 2.4]. Alternative "algebraizations" of topology do not satisfy such an exact functorial condition. See the introduction of [7] for a discussion.

More or less in disguise and in different terminology, specialization semilattices appear in domain theory, in algebraic geometry, in lattice theory, in set theory (for example if we take inclusion modulo finite as a "specialization"), in algebraic logic,

2020 Mathematics Subject Classification: 06A12, 54A05, 18A25
Keywords and phrases: Specialization semilattice; specialization poset; epimorphism; semilattice with a congruence; compatible preorder; arrow category of epimorphisms.

in the study of tolerance spaces, in measure theory and as *Complete Implicational Systems*, motivated by relational data bases, data analysis and artificial intelligence. See [7, Section 4] for details and references.

Besides the topological construction hinted above, there is another way of representing specialization semilattices. If $(S, \vee_S)$, $(T, \vee_T)$ are semilattices, φ is a homomorphism from $\mathbf{S}$ to $\mathbf{T}$ and in S we set

$$a \sqsubseteq_\varphi b \text{ if } \varphi(a) \leq_T \varphi(b), \tag{2}$$

then $(S, \vee_S, \sqsubseteq_\varphi)$ is a specialization semilattice. In the formula (2) $\leq_T$ is the partial order on T induced by the semilattice operation $\vee_T$.

In Corollary 2.4 below we will show that every specialization semilattice can be represented in the above way. Thus specialization semilattices are at the same time "substructures" of topological spaces and "kernels" of semilattice homomorphisms. Together with the mentioned abundance of examples occurring in varied mathematical contexts, this double kind of representation confirms the naturalness of the notion.

We will also show that, given a semilattice $\mathbf{S}$, the set of preorders making $\mathbf{S}$ a specialization semilattice is in a bijective correspondence with the set of congruences on $\mathbf{S}$; actually, the correspondence is a complete lattice isomorphism (see Theorem 2.1). Put in another way, the set of those preorders which make $\mathbf{S}$ a specialization semilattice is in a bijective correspondence with the class of surjective homomorphisms with domain $\mathbf{S}$, considered up to isomorphisms keeping $\mathbf{S}$ fixed.

As another reformulation, the category of specialization semilattices, with the standard notion of morphism, is isomorphic to the category of semilattices with a congruence, where in the latter category morphisms are semilattice morphisms which respect the congruence (see Corollary 2.5). It is then standard to see that the category of specialization semilattices is equivalent to the category $\mathcal{S}\mathfrak{Epi}$ of semilattice epimorphisms. Objects of $\mathcal{S}\mathfrak{Epi}$ are semilattice epimorphisms and morphisms of $\mathcal{S}\mathfrak{Epi}$ are commuting squares of semilattice morphisms, where the additional morphisms in the square are not necessarily assumed to be epi.

Let us now consider a weaker notion. A *specialization poset* is a partially ordered set (henceforth, poset, for short) together with a coarser preorder. Again, examples of specialization posets abound in the mathematical literature [7, Section 4]. The connection with topological spaces goes the same way as for specialization semilattices. If X is a topological space, then $(\mathcal{P}(X), \subseteq, \sqsubseteq)$ is a specialization poset; every specialization poset can be represented as a substructure of such a topological specialization poset (see [7, Proposition 5.10]). As for "quotient" representations, if $(P, \leq_P)$, $(Q, \leq_Q)$ are posets and φ is an order preserving map, then we can define $\sqsubseteq_\varphi$ as in (2). Then $(P, \leq_P, \sqsubseteq_\varphi)$ turns out to be a specialization poset. Every specialization poset can be represented this way (see Example 3.1). As above, the category of specialization posets is equivalent to the category $\mathcal{POS}\mathfrak{Surj}$ of surjective order preserving maps between posets, where, again, morphisms of $\mathcal{POS}\mathfrak{Surj}$ are commuting squares and the additional poset morphisms are not assumed to be surjective. However, it is standard to see that surjective order preserving maps between posets cannot be characterized "internally" by an equivalence relation, see Remark 3.3.

Hence an "internal" characterization of surjective order preserving maps between posets is better provided by a binary relation (a coarser preorder) which is not necessarily an equivalence relation. Put in another way, specializations in posets can be used exactly as congruences in algebraic structures, where we have only operations and no relation, apart from equality.

In Section 3 we recall a similar characterization for models for an arbitrary language $\mathscr{L}$, allowed to contain both relation and function symbols. In this sense, the "kernel" of some homomorphism φ is given not only by an equivalence relation Θ, defined as usual by $a \ \Theta \ b$ if $\varphi(a) = \varphi(b)$, but also by a set of further relations, one additional relation for each relation symbol in $\mathscr{L}$.

2. Specialization semilattices "are" semilattices with a congruence

Recall the definition of a specialization semilattice from the introduction. It is elementary to see that a specialization semilattice satisfies

$$a \sqsubseteq b \ \& \ a_1 \sqsubseteq b_1 \Rightarrow a \vee a_1 \sqsubseteq b \vee b_1 \tag{3}$$

(see [7, Remark 3.5(a)]).

If $\mathbf{S} = (S, \vee)$ is a join semilattice, we say that a preorder $\sqsubseteq$ on S is *compatible* if $(S, \vee, \sqsubseteq)$ is a specialization semilattice, namely, $\sqsubseteq$ is coarser than $\leq$ (the order induced by $\vee$) and (1) is satisfied. As remarked in [7, Remark 7.1], the set of all the compatible preorders on $\mathbf{S}$ is a complete lattice, with minimum $\leq$, maximum the total binary relation and meet as intersection of relations, namely, a and b are related by $\bigcap_{i \in I} \sqsubseteq_i$ if $a \sqsubseteq_i b$, for every $i \in I$.

Recall that a *congruence* on a semilattice $\mathbf{S}$ is an equivalence relation Θ on S such that $a \ \Theta \ b$ implies $a \vee c \ \Theta \ b \vee c$ for all $a, b, c \in S$. This is a special case of a general notion which applies to arbitrary algebraic structures [2]. The set of congruences of some algebraic structure, in particular, of a semilattice, is a complete lattice, as well.

THEOREM 2.1. *Suppose that $\mathbf{S} = (S, \vee)$ is a join semilattice.*

The lattice of compatible preorders on $\mathbf{S}$ is isomorphic, as a complete lattice, to the lattice of congruences on $\mathbf{S}$.

An isomorphism is given by the function $\Psi_{\mathbf{S}}$ which assigns to a compatible preorder $\sqsubseteq$ the congruence $\Psi_{\mathbf{S}}(\sqsubseteq) = \Theta_{\sqsubseteq}$ defined by $a \ \Theta_{\sqsubseteq} \ b$ if both $a \sqsubseteq b$ and $b \sqsubseteq a$.

The inverse of $\Psi_{\mathbf{S}}$ is the function $\Omega_{\mathbf{S}}$ which assigns to a congruence Θ the preorder $\sqsubseteq_\Theta$ such that $a \sqsubseteq_\Theta b$ if $a \vee b \ \Theta \ b$.

Proof. If $\sqsubseteq$ is a compatible preorder, then $\Theta_{\sqsubseteq}$ is a congruence on $\mathbf{S}$, by (3) and transitivity and reflexivity of $\sqsubseteq$. Thus the codomain of $\Psi_{\mathbf{S}}$ is as wanted. Henceforth we will drop the subscripts in $\Psi_{\mathbf{S}}$ and $\Omega_{\mathbf{S}}$.

Conversely, let Θ be a congruence. If $a \leq b$, that is, $a \vee b = b$, then $a \vee b \ \Theta \ b$ holds, thus $a \sqsubseteq_\Theta b$. This shows that $\sqsubseteq_\Theta$ is coarser than $\leq$. In particular, $\sqsubseteq_\Theta$ is reflexive. If $a \sqsubseteq_\Theta b$ and $b \sqsubseteq_\Theta c$, then $a \vee b \ \Theta \ b$ and $b \vee c \ \Theta \ c$, thus $a \vee c \ \Theta \ a \vee (b \vee c) = (a \vee b) \vee c \ \Theta \ b \vee c \ \Theta \ c$, since Θ is a congruence, hence $a \sqsubseteq_\Theta c$, since Θ is transitive. This shows that $\sqsubseteq_\Theta$ is transitive. If $a \sqsubseteq_\Theta b$ and $a_1 \sqsubseteq_\Theta b$, then $a \vee b \ \Theta \ b$ and $a_1 \vee b \ \Theta \ b$, thus $a \vee a_1 \vee b \ \Theta \ b \vee b = b$, hence $a \vee a_1 \sqsubseteq_\Theta b$. We have showed that $\sqsubseteq_\Theta$ is a compatible preorder satisfying (1), hence the codomain of Ω is appropriate, as well.

If $\sqsubseteq = \bigcap_{i \in I} \sqsubseteq_i$, then $a \ \Theta_\sqsubseteq \ b$ if and only if $a \sqsubseteq_i b$ and $b \sqsubseteq_i a$, for every $i \in I$, if and only if $a \ \Theta_{\sqsubseteq_i} \ b$ for every $i \in I$. Thus $\Psi(\bigcap_{i \in I} \sqsubseteq_i) = \bigcap_{i \in I} \Psi(\sqsubseteq_i)$, that is, Ψ is a complete meet homomorphism.

If we show that Ω is the inverse of Ψ, then Ψ is a bijection, hence an isomorphism. Note that a complete meet isomorphism is also a complete join isomorphism. If $\sqsubseteq$ is a compatible preorder and $\sqsubseteq^* = \Omega(\Psi(\sqsubseteq))$, then $a \sqsubseteq^* b$ if and only if $a \vee b \ \Theta_\sqsubseteq \ b$, if and only if both $a \vee b \sqsubseteq b$ and $b \sqsubseteq a \vee b$. The latter inclusion is always true, since $\sqsubseteq$ is coarser than $\leq$; the former inclusion holds if and only if $a \sqsubseteq b$, in view of (1) and since $\sqsubseteq$ is transitive and coarser than $\leq$. Thus $a \sqsubseteq^* b$ if and only if $a \sqsubseteq b$, hence $\Psi \circ \Omega$ is the identity.

Conversely, if Θ is a congruence and $\Theta^* = \Psi(\Omega(\Theta))$, then $a \ \Theta^* \ b$ if and only if both $a \sqsubseteq_\Theta b$ and $b \sqsubseteq_\Theta a$, if and only if both $a \vee b \ \Theta \ b$ and $b \vee a \ \Theta \ a$. This implies $a \ \Theta \ b$, since $a \vee b = b \vee a$ and Θ is transitive. Conversely, if $a \ \Theta \ b$, then $a \vee b \ \Theta \ b$ and $b \vee a \ \Theta \ a$. Thus $a \ \Theta^* \ b$ if and only if $a \ \Theta \ b$. □

Definition 2.2. If $\mathbf{S}$ is a semilattice, a *quotient* of $\mathbf{S}$ is a pair $(\varphi, \mathbf{T})$, where $\mathbf{T}$ is a semilattice and φ is a surjective homomorphism from $\mathbf{S}$ to $\mathbf{T}$. Two quotients $(\varphi, \mathbf{T})$ and $(\varphi', \mathbf{T}')$ of $\mathbf{S}$ are *isomorphic* if there is an isomorphism $\chi : \mathbf{T} \to \mathbf{T}'$ such that $\varphi \circ \chi = \varphi'$. Of course, the above definition applies to any kind of algebraic or relational structure.

From Theorem 2.1 and the Fundamental Homomorphism Theorem [2, Theorem 1.22] we immediately get the following corollary.

Corollary 2.3. *Let $\mathbf{S}$ be a semilattice and consider the function Γ which sends a compatible preorder $\sqsubseteq$ to the pair $(\pi_\sqsubseteq, \mathbf{S}/\Theta_\sqsubseteq)$, where $\Theta_\sqsubseteq$ is defined in Theorem 2.1 and $\pi_\sqsubseteq$ is the canonical projection from S to $S/\Theta_\sqsubseteq$.*

Then Γ is a bijection from the set of compatible preorders on $\mathbf{S}$ to the set of equivalence classes of quotients of $\mathbf{S}$ up to isomorphism.

As another corollary of Theorem 2.1, we get the alternative representation theorem for specialization semilattices mentioned in the introduction.

Corollary 2.4. *Suppose that $(S, \vee_S)$ is a semilattice and $\sqsubseteq$ is a binary relation on S.*

Then $(S, \vee_S, \sqsubseteq)$ is a specialization semilattice if and only if there are a semilattice $(T, \vee_T)$ and a semilattice homomorphism $\varphi : (S, \vee_S) \to (T, \vee_T)$ such that $a \sqsubseteq b$ if and only if $\varphi(a) \leq_T \varphi(b)$, for all $a, b \in S$. Namely, $\sqsubseteq$ is $\sqsubseteq_\varphi$, as given by (2).

Proof. The "if" condition is straightforward.

Conversely, if $(S, \vee_S, \sqsubseteq)$ is a specialization semilattice, then $\Theta_\sqsubseteq$ is a congruence of $(S, \vee_S)$, by Theorem 2.1. If $(T, \vee_T)$ is the quotient of $(S, \vee_S)$ modulo $\Theta_\sqsubseteq$ with φ the canonical projection, then, for all $a, b \in S$, we have $\varphi(a) \leq_T \varphi(b)$ if and only if

$\varphi(a \vee_S b) = \varphi(a) \vee_T \varphi(b) = \varphi(b)$, if and only if $a \vee_S b \ \Theta_{\sqsubseteq} \ b$. Thus $\sqsubseteq_\varphi = \Omega(\Psi(\sqsubseteq)) = \sqsubseteq$ by Theorem 2.1. $\square$

We now reformulate the above results in a categorical framework. We refer to [1] for basic notions about categories.

Specialization semilattices form a category with the standard model theoretical notion of homomorphism, namely, a homomorphism between two specialization semilattices is a semilattice homomorphism φ such that $a \sqsubseteq b$ implies $\varphi(a) \sqsubseteq \varphi(b)$. The category $\mathcal{S}\mathfrak{Con}$ of *semilattices with a congruence* has objects those triplets $(S, \vee, \Theta)$ such that $(S, \vee)$ is a semilattice and Θ is a congruence on $(S, \vee)$. A morphisms φ in $\mathcal{S}\mathfrak{Con}$ from $(S, \vee_S, \Theta)$ to $(T, \vee_T, \Gamma)$ is a semilattice homomorphism such that $a \ \Theta \ b$ implies $\varphi(a) \ \Gamma \ \varphi(b)$. Of course, semilattices play no special role in the above definition: for every variety $\mathcal{V}$ of algebraic structures, exactly in the same way, we can define the category $\mathcal{V}\mathfrak{Con}$ of algebras in $\mathcal{V}$ with a congruence.

Corollary 2.5. *The category of specialization semilattices is isomorphic to the category of semilattices with a congruence.*

The correspondence between objects is given by the functions $\Psi_{\mathbf{S}}$ from Theorem 2.1 and by the identity for morphisms.

Proof. In view of Theorem 2.1, it is enough to show that some function φ is a morphism for specialization semilattices if and only if φ is a morphism for the corresponding semilattices with a congruence.

Indeed, if φ is a morphism from $(S, \vee_S, \sqsubseteq_S)$ to $(T, \vee_T, \sqsubseteq_T)$ and $a \ \Theta_{\sqsubseteq_S} \ b$, then $a \sqsubseteq_S b$ and $b \sqsubseteq_S a$, thus $\varphi(a) \sqsubseteq_T \varphi(b)$ and $\varphi(b) \sqsubseteq_T \varphi(a)$, hence $\varphi(a) \ \Theta_{\sqsubseteq_T} \ \varphi(b)$. Conversely, if φ is a morphism from $(S, \vee_S, \Theta)$ to $(T, \vee_T, \Gamma)$ and $a \sqsubseteq_\Theta b$, then $a \vee b \ \Theta \ b$, thus $\varphi(a) \vee \varphi(b) = \varphi(a \vee b) \ \Gamma \ \varphi(b)$, since φ is a morphism of semilattices with a congruence, in particular, a semilattice homomorphism. Hence $\varphi(a) \sqsubseteq_\Gamma \varphi(b)$. $\square$

By a *concrete* category we mean a *construct* in the terminology from [1], namely, a concrete category over **Set**. If $\mathfrak{C}$ is a (concrete) category, the category $\mathfrak{C}\mathfrak{Epi}$ of $\mathfrak{C}$-epimorphisms (the category $\mathfrak{C}\mathfrak{Surj}$ of $\mathfrak{C}$-surjections) is the category whose objects are epimorphisms of $\mathfrak{C}$ (surjective morphisms of $\mathfrak{C}$). The morphisms of $\mathfrak{C}\mathfrak{Epi}$ ($\mathfrak{C}\mathfrak{Surj}$) are commuting squares of morphisms of $\mathfrak{C}$. Thus $\mathfrak{C}\mathfrak{Epi}$ and $\mathfrak{C}\mathfrak{Surj}$ are full subcategories of the arrow category of $\mathfrak{C}$; in $\mathfrak{C}\mathfrak{Epi}$ ($\mathfrak{C}\mathfrak{Surj}$) only epimorphisms (surjective morphisms) are considered as "arrows", but the other morphisms in the commuting squares are not necessarily assumed to be epi or surjective.

Remark 2.6. Congruences in general algebraic systems are just an internal description of homomorphic images. Restated from a categorical point of view, if $\mathcal{V}$ is a variety, then the category of algebras in $\mathcal{V}$ with a congruence is equivalent to the category $\mathcal{V}\mathfrak{Surj}$ of $\mathcal{V}$-surjections. Indeed, consider the functor F which associates to an algebra with a congruence $(\mathbf{A}, \Theta)$ the canonical quotient morphism $\pi_\Theta : \mathbf{A} \to \mathbf{A}/\Theta$. If $\varphi : A \to B$ is a function, then φ is a morphism of $\mathcal{V}\mathfrak{Con}$ from $(\mathbf{A}, \Theta)$ to $(\mathbf{B}, \Gamma)$ if and only if φ is a morphism in $\mathcal{V}$ and there exists a $\mathcal{V}$-morphism $\hat{\varphi}$ such that $\varphi \circ \pi_\Gamma = \pi_\Theta \circ \hat{\varphi}$, thus necessarily $\hat{\varphi}$ is defined by $a/\Theta \mapsto \varphi(a)/\Gamma$. In fact, this assignment provides a good definition exactly when φ is a $\mathcal{V}\mathfrak{Con}$-morphism from $(\mathbf{A}, \Theta)$

to $(\mathbf{B}, \Gamma)$. Thus if we set $F(\varphi) = (\varphi, \hat{\varphi})$ on morphisms, then F is full and faithful from $\mathcal{V}\mathfrak{Con}$ to $\mathcal{V}\mathfrak{Surj}$. If $\psi : \mathbf{A} \to \mathbf{B}$ is surjective, then ψ is isomorphic to the canonical projection $\pi_\Theta : \mathbf{A} \to \mathbf{A}/\Theta$ in $\mathcal{V}\mathfrak{Surj}$, where Θ is the kernel of ψ, namely $\Theta = \{\, (a, b) \mid \psi(a) = \psi(b) \,\}$. This means that F is *isomorphism-dense* [1, Definition 3.33]. Thus F witnesses that the category of algebras in $\mathcal{V}$ with a congruence is equivalent to the category of $\mathcal{V}$-surjections.

As we will see in Section 3, the situation is more delicate in the context of relational structures. For arbitrary categories, additional problems might arise; see, e. g., the section "Quotient objects" in [1].

The above considerations apply, in particular, to the variety of semilattices. Noticing that in the category of semilattices epimorphisms are exactly surjective morphisms [6], from Corollary 2.5 we get the following corollary.

COROLLARY 2.7. *The category of specialization semilattices is equivalent to the category of semilattice epimorphisms.*

REMARK 2.8. There is a subtle difference between Corollary 2.7 and Corollary 2.3. For example, let $\mathbf{S}$ be a semilattice with five elements a_1, a_2, b_1, b_2, c, with $a_1 < a_2 < c$, $b_1 < b_2 < c$ and $a_i \vee b_j = c$, for $i, j \in \{1, 2\}$. Let Θ (resp. Θ') be the equivalence relation such that a_1 and a_2 are related (resp., b_1 and b_2 are related) and no other pair of distinct elements are related. Then the corresponding quotients $(\pi, \mathbf{T})$ and $(\pi', \mathbf{T}')$ are not isomorphic. Indeed, the only semilattice isomorphism χ from $\mathbf{T}$ to $\mathbf{T}'$ sends the class $\{a_1, a_2\}$ to the class $\{b_1, b_2\}$, but then $\pi \circ \chi = \pi'$ fails.

On the other hand, the objects $\mathbf{S} \xrightarrow{\pi} \mathbf{T}$ and $\mathbf{S} \xrightarrow{\pi'} \mathbf{T}'$ are isomorphic in $\mathcal{S}\mathfrak{Epi}$.

3. An internal characterization of surjective homomorphisms between relational systems

We now generalize some arguments from the previous section. We begin with an example.

EXAMPLE 3.1. A *specialization poset* is a partially ordered set together with a preorder coarser than the order. If X is a topological space, then $(\mathcal{P}(X), \subseteq, \sqsubseteq)$ is a specialization poset. As recalled in the introduction, and parallel to the case of specialization semilattices, every specialization poset can be embedded into some "topological" specialization poset as above; moreover, continuous functions correspond exactly to specialization homomorphisms [7, Propositions 5.10 and 2.4].

If $\mathbf{P} = (P, \leq_P)$ and $\mathbf{Q} = (Q, \leq_Q)$ are posets, $\varphi : \mathbf{P} \to \mathbf{Q}$ is an order preserving function and, for $a, b \in P$, we set $a \sqsubseteq_\varphi b$ if $\varphi(a) \leq_Q \varphi(b)$, as in (2), then $(P, \leq_P, \sqsubseteq_\varphi)$ is a specialization poset. As in the previous section, every specialization poset can be constructed in this way, but in this case the argument is much easier. Indeed, if $\sqsubseteq$ is a preorder on some set P, then, classically, $\sqsubseteq$ induces a partial order on $P/{\sim}$, where $\sim$ is the equivalence relation on P defined by $a \sim b$ if both $a \sqsubseteq b$ and $b \sqsubseteq a$. If $\leq$

is an order on P and $\leq$ is finer than $\sqsubseteq$, then the projection π which sends a to $a/{\sim}$ is order preserving from $(P, \leq)$ to $(P/{\sim}, \sqsubseteq/{\sim})$. Then $\sqsubseteq_\pi$ coincides with the original $\sqsubseteq$, hence we have got the desired representation.

By using some arguments from Remark 2.6, we then get the following proposition.

PROPOSITION 3.2. *The category of specialization posets is equivalent to the category $\mathcal{POS}\mathfrak{Surj}$, whose objects are surjective order preserving maps between posets. Morphisms of $\mathcal{POS}\mathfrak{Surj}$ are commuting squares of order preserving maps.*

REMARK 3.3. Notice a difference in comparison with the previous section. Here we cannot do simply with an equivalence relation. Consider the poset $\mathbf{P}$ with just two incomparable elements a and b. Up to isomorphism, we have four surjective order preserving functions from $\mathbf{P}$ to some other poset: the isomorphism, the quotient which identifies a and b and the two bijections onto a poset in which a and b become comparable[1]. On the other hand, there are only two equivalence relations on a set with two elements, hence surjective order preserving maps cannot be characterized just by equivalence relations. See [9] for a survey of what can be done with just an equivalence relation. Note also that the image of a poset under some surjective function need not be transitive, neither antisymmetric (see again [9]).

We now see that Example 3.1 can be generalized to arbitrary models. For the rest of this section we assume that the reader has some familiarity with the basic notions of model theory (see e. g. [5]).

DEFINITION 3.4. (a) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are models for the same language $\mathscr{L}$ and $\varphi : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism. We say that φ is *into classes* if there is some equivalence relation Θ on A such that $B = \{\, a/\Theta \mid a \in A \,\}$, as a set, and moreover $\varphi(a) = a/\Theta$, for every $a \in A$. Of course, if this is the case, then $\Theta = \{\, (a, a_1) \mid \varphi(a) = \varphi(a_1) \,\}$. Clearly, every surjective homomorphism is isomorphic (in the sense of Definition 2.2) to some surjective homomorphism into classes.

(b) Suppose that $\mathbf{A}$ is a model for the language $\mathscr{L}$. An expansion $\mathbf{A}^\star$ of $\mathbf{A}$ is *appropriate for describing a surjective homomorphic image*, for short, simply *appropriate*, if the following conditions hold.

(i) $\mathbf{A}^\star$ is a model for the language $\mathscr{L}^\star$, where $\mathscr{L}^\star$ is obtained from $\mathscr{L}$ by adding, for every relation symbol R of $\mathscr{L}$, a new relation symbol $R^\star$ of the same arity. This is applied also to the equality symbol $\equiv$, and we will write Θ for $\equiv^\star$. Notice that $\mathscr{L}$ is allowed to contain constant and function symbols, but no new constant nor function symbol is added to $\mathscr{L}^\star$. In particular, if $\mathscr{L}$ contains no relation symbol, apart from equality, then Θ is the only additional relation symbol in $\mathscr{L}^\star$.

(ii) The interpretation of Θ in $\mathbf{A}^\star$ is an equivalence relation and, furthermore, the universal closures of the following formulas are satisfied in $\mathbf{A}^\star$

$$R(x_1, \dots, x_n) \Rightarrow R^\star(x_1, \dots, x_n) \tag{4}$$

[1]These last two are better considered to be distinct, since we consider surjective functions up to isomorphism, but keeping $\mathbf{P}$ fixed, thus the elements of P are considered to "have a name". However, the remark in the present footnote is irrelevant for what follows.

$$x_1 \; \Theta \; y_1 \wedge \cdots \wedge x_n \; \Theta \; y_n \Rightarrow f(x_1, \dots, x_n) \; \Theta \; f(y_1, \dots, y_n), \tag{5}$$

$$x_1 \; \Theta \; y_1 \wedge \cdots \wedge x_n \; \Theta \; y_n \wedge R^\star(x_1, \dots, x_n) \Rightarrow R^\star(y_1, \dots, y_n), \tag{6}$$

for every function symbol f and every relation symbol R in $\mathscr{L}$, where we implicitly assume that indexes are appropriate for arities.

In the next proposition the symbols for the $R^\star$s are kept fixed; otherwise, the result holds just up to renaming.

PROPOSITION 3.5. *Let* $\mathbf{A}$ *be a model for the language* $\mathscr{L}$. *There is a bijection between the set of appropriate expansions* $\mathbf{A}^\star$ *of* $\mathbf{A}$ *and the set of surjective homomorphisms into classes with domain* $\mathbf{A}$ *or, put in an equivalent way, the set of equivalence classes of surjective homomorphisms with domain* $\mathbf{A}$, *up to isomorphism as in Definition 2.2.*

Proof. If $\varphi : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism, not necessarily into classes, expand $\mathbf{A}$ to an $\mathscr{L}^\star$ model by interpreting Θ as $\{ (a, a_1) \mid \varphi(a) = \varphi(a_1) \}$ and, for each relation symbol R in $\mathscr{L}$, letting $R^\star(a_1, \dots, a_n)$ hold in $\mathbf{A}^\star$ if $R(\varphi(a_1), \dots, \varphi(a_n))$ holds in $\mathbf{B}$. Then $\mathbf{A}^\star$ is appropriate; indeed, (4) and (5) follow from the assumption that φ is a homomorphism, while (6) holds by construction.

Conversely, if $\mathbf{A}^\star$ is an appropriate expansion of $\mathbf{A}$, let $B = B/\Theta$ and, for every relation symbol $\mathcal{R} \in \mathscr{L}$, let $R(a_1/\Theta, \dots, a_n/\Theta)$ hold in $\mathbf{B}$ if $R^\star(a_1, \dots, a_n)$ holds in $\mathbf{A}^\star$. This is a good definition in view of (6). As standard, for function symbols, let $f(a_1/\Theta, \dots, a_n/\Theta) = f(a_1, \dots, a_n)/\Theta$ in $\mathbf{B}$. This is well-defined because of (5). The projection π sending $a \in A$ to a/Θ is then a surjective homomorphisms into classes, by construction and (4).

If φ in the first paragraph of the proof is actually into classes, then the above constructions are one the inverse of the other, hence they provide a bijection as desired. The last statement is immediate from the last comment in Definition 3.4 (b). $\square$

If $\mathscr{L}$ contains no relation symbol apart from equality, then the only additional relation in $\mathbf{A}^\star$ is Θ. In this case Θ is interpreted as a congruence on the algebraic structure $\mathbf{A}$, hence Definition 3.4(a) generalizes the notion of a congruence and Proposition 3.5 generalizes the basic property of a congruence. Note also that, in the special case of posets, the content of Proposition 3.5 does not correspond exactly to the content of Example 3.1, since, in this special case of Proposition 3.5, $\leq^\star$ is not assumed to be transitive, namely, $\mathbf{B}$ does not necessarily become a poset. Compare the last sentence in Remark 3.3.

Proposition 3.5 is essentially a reformulation of known ideas; see [4, Section 1.4], whose results and methods, in our modest opinion, have not yet received the full attention they deserve. Indeed, very special cases have been subsequently and independently discovered. The special case of graphs has been dealt with in [3], with many applications. The special case of ordered algebras in which operations are either monotone or antitone at each place has been dealt with in [8] ([8] deals only with $\leq^\star$, in the present notation, but, under the assumptions, Θ can be defined in terms of $\leq^\star$). Note that, in the case of posets, if one wants to deal with only an equivalence relation and no auxiliary relation, the situation appears much more involved and with many facets [9]. Many more references can be found in the quoted works.

Acknowledgement. Work performed under the auspices of G.N.S.A.G.A. The author acknowledges the MIUR Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.

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(received 11.07.2024; in revised form 11.07.2025; available online 06.09.2026)

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