

PERIMETRIC CONTRACTIONS OF KANNAN AND CHATTERJEA TYPE IN QUADRILATERALS

Anish Banerjee, Pratikshan Mondal and Lakshmi Kanta Dey

Abstract. This article introduces Kannan type and Chatterjea type perimetric contractions on quadrilaterals and explores their key properties. It provides sufficient conditions for these mappings to achieve fixed points and identifies scenarios ensuring the uniqueness of such fixed points within a complete metric space. Furthermore, the study examines the relationships between these mapping classes and other well-established ones, offering additional insights. To support the theoretical results, several illustrative examples are included. Finally, we provide a remark showing that the arguments in [Karapinar, RNA, 8(1):115–123, 2025] contain a gap and therefore cannot be applied to our results.

1. Introduction

The development of fixed point theory started with the elementary idea of S. Banach [1], submitted in his dissertation in 1922. This work ensures the presence of unique fixed point for contraction mappings in a complete metric space. The study of fixed point theory has advanced over the years, yielding some amazing findings. Most of the works in this field are accomplished by introducing new contraction conditions, considering various underlying spaces, and providing several approaches for attaining the fixed point. For few classical results in the literature of fixed point theory see [3, 6, 7, 9].

Recently, extending Banach's result to the three point analogue, Petrov [12] introduced a class of mappings that contracts the perimeters of triangles. Let us recall that.

DEFINITION 1.1 ([12]). Let (R, d) be a metric space with at least three points. Then a mapping $T : R \rightarrow R$ is defined as contracting perimeters of triangles if there is an $\alpha \in [0, 1)$ such that

$$d(Tp, Tq) + d(Tq, Tr) + d(Tr, Tp) \leq \alpha(d(p, q) + d(q, r) + d(r, p)) \quad (1)$$

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for three pairwise distinct points $p, q, r \in R$.

Such a mapping admits fixed points in a complete metric space if and only if it does not contain any periodic point of prime period 2. Moreover, the number of fixed points is at most two.

Following that, Petrov and Bisht [14] extended the Kannan type mappings to a three point analogue, and developed fixed point results.

DEFINITION 1.2 ([14]). Let (R, d) be a metric space with at least three points. Then $T : R \rightarrow R$ is a generalized Kannan type mapping on R if there is a $0 \leq \delta < \frac{2}{3}$ such that

$$d(Tp, Tq) + d(Tq, Tr) + d(Tr, Tp) \leq \delta(d(p, Tp) + d(q, Tq) + d(r, Tr)) \quad (2)$$

for any three pairwise distinct points $p, q, r \in R$.

Thereafter, Bisht and Petrov [5] introduced the three point analogue of the Chatterjea type mappings. Later, Păcurar and Popescu [11] modified this class of mappings and obtained related fixed point results.

DEFINITION 1.3. ([11]) Let (R, d) be a metric space with at least three points. Then $T : R \rightarrow R$ is a generalized Chatterjea type mapping on R if there is $0 \leq \lambda < \frac{1}{2}$ such that

$$\begin{aligned} d(Tp, Tq) + d(Tq, Tr) + d(Tr, Tp) \\ \leq \lambda[d(p, Tq) + d(p, Tr) + d(q, Tp) + d(q, Tr) + d(r, Tp) + d(r, Tq)] \end{aligned} \quad (3)$$

for any three pairwise distinct points $p, q, r \in R$.

In a complete metric space, both a generalized Kannan type mapping and a generalized Chatterjea type mapping attain fixed points, provided they do not admit periodic points of prime period 2. In both cases, the number of fixed points is at most two.

Recently, Banerjee et al. in [2] introduced the four point analogue of the contraction mapping.

DEFINITION 1.4 ([2]). Let (R, d) be a metric space with at least four points. Then a mapping $T : R \rightarrow R$ is said to be a perimetric contraction on quadrilaterals on R if there is a $\beta \in [0, 1)$ such that for all distinct points $p, q, r, s \in R$ the following holds:

$$d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp) \leq \beta(d(p, q) + d(q, r) + d(r, s) + d(s, p)). \quad (4)$$

In a complete metric space, perimetric contraction on quadrilaterals attains fixed points if it does not contain periodic points of prime period 2 and 3. In this case, it can have at most three fixed points. This study has been further pursued in [4, 8, 13, 15].

E. Karapınar [10] recently showed that the fixed point results of mapping contracting perimeters of triangles, generalized Kannan type mapping and generalized Chatterjea type mapping in metric spaces are equivalent to a derived version of these contractions in the context of G -metric spaces. The author further expressed doubts regarding the originality of fixed point results obtained from similar kinds of contraction mappings. However, this result contains a gap, which is demonstrated in the following.

REMARK 1.5. In [10, p. 5, Theorem 3.1], it is shown that Theorem 2.1 follows from Theorem 2.2, which is not true. In Theorem 2.2, the contraction condition holds for all $x, y, z \in X$, but in Theorem 2.1, the contraction condition holds for all pairwise distinct points $x, y, z \in X$. Thus, we can not apply Theorem 2.2 after satisfying the contraction condition for all pairwise distinct points $x, y, z \in X$. Therefore, the existence and consequently the uniqueness of fixed point can not be ensured. Lastly, for the points to be distinct in the contraction condition, we need to avoid the periodic points of prime period 2. But the matter of periodic points of prime period 2 is skipped throughout Theorem 3.1.

In this article, we focus on an investigation into the four point extension of Kannan type and Chatterjea type mappings to establish adequate conditions ensuring the existence and uniqueness of fixed points. Additionally, we also want to compare these various classes of mapping and find any connections between them.

In Section 2, Kannan type perimetric contraction on quadrilaterals is introduced and a fixed point theorem for these mappings is established. In addition, some inter-relations among those classes of mappings are obtained. Moreover, several non-trivial examples are provided.

In Section 3, Chatterjea type perimetric contraction on quadrilaterals is introduced and a necessary condition is derived for the existence of fixed points of these mappings. Connections among the classes of mappings are explored. Moreover, few examples are provided supporting the theoretical findings.

Throughout the paper, we denote as (R, d) a metric space, $\mathbb{R}$ the set of all real numbers, $\text{Fix}(T)$ the collection of all fixed points of a mapping T and $|X|$ the cardinality of the set X .

Let T be a mapping on the metric space R . A point $y \in R$ is said to be a *periodic point of prime period p* if p is the least positive integer for which $T^p y = y$.

2. Kannan type perimetric contraction on quadrilaterals

We begin the section by introducing the four point analogue of the Kannan type contraction in the following way.

DEFINITION 2.1. Let (R, d) be a metric space with at least four points. A mapping $T : R \rightarrow R$ is said to be a Kannan type perimetric contraction on quadrilaterals if there exists a $\delta \in [0, \frac{1}{2})$ such that

$$\begin{aligned} d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp) \\ \leq \delta(d(p, Tp) + d(q, Tq) + d(r, Tr) + d(s, Ts)) \end{aligned} \quad (5)$$

for all distinct points $p, q, r, s \in R$.

Now, we would like to look into the continuity of these mappings.

PROPOSITION 2.2. *Kannan type perimetric contraction on quadrilaterals are continuous at fixed points.*

Proof. Let (R, d) be a metric space with at least four points and let $T : R \rightarrow R$ be a Kannan type perimetric contraction on quadrilaterals. Suppose that, $a^* \in \text{Fix}(T)$. Let us consider a sequence $\{a_n\}$ in R such that $a_n \rightarrow a^*$ as $n \rightarrow \infty$ with $a_n \neq a_{n+1}, a_n \neq a_{n+2}$ and $a_n \neq a^*$ for all $n \in \mathbb{N}$. We have to show that $Ta_n \rightarrow Ta^*$ as $n \rightarrow \infty$.

Now, from (5) we have,

$$\begin{aligned}
& d(Ta^*, Ta_n) + d(Ta_n, Ta_{n+1}) + d(Ta_{n+1}, Ta_{n+2}) + d(Ta_{n+2}, Ta^*) \\
& \leq \delta[d(a^*, Ta^*) + d(a_n, Ta_n) + d(a_{n+1}, Ta_{n+1}) + d(a_{n+2}, Ta_{n+2})] \\
\implies & d(Ta^*, Ta_n) + d(Ta_n, Ta_{n+1}) + d(Ta_{n+2}, Ta^*) \\
& \leq \delta[d(a_n, a^*) + d(a^*, Ta_n) + d(a_{n+1}, a^*) + d(a^*, Ta_{n+1}) + d(Ta_n, Ta_{n+1}) \\
& \quad + d(a_{n+2}, a^*) + d(a^*, Ta_{n+2})] \\
\implies & d(Ta^*, Ta_n) + d(Ta_n, Ta_{n+1}) + d(Ta_{n+2}, Ta^*) \\
& \leq \delta[d(a_n, a^*) + d(a_{n+1}, a^*) + d(a_{n+2}, a^*)] \\
& \quad + 2\delta[d(Ta^*, Ta_n) + d(Ta_n, Ta_{n+1}) + d(Ta^*, Ta_{n+2})].
\end{aligned}$$

This implies that,

$$\begin{aligned}
& d(Ta^*, Ta_n) + d(Ta_n, Ta_{n+1}) + d(Ta_{n+2}, Ta^*) \\
& \leq \frac{\delta}{1 - 2\delta} [d(a_n, a^*) + d(a_{n+1}, a^*) + d(a_{n+2}, a^*)].
\end{aligned}$$

According to the assumption, $d(a_n, a^*) \rightarrow 0$, $d(a_{n+1}, a^*) \rightarrow 0$ and $d(a_{n+2}, a^*) \rightarrow 0$ as $n \rightarrow \infty$. Thus, $d(Ta_n, Ta^*) \rightarrow 0$ as $n \rightarrow \infty$ and therefore, $Ta_n \rightarrow Ta^*$ as $n \rightarrow \infty$.

Now, consider a sequence $\{a_n\}$ in R such that $a_n \rightarrow a^*$ as $n \rightarrow \infty$ with $a_n \neq a^*$ for all $n \in \mathbb{N}$, but $a_n = a_{n+1}$ and $a_n = a_{n+2}$ may occur. Then, we construct a subsequence, $\{a_{n_p}\}$ by considering those terms for which three consecutive terms are distinct. Then clearly $a_{n_p} \rightarrow a^*$ and $Ta_{n_p} \rightarrow Ta^*$ as $p \rightarrow \infty$. Hence, $Ta_n \rightarrow Ta^*$ as $n \rightarrow \infty$, as $\{Ta_n\}$ is obtained by adding those deleted terms with $\{Ta_{n_p}\}$.

Again, let us consider a sequence $\{a_n\}$ with $a_n = a^*$ for some $n > N$, where $N \in \mathbb{N}$ be arbitrary. Then clearly, $Ta_n \rightarrow Ta^*$ as $n \rightarrow \infty$.

Finally, consider any arbitrary sequence $\{a_n\}$ such that $a_n \rightarrow a^*$ as $n \rightarrow \infty$. Then consider a subsequence, $\{a_{n_p}\}$ by deleting the terms a^* in $\{a_n\}$. Then clearly $a_{n_p} \rightarrow a^*$ as $p \rightarrow \infty$ and by previous result, $Ta_{n_p} \rightarrow Ta^*$ as $p \rightarrow \infty$. Hence, $Ta_n \rightarrow Ta^*$ as $n \rightarrow \infty$ and the result follows. $\square$

The definitions directly lead to the following outcome.

PROPOSITION 2.3. *Let (R, d) be a metric space with at least four points and let $T : R \rightarrow R$ be a Kannan type mapping with $\delta \in [0, \frac{1}{4})$. Then T is a Kannan type perimetric contraction on quadrilaterals.*

In the following result, we show that a sub-collection of perimetric contraction on quadrilaterals is contained in the collection of Kannan type perimetric contraction on quadrilaterals.

PROPOSITION 2.4. *Let (R, d) be a metric space with at least four points and let $T : R \rightarrow R$ be a perimetric contraction on quadrilaterals with $\delta \in [0, \frac{1}{5})$. Then T is a Kannan type perimetric contraction on quadrilaterals.*

Proof. Let T be a perimetric contraction on quadrilaterals with $\delta \in [0, \frac{1}{5})$ and let $p, q, r, s \in R$ be distinct. Then using (5), we have

$$\begin{aligned} & d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp) \\ & \leq \delta[d(p, q) + d(q, r) + d(r, s) + d(s, p)] \\ & \leq \delta[d(p, Tp) + d(Tp, Tq) + d(Tq, q) + d(q, Tq) + d(Tq, Tr) + d(Tr, r) \\ & \quad + d(r, Tr) + d(Tr, Ts) + d(Ts, s) + d(s, Ts) + d(Ts, Tp) + d(Tp, p)]. \end{aligned}$$

Consequently,

$$\begin{aligned} & d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp) \\ & \leq \frac{2\delta}{1-\delta} [d(p, Tp) + d(q, Tq) + d(r, Tr) + d(s, Ts)]. \end{aligned}$$

Thus, T is a Kannan type perimetric contraction on quadrilaterals as $\frac{2\delta}{1-\delta} \in [0, \frac{1}{2})$. $\square$

Now, we obtain a sufficient condition for the existence of fixed point(s) of Kannan type perimetric contraction on quadrilaterals.

THEOREM 2.5. *Let (R, d) be a complete metric space with at least four points, and let $T : R \rightarrow R$ be a Kannan type perimetric contraction on quadrilaterals. Then T admits a fixed point if T does not possess periodic points of prime period 2 and 3. Moreover, $|\text{Fix}(T)| \leq 3$.*

Proof. Suppose that T have no periodic points of prime period 2 and 3. Let $a_0 \in R$ be chosen arbitrarily. Define $Ta_0 = a_1, Ta_1 = a_2, \dots, Ta_n = a_{n+1}, \dots$. If a_i is a fixed point of T for some i , then there is nothing to show.

Suppose that a_i is not a fixed point of T for any i . Then $a_0 \neq a_1, a_1 \neq a_2$ and so on. Since T has no periodic points of prime period 2, then $a_0 \neq a_2, a_1 \neq a_3$ and so on. Also, since T has no periodic points of prime period 3, then $a_0 \neq a_3$, and so on. Therefore, any four consecutive elements of $\{a_n\}$ are distinct. Now, we have

$$\begin{aligned} & d(Ta_i, Ta_{i+1}) + d(Ta_{i+1}, Ta_{i+2}) + d(Ta_{i+2}, Ta_{i+3}) + d(Ta_{i+3}, Ta_i) \\ & \leq \delta(d(a_i, Ta_i) + d(a_{i+1}, Ta_{i+1}) + d(a_{i+2}, Ta_{i+2}) + d(a_{i+3}, Ta_{i+3})) \\ \implies & d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3}) + d(a_{i+3}, a_{i+4}) + d(a_{i+4}, a_{i+1}) \\ & \leq \delta(d(a_i, a_{i+1}) + d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3}) + d(a_{i+3}, a_{i+4})) \\ \implies & (1-\delta)d(a_{i+3}, a_{i+4}) \leq \delta(d(a_i, a_{i+1}) + d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3})) \\ & \quad - (d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3}) + d(a_{i+4}, a_{i+1})) \\ \implies & (1-\delta)d(a_{i+3}, a_{i+4}) \leq \delta(d(a_i, a_{i+1}) + d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3})) - d(a_{i+3}, a_{i+4}) \\ \implies & (2-\delta)d(a_{i+3}, a_{i+4}) \leq \delta(d(a_i, a_{i+1}) + d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3})) \\ \implies & d(a_{i+3}, a_{i+4}) \leq \frac{\delta}{2-\delta} (d(a_i, a_{i+1}) + d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3})) \\ \implies & d(a_{i+3}, a_{i+4}) \leq \frac{3\delta}{2-\delta} \max\{d(a_i, a_{i+1}), d(a_{i+1}, a_{i+2}), d(a_{i+2}, a_{i+3})\}. \end{aligned}$$

Let $\lambda = \frac{3\delta}{2-\delta}$. Then $\lambda \in [0, 1)$ as $\delta \in [0, \frac{1}{2})$. Also, let $k_i = d(a_i, a_{i+1}), i \in \mathbb{N} \cup \{0\}$ and $k = \max\{k_1, k_2, k_3\}$. Then, for any $i \in \mathbb{N}$, we have

$$\begin{aligned} d(a_{i+3}, a_{i+4}) &\leq \lambda \max\{d(a_i, a_{i+1}), d(a_{i+1}, a_{i+2}), d(a_{i+2}, a_{i+3})\} \\ \implies k_{i+3} &\leq \lambda \max\{k_i, k_{i+1}, k_{i+2}\}. \end{aligned}$$

Consequently, we have

$$k_1 \leq k, k_2 \leq k, k_3 \leq k, k_4 \leq \lambda k, k_5 \leq \lambda k, k_6 \leq \lambda k, k_7 \leq \lambda^2 k, \dots$$

Since $\lambda < 1$, so we have

$$\begin{aligned} k_1 &\leq k, k_2 \leq k, k_3 \leq k, k_4 \leq \lambda^{\frac{1}{3}} k, k_5 \leq \lambda^{\frac{2}{3}} k, k_6 \leq \lambda k, k_7 \leq \lambda^{\frac{4}{3}} k, \dots \\ \implies k_n &\leq k \lambda^{\frac{n}{3}-1}, \text{ for all } n \in \mathbb{N} \text{ with } n \geq 4. \end{aligned}$$

Now, for all $n \in \mathbb{N}$ with $n \geq 4$ and for any $p = 1, 2, \dots$, we have

$$\begin{aligned} d(a_n, a_{n+p}) &\leq d(a_n, a_{n+1}) + d(a_{n+1}, a_{n+2}) + \dots + d(a_{n+p-1}, a_{n+p}) \\ &\leq k_n + k_{n+1} + \dots + k_{n+p-1} \\ &\leq k(\lambda^{\frac{n}{3}-1} + \lambda^{\frac{n+1}{3}-1} + \dots + \lambda^{\frac{n+p-1}{3}-1}) \leq k \lambda^{\frac{n}{3}-1} \frac{1 - \lambda^{\frac{p}{3}}}{1 - \lambda^{\frac{1}{3}}}. \end{aligned}$$

Therefore, $d(a_n, a_{n+p}) \rightarrow 0$ as $n \rightarrow \infty$ and for any $p = 1, 2, \dots$. Thus, $\{a_n\}$ is a Cauchy sequence in R and consequently, by completeness of R , there exists an $a^* \in R$ such that $a_n \rightarrow a^*$ as $n \rightarrow \infty$. Now,

$$\begin{aligned} d(a^*, Ta^*) &\leq d(a^*, a_n) + d(a_n, Ta^*) = d(a^*, a_n) + d(Ta_{n-1}, Ta^*) \\ &\leq d(a^*, a_n) + d(Ta^*, Ta_{n-1}) + d(Ta_{n-1}, Ta_{n-2}) + d(Ta_{n-2}, Ta_n) + d(Ta_n, Ta^*) \\ &\leq d(a^*, a_n) + \delta(d(a^*, Ta^*) + d(a_{n-1}, Ta_{n-1}) + d(a_{n-2}, Ta_{n-2}) + d(a_n, Ta_n)), \end{aligned}$$

which implies that

$$(1 - \delta)d(a^*, Ta^*) \leq d(a^*, a_n) + \delta(d(a_{n-1}, a_n) + d(a_{n-2}, a_{n-1}) + d(a_n, a_{n+1})).$$

Taking limit as $n \rightarrow \infty$, we get $Ta^* = a^*$ i.e., $a^* \in \text{Fix}(T)$.

If possible, suppose that T has four distinct fixed points say $p, q, r, s \in R$. Then from (5), we have $d(p, q) + d(q, r) + d(r, s) + d(s, p) \leq 0$, which contradicts the fact that p, q, r, s all are distinct. Hence, $|\text{Fix}(T)| \leq 3$ and this completes the proof. $\square$

Below, we provide an example to show the existence of a Kannan type perimetric contraction on quadrilaterals with exactly three fixed points.

EXAMPLE 2.6. Let $R = \{a, b, c, d\}$ and let the metric d be defined on R by $d(a, b) = d(b, c) = d(a, c) = 1$ and $d(a, d) = d(b, d) = d(c, d) = 10$. Then (R, d) is a complete metric space. Define a mapping $T : R \rightarrow R$ by $Ta = a, Tb = b, Tc = c, Td = a$. Then, T is a Kannan type perimetric contraction on quadrilaterals. Also, T does not contain any periodic points of prime period 2 and 3. Therefore, by Theorem 2.5, $\text{Fix}(T) \neq \emptyset$. Note that, $\text{Fix}(T) = \{a, b, c\}$.

Although the converse of Theorem 2.5 does not necessarily hold in general, in the following result we establish a partial converse of the theorem whose proof is straightforward and hence omitted.

THEOREM 2.7. *Let (R, d) be a complete metric space with at least four points, and let $T : R \rightarrow R$ be a Kannan type perimetric contraction on quadrilaterals. If $\text{Fix}(T) \neq \emptyset$, then T does not possess periodic points of prime period 3.*

It is obvious from Example 2.6 that fixed points of Kannan type perimetric contraction on quadrilaterals may not be unique. On the other hand, we can create a situation where the uniqueness of the fixed point is guaranteed.

PROPOSITION 2.8. *Let (R, d) be a complete metric space. Let $T : R \rightarrow R$ be a Kannan type perimetric contraction on quadrilaterals having no periodic points of prime period 2 and 3. If R contains infinitely many points such that for $a_0 \in R$, the iterative sequence $\{a_0, a_1 = Ta_0, a_2 = Ta_1, \dots\}$ converges to a point $a^* \in R$ with $a^* \neq a_n$; for all $n \in \mathbb{N} \cup \{0\}$, then $\text{Fix}(T) = \{a^*\}$.*

Proof. It follows from Theorem 2.5 that $a^* \in \text{Fix}(T)$. Let $b^* \in \text{Fix}(T)$ such that $a^* \neq b^*$. Then $b^* \neq a_n$, for all $n \in \mathbb{N} \cup \{0\}$, otherwise we have $a^* = b^*$. Therefore, a^*, b^* , and a_n are all distinct, for all $n \in \mathbb{N} \cup \{0\}$. Now, for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} d(Ta^*, Ta_n) + d(Ta_n, Tb^*) + d(Tb^*, Ta_{n+1}) + d(Ta_{n+1}, Ta^*) \\ \leq \delta(d(a^*, Ta^*) + d(a_n, Ta_n) + d(b^*, Tb^*) + d(a_{n+1}, Ta_{n+1})). \end{aligned}$$

This implies that,

$$d(a^*, a_{n+1}) + d(a_{n+1}, b^*) + d(b^*, a_{n+2}) + d(a_{n+2}, a^*) \leq \delta(d(a_n, a_{n+1}) + d(a_{n+1}, a_{n+2})).$$

Now, letting $n \rightarrow \infty$, we get $2d(a^*, b^*) \leq 0$, which is a contradiction to $a^* \neq b^*$. Therefore, $\text{Fix}(T) = \{a^*\}$. $\square$

Below, we mention an example of [14, p. 5, Example 2.5] to distinguish the classes of Kannan type mappings, generalized Kannan type mappings, and Kannan type perimetric contraction on quadrilaterals.

EXAMPLE 2.9. Let $R = [0, 1]$ and let d be the Euclidean metric on R . Consider a mapping $T : R \rightarrow R$ defined by $T(x) = \frac{x}{k}$ for all $x \in R$ and for some $k > 1$.

In [14], it is shown that T is a Kannan type mapping for $k > 3$ and T is a generalized Kannan type mapping for $k > 4$.

Without loss of generality, let us suppose that $a, b, c, d \in [0, 1]$ with $a > b > c > d$. Therefore, by (5), we have

$$\begin{aligned} \frac{1}{k}(a - b + b - c + c - d + a - d) &\leq \lambda \left(1 - \frac{1}{k}\right) (a + b + c + d) \\ \implies 2(a - d) &\leq \lambda(k - 1)(a + b + c + d) \implies a - d \leq \frac{\lambda}{2}(k - 1)(a + b + c + d). \end{aligned}$$

The above inequality holds for all $a, b, c, d \in [0, 1]$ with $a > b > c > d$ if and only if $\frac{\lambda}{2}(k - 1) \geq 1$. Thus, T is a Kannan type perimetric contraction on quadrilaterals if and only if $\lambda \in [0, \frac{1}{2})$. Consequently, $\frac{1}{2} > \lambda \geq \frac{2}{k-1}$.

As a result, T is a Kannan type perimetric contraction on quadrilaterals for $k > 5$.

Now, we give an example in support of Theorem 2.5.

EXAMPLE 2.10. Let $R = \{a, b, c, d\}$ and the metric d be defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } (x, y) = (a, b) \text{ or } (x, y) = (b, a), \\ 3, & \text{if } (x, y) = (a, c) \text{ or } (x, y) = (c, a) \text{ or } (x, y) = (b, c) \text{ or } (x, y) = (c, b), \\ 6, & \text{otherwise.} \end{cases}$$

Then, (R, d) is a complete metric space. Consider $T : R \rightarrow R$ be defined by $Ta = a$, $Tb = b$, $Tc = b$, $Td = a$. Taking $x = a$ and $y = b$, it can be shown that T is neither a Kannan type mapping nor a Chatterjea type mapping. If T is a generalized Kannan type mapping, then we have

$$\begin{aligned} d(Ta, Tb) + d(Tb, Tc) + d(Tc, Ta) &\leq \delta(d(a, Ta) + d(b, Tb) + d(c, Tc)) \\ \implies d(a, b) + d(b, b) + d(b, a) &\leq \delta(d(a, a) + d(b, b) + d(c, b)) \\ \implies 2d(a, b) &\leq \delta d(b, c) \implies 2 \leq 3\delta, \end{aligned}$$

which is a contradiction to (2). Thus, T is not a generalized Kannan type mapping. However, T is a Kannan type perimetric contraction on quadrilaterals with $\delta \in [\frac{4}{9}, \frac{1}{2})$. Note that T does not contain any periodic point of prime period 2 and 3. Therefore by Theorem 2.5, $\text{Fix}(T) \neq \emptyset$. Note that $\text{Fix}(T) = \{a, b\}$.

Below, we provide an example to show that containing periodic points of prime period 2 and 3 restricts from attaining a fixed point of the mapping.

EXAMPLE 2.11. Let (R, d) be a metric space with $R = \{p, q, r, s\}$ and the metric d be defined as

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } (x, y) = (p, r) \text{ or } (x, y) = (r, p), \\ 4, & \text{otherwise.} \end{cases}$$

Define a mapping $T : R \rightarrow R$ by $Tp = r$, $Tq = p$, $Tr = p$, $Ts = r$. Then, T is a Kannan type perimetric contraction on quadrilaterals with $\delta \in [\frac{1}{3}, \frac{1}{2})$. Also, p and r are two periodic points of T of prime period 2. Note that, $\text{Fix}(T) = \emptyset$.

The next example shows that a *discontinuous* Kannan type perimetric contraction on quadrilaterals can indeed have a fixed point.

EXAMPLE 2.12. Consider the Euclidean metric space $(\mathbb{R}, d)$. The mapping $T : \mathbb{R} \rightarrow \mathbb{R}$ is defined as:

$$Ty = \begin{cases} 0, & \text{if } y < 6, \\ 1, & \text{if } y \geq 6. \end{cases}$$

It is easy to show that T is neither a mapping contracting perimeter of triangles nor a perimetric contraction on quadrilaterals.

Without loss of generality, suppose that $p < q < r < s$ and let,

$$U(p, q, r, s) = d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp)$$

and

$$V(p, q, r, s) = d(p, Tp) + d(q, Tq) + d(r, Tr) + d(s, Ts)$$

Now, if $p < q < r < s < 6$, then $U(p, q, r, s) = 0$.

For $p < q < r < 6 \leq s$, we get, $U(p, q, r, s) = 2$ and $V(p, q, r, s) = |p| + |q| + |r| + |s - 1| \geq 5$.

For $p < q < 6 \leq r < s$, we get,

$$U(p, q, r, s) = 2, \quad V(p, q, r, s) = |p| + |q| + |r - 1| + |s - 1| \geq 10$$

and $U(p, r, q, s) = 4, \quad V(p, q, r, s) = |p| + |q| + |r - 1| + |s - 1| \geq 10$.

For $p < 6 \leq q < r < s$, we get, $U(p, q, r, s) = 2$ and $V(p, q, r, s) = |p| + |q - 1| + |r - 1| + |s - 1| \geq 15$.

Now, if $6 \leq p < q < r < s$, then $U(p, q, r, s) = 0$.

Thus, $U(p, q, r, s) \leq \frac{2}{5}V(p, q, r, s)$, for all distinct $p, q, r, s \in \mathbb{R}$ and consequently, T is a Kannan type perimetric contraction on quadrilaterals. Moreover, T has no periodic points of prime period 2 and 3. Hence, by Theorem 2.5, $\text{Fix}(T) \neq \phi$. Note that, $\text{Fix}(T) = \{0\}$.

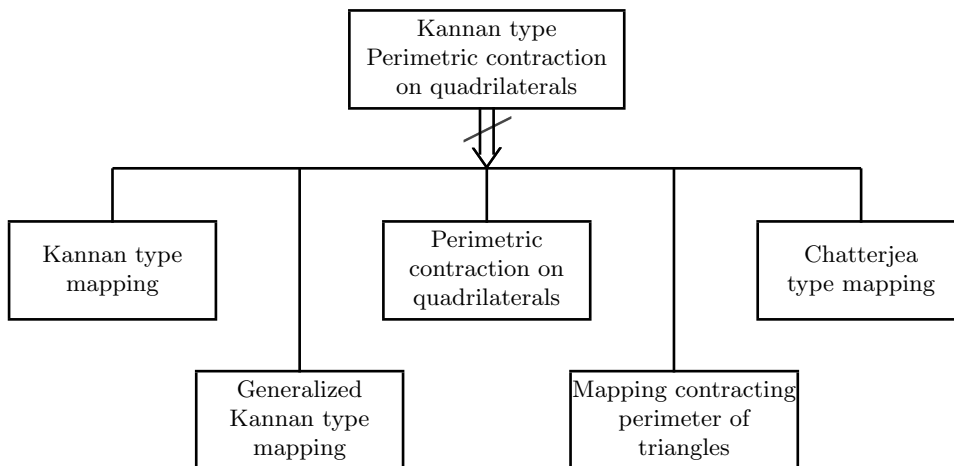


Figure 1: Summary of example implications

Open problem 1. Whether metric completeness is characterized by Kannan type perimetric contraction on quadrilaterals is still to be investigated.

3. Chatterjea type perimetric contraction on quadrilaterals

In this section, we introduce and study the Chatterjea type perimetric contraction on quadrilaterals. We start with the definitions of such mappings.

DEFINITION 3.1. Let (R, d) be a metric space with at least four points. Then a mapping $T : R \rightarrow R$ is said to be a Chatterjea type perimetric contraction on quadrilaterals if there exists a $\lambda \in [0, \frac{1}{3})$ such that

$$d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp)$$

$$\leq \lambda(d(p, Tq) + d(p, Ts) + d(q, Tp) + d(q, Tr) + d(r, Tq) + d(r, Ts) + d(s, Tr) + d(s, Tp)) \quad (6)$$

for all distinct points $p, q, r, s \in R$.

Now, we obtain that these mappings are continuous at their fixed points. The proof is analogous to that of Proposition 2.2 and is therefore omitted.

PROPOSITION 3.2. *Chatterjea type perimetric contraction on quadrilaterals are continuous at fixed points.*

The following result is a direct consequence of the definitions:

PROPOSITION 3.3. *Let (R, d) be a metric space with at least four points and let $T : R \rightarrow R$ be a Chatterjea type mapping with $\lambda \in [0, \frac{1}{3})$. Then T is a Chatterjea type perimetric contraction on quadrilaterals.*

Below, we establish a result to show that a sub-collection of perimetric contraction on quadrilaterals is contained in the collection of Chatterjea type perimetric contraction on quadrilaterals.

PROPOSITION 3.4. *Let (R, d) be a metric space with at least four points and let $T : R \rightarrow R$ be a perimetric contraction on quadrilaterals with $\lambda \in [0, \frac{1}{4})$. Then T is a Chatterjea type perimetric contraction on quadrilaterals.*

Proof. Let $p, q, r, s \in R$ be distinct. Then, from (4) we get

$$\begin{aligned} & d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp) \\ & \leq \lambda(d(p, q) + d(q, r) + d(r, s) + d(s, p)) \\ & \leq \lambda(d(p, Tq) + d(Tq, Tp) + d(Tp, q) + d(q, Tr) + d(Tr, Tq) + d(Tq, r) \\ & \quad + d(r, Ts) + d(Ts, Tr) + d(Tr, s) + d(s, Tp) + d(Tp, Ts) + d(Ts, p)). \end{aligned}$$

Consequently,

$$\begin{aligned} & d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp) \\ & \leq \frac{\lambda}{1-\lambda}(d(p, Tq) + d(p, Ts) + d(q, Tp) + d(q, Tr) \\ & \quad + d(r, Tq) + d(r, Ts) + d(s, Tr) + d(s, Tp)). \end{aligned}$$

As $\frac{\lambda}{1-\lambda} \in [0, \frac{1}{3})$, T is a Chatterjea type perimetric contraction on quadrilaterals. $\square$

Next, we show that a sub-collection of Chatterjea type perimetric contraction on quadrilaterals is contained in the collection of Kannan type perimetric contraction on quadrilaterals.

PROPOSITION 3.5. *Let (R, d) be a metric space with at least four points and let $T : R \rightarrow R$ be a Chatterjea type perimetric contraction on quadrilaterals with $\lambda \in [0, \frac{1}{6})$. Then T is a Kannan type perimetric contraction on quadrilaterals.*

Proof. Let $p, q, r, s \in R$ be distinct. Then, from (4) we get

$$d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp)$$

$$\begin{aligned}
&\leq \lambda(d(p, Tq) + d(p, Ts) + d(q, Tp) + d(q, Tr) \\
&\quad + d(r, Tq) + d(r, Ts) + d(s, Tr) + d(s, Tp)) \\
&\leq \lambda(d(p, Tp) + d(Tp, Tq) + d(p, Tp) + d(Tp, Ts) \\
&\quad + d(q, Tq) + d(Tq, Tp) + d(q, Tq) + d(Tq, Tr) + d(r, Tr) + d(Tr, Tq) \\
&\quad + d(r, Tr) + d(Tr, Ts) + d(s, Ts) + d(Ts, Tr) + d(s, Ts) + d(Ts, Tp)).
\end{aligned}$$

Subsequently, we have

$$\begin{aligned}
&d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp) \\
&\leq \frac{2\lambda}{1-2\lambda}(d(p, Tp) + d(q, Tq) + d(r, Tr) + d(s, Ts)).
\end{aligned}$$

As $\frac{2\lambda}{1-2\lambda} \in [0, \frac{1}{2})$, T is a Kannan type perimetric contraction on quadrilaterals. $\square$

We present the following result that ensures the existence of fixed point(s) of Chatterjea type perimetric contraction on quadrilaterals:

THEOREM 3.6. *Let (R, d) be a complete metric space with at least four points, and let $T : R \rightarrow R$ be a Chatterjea type perimetric contraction on quadrilaterals. Then T attains a fixed point if T does not possess periodic points of prime period 2 and 3. Moreover, $|\text{Fix}(T)| \leq 3$.*

Proof. Assume that T has no periodic points of prime period 2 and 3. Let $a_0 \in R$ be chosen arbitrarily. Define $Ta_0 = a_1, Ta_1 = a_2, \dots, Ta_n = a_{n+1}, \dots$. If a_i is a fixed point of T for any $i \in \mathbb{N} \cup \{0\}$, then we are done.

Since a_i is not a fixed point of T for any i and since T has no periodic points of prime period 2 and 3, proceeding similarly as in Theorem 2.5, we can show that any four consecutive elements of the sequence $\{a_n\}$ are distinct. Now, for any $i \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
&d(Ta_i, Ta_{i+1}) + d(Ta_{i+1}, Ta_{i+2}) + d(Ta_{i+2}, Ta_{i+3}) + d(Ta_{i+3}, Ta_i) \\
&\leq \lambda(d(a_i, Ta_{i+1}) + (d(a_i, Ta_{i+3}) + d(a_{i+1}, Ta_i) + d(a_{i+1}, Ta_{i+2}) \\
&\quad + d(a_{i+2}, Ta_{i+1}) + d(a_{i+2}, Ta_{i+3}) + d(a_{i+3}, Ta_{i+2}) + d(a_{i+3}, Ta_i)).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3}) + d(a_{i+3}, a_{i+4}) + d(a_{i+4}, a_{i+1}) \\
&\leq \lambda(d(a_i, a_{i+2}) + d(a_i, a_{i+4}) + d(a_{i+1}, a_{i+1}) + d(a_{i+1}, a_{i+3}) \\
&\quad + d(a_{i+2}, a_{i+2}) + d(a_{i+2}, a_{i+4}) + d(a_{i+3}, a_{i+3}) + d(a_{i+3}, a_{i+1})) \\
&\leq \lambda(d(a_i, a_{i+1}) + d(a_{i+1}, a_{i+2}) + d(a_i, a_{i+3}) + d(a_{i+3}, a_{i+4}) + d(a_{i+1}, a_{i+2}) \\
&\quad + d(a_{i+2}, a_{i+3}) + d(a_{i+2}, a_{i+3}) + d(a_{i+3}, a_{i+4}) + d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3})) \\
&\leq \lambda(d(a_i, a_{i+1}) + d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3}) + d(a_{i+3}, a_i)) \\
&\quad + 2\lambda(d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3}) + d(a_{i+3}, a_{i+4}) + d(a_{i+4}, a_{i+1})).
\end{aligned}$$

Hence,

$$d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3}) + d(a_{i+3}, a_{i+4}) + d(a_{i+4}, a_{i+1})$$

$$\leq \frac{\lambda}{1-2\lambda}(d(a_i, a_{i+1}) + d(a_{i+1}, a_{i+2}) + d(a_{i+2}, a_{i+3}) + d(a_{i+3}, a_i)).$$

Let $\lambda_n = d(a_n, a_{n+1}) + d(a_{n+1}, a_{n+2}) + d(a_{n+2}, a_{n+3}) + d(a_{n+3}, a_n)$ so that $\lambda_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Again, let, $\delta = \frac{\lambda}{1-2\lambda}$. Then $\delta \in [0, 1)$ as $\lambda \in [0, \frac{1}{3})$.

Now, for any $n \in \mathbb{N}$, we have $\lambda_n \leq \delta\lambda_{n-1}$. Then, it is clear that

$$d(a_0, a_1) \leq \lambda_0, \quad d(a_1, a_2) \leq \lambda_1 \leq \delta\lambda_0, \quad \dots, \quad d(a_n, a_{n+1}) \leq \lambda_n \leq \delta^n \lambda_0.$$

So, for all $n \in \mathbb{N} \cup \{0\}$ and for any $p = 1, 2, 3, \dots$, we have

$$\begin{aligned} d(a_n, a_{n+p}) &\leq d(a_n, a_{n+1}) + d(a_{n+1}, a_{n+2}) + \dots + d(a_{n+p-1}, a_{n+p}) \\ &\leq \delta^n \lambda_0 + \delta^{n+1} \lambda_0 + \dots + \delta^{n+p-1} \lambda_0 \leq \delta^n (1 + \delta + \dots + \delta^{p-1}) \lambda_0 \leq \delta^n \frac{1 - \delta^p}{1 - \delta} \lambda_0. \end{aligned}$$

Thus, $d(a_n, a_{n+p}) \rightarrow 0$ as $n \rightarrow \infty$ and for any $p = 1, 2, \dots$. Therefore, $\{a_n\}$ is a Cauchy sequence and hence is convergent as R is complete. Let $a_n \rightarrow a^* \in R$. Now,

$$\begin{aligned} d(a^*, Ta^*) &\leq d(a^*, a_n) + d(a_n, Ta^*) \\ &\leq d(a^*, a_n) + d(Ta^*, Ta_{n-1}) + d(Ta_{n-1}, Ta_{n-2}) + d(Ta_{n-2}, Ta_n) + d(Ta_n, Ta^*) \\ &\leq d(a^*, a_n) + \lambda(d(a^*, Ta_{n-1}) + d(a^*, Ta_n) + d(a_{n-1}, Ta^*) + d(a_{n-1}, Ta_{n-2}) \\ &\quad + d(a_{n-2}, Ta_{n-1}) + d(a_{n-2}, Ta_n) + d(a_n, Ta_{n-2}) + d(a_n, Ta^*)) \\ &\leq d(a^*, a_n) + \lambda(d(a^*, a_n) + d(a^*, a_{n+1}) + d(a_{n-1}, Ta^*) + d(a_{n-1}, a_{n-1}) \\ &\quad + d(a_{n-2}, a_n) + d(a_{n-2}, a_{n+1}) + d(a_n, a_{n-1}) + d(a_n, Ta^*)). \end{aligned}$$

Letting $n \rightarrow \infty$, we get $(1 - 2\lambda)d(a^*, Ta^*) \leq 0$. This implies that $Ta^* = a^*$.

If possible, suppose that T has four distinct fixed points, say, $p, q, r, s \in R$. Then from (6), we have

$$\begin{aligned} d(p, q) + d(q, r) + d(r, s) + d(s, p) &\leq 2\lambda(d(p, q) + d(q, r) + d(r, s) + d(s, p)) \\ \implies (1 - 2\lambda)(d(p, q) + d(q, r) + d(r, s) + d(s, p)) &\leq 0, \end{aligned}$$

which contradicts the fact that p, q, r, s are all distinct, since $(1 - 2\lambda) > 0$. Thus, $|\text{Fix}(T)| \leq 3$. This completes the proof. $\square$

Now, we provide an example of a Chatterjea type perimetric contraction on quadrilaterals with three fixed points.

EXAMPLE 3.7. Let $R = \{w, x, y, z\}$ be equipped with the metric d defined on R by, $d(x, y) = d(y, z) = d(z, x) = 1$ and $d(x, w) = d(y, w) = d(z, w) = 5$. Then, (R, d) is a complete metric space. Define a mapping $T : R \rightarrow R$ by $Tw = x$, $Tx = x$, $Ty = y$, $Tz = z$. Then, T is a Chatterjea type perimetric contraction on quadrilaterals with $\lambda \in [\frac{4}{15}, \frac{1}{3})$ having no periodic points of prime period 2 and 3. Therefore, by Theorem 3.6, $\text{Fix}(T) \neq \emptyset$. Note that, $\text{Fix}(T) = \{x, y, z\}$.

Below we present a partial converse of Theorem 3.6 whose proof is straightforward and therefore omitted.

THEOREM 3.8. *Let (R, d) be a complete metric space with at least four points, and let $T : R \rightarrow R$ be a Chatterjea type perimetric contraction on quadrilaterals. If $\text{Fix}(T) \neq \emptyset$, then T does not contain periodic points of prime period 3.*

In Example 3.7, we saw that a Chatterjea type perimetric contraction on quadrilaterals may have more than one fixed point. However, uniqueness can also be guaranteed under suitable conditions.

PROPOSITION 3.9. *Let (R, d) be a complete metric space, and let $T : R \rightarrow R$ be a Chatterjea type perimetric contraction on quadrilaterals having no periodic point of prime period 2 and 3. If R contains infinitely many points such that for $a_0 \in R$, the iterative sequence $\{a_0, a_1 = Ta_0, a_2 = Ta_1, \dots\}$, converges to a point $a^* \in R$ with $a^* \neq a_i$; for all $i \in \mathbb{N} \cup \{0\}$, then $\text{Fix}(T) = \{a^*\}$.*

Proof. Theorem 3.6 assures that $a^* \in \text{Fix}(T)$. Let b^* be another fixed point of T . Then $b^* \neq a_i$, for all $i \in \mathbb{N} \cup \{0\}$, otherwise we have $a^* = b^*$. Therefore, a^*, b^*, a_i are all distinct, for all $i \in \mathbb{N} \cup \{0\}$.

Now, for all $i \in \mathbb{N} \cup \{0\}$, let

$$\lambda_i = \frac{d(Ta^*, Ta_i) + d(Ta_i, Tb^*) + d(Tb^*, Ta_{i+1}) + d(Ta_{i+1}, Ta^*)}{d(a^*, Ta_i) + d(a^*, Ta_{i+1}) + d(a_i, Ta^*) + d(a_i, Tb^*) + d(b^*, Ta_i) + d(b^*, Ta_{i+1}) + d(a_{i+1}, Tb^*) + d(a_{i+1}, Ta^*)}.$$

Consequently,

$$\lambda_i = \frac{d(a^*, a_{i+1}) + d(a_{i+1}, b^*) + d(b^*, a_{i+2}) + d(a_{i+2}, a^*)}{d(a^*, a_{i+1}) + d(a^*, a_{i+2}) + d(a_i, a^*) + d(a_i, b^*) + d(b^*, a_{i+1}) + d(b^*, a_{i+2}) + d(a_{i+1}, b^*) + d(a_{i+1}, a^*)}.$$

Then $\lambda_i \leq \lambda$ for all $i \in \mathbb{N} \cup \{0\}$. Now, letting $i \rightarrow \infty$, we get $\lambda_i \rightarrow 1$ - which contradicts (6). Therefore, $\text{Fix}(T) = \{a^*\}$. $\square$

EXAMPLE 3.10. Let $R = \{p, q, r, s\}$ and let d be defined on R as $d(r, s) = 1$, $d(q, r) = d(q, s) = 2$, and $d(p, q) = d(p, r) = d(p, s) = 3$ so that (R, d) becomes a complete metric space. Define $T : R \rightarrow R$ by $Tp = s$, $Tq = r$, $Tr = r$, $Ts = s$.

Now, T is not a Kannan type mapping as $d(Tq, Ts) = \frac{1}{2}(d(q, Tq) + d(s, Ts))$. Also, T is not a Chatterjea type mapping as $d(Tr, Ts) = \frac{1}{2}(d(r, Ts) + d(s, Tr))$. Moreover, T is not a generalized Kannan type mapping as $d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tq) = d(q, Tq) + d(r, Tr) + d(s, Ts)$. Finally, T is not a Kannan type perimetric contraction on quadrilaterals as $d(Tp, Tq) + d(Tq, Ts) + d(Ts, Tr) + d(Tr, Tp) = 4$ and $d(p, Tp) + d(q, Tq) + d(s, Ts) + d(r, Tr) = 5$.

Suppose,

$$\begin{aligned} U(p, q, r, s) &= d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp) \\ V(p, q, r, s) &= d(p, Tq) + d(p, Ts) + d(q, Tp) \\ &\quad + d(q, Tr) + d(r, Tq) + d(r, Ts) + d(r, Tr) + d(s, Ts). \end{aligned}$$

Now, we have,

$$\begin{aligned} U(p, q, r, s) &= U(p, r, q, s) = 2, & U(p, q, s, r) &= 4, \\ V(p, q, r, s) &= V(p, r, q, s) = 12, & V(p, q, s, r) &= 14. \end{aligned}$$

Thus, $U(p, q, r, s) \leq \frac{2}{7}V(p, q, r, s)$, for all distinct $p, q, r, s \in R$. So, T is a Chatterjea type perimetric contraction on quadrilaterals. Moreover, T has no periodic point of

prime period 2 or 3. Thus, by Theorem 3.6, T has at least one fixed point. In this case, $\text{Fix}(T) = \{r, s\}$.

In the following, we provide examples of Chatterjea type perimetric contraction on quadrilaterals, demonstrating its distinctness from some well-known classes of mappings.

EXAMPLE 3.11. Let us consider the Euclidean metric space $(\mathbb{R}, d)$. Consider the mapping $T : \mathbb{R} \rightarrow \mathbb{R}$ as:

$$Ta = \begin{cases} 0, & \text{if } a < 4, \\ 1, & \text{if } a \geq 4. \end{cases}$$

It is easy to see that T is neither a mapping contracting perimeter of triangles nor a perimetric contraction on quadrilaterals.

Without loss of generality, suppose that $p < q < r < s$ and let,

$$U(p, q, r, s) = d(Tp, Tq) + d(Tq, Tr) + d(Tr, Ts) + d(Ts, Tp)$$

and

$$V(p, q, r, s) = d(p, Tq) + d(p, Ts) + d(q, Tp) + d(q, Tr) \\ + d(r, Tq) + d(r, Ts) + d(s, Tr) + d(s, Tp).$$

Applying the same technique as in Example 2.12, it is easy to see that $U(p, q, r, s) \leq \frac{1}{4}V(p, q, r, s)$, for all distinct $p, q, r, s \in \mathbb{R}$. Consequently, T is a Chatterjea type perimetric contraction on quadrilaterals. Moreover, T has no periodic point of prime period 2 and 3. Hence, by Theorem 3.6, $\text{Fix}(T) \neq \phi$. Note that here, $\text{Fix}(T) = \{0\}$.

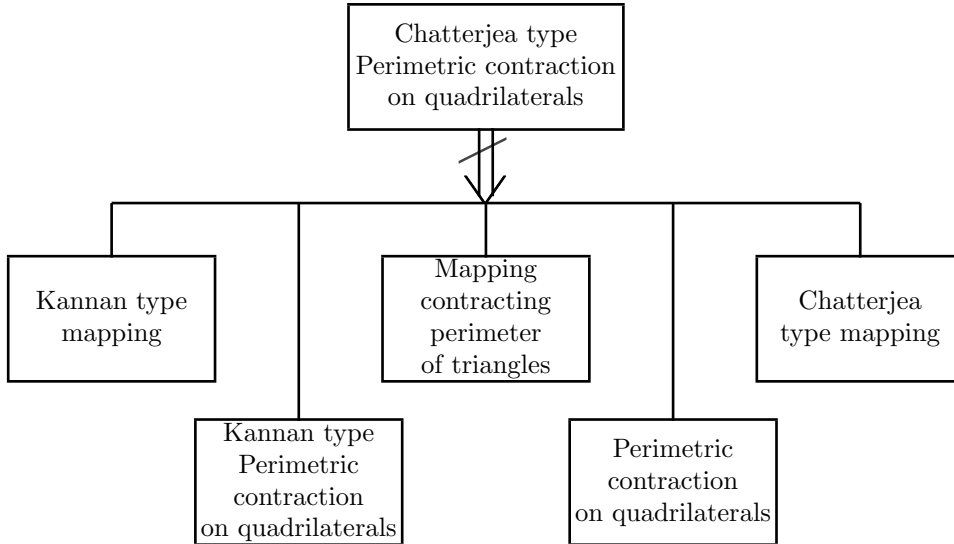


Figure 2: Flowchart of implications from the examples

Open problem 2. One can look into the classes of generalized Chatterjea type mapping and Chatterjea type perimetric contraction on quadrilaterals to check whether these classes are interrelated.

4. Conclusion

In this paper, we have extended the classical Kannan and Chatterjea type mappings to a four-point analogue by introducing the notions of *Kannan type perimetric contraction on quadrilaterals* and *Chatterjea type perimetric contraction on quadrilaterals*, formulated through the concept of mappings that contract the perimeter of quadrilaterals. Fixed point results for these mappings have been established in complete metric spaces.

Our analysis shows that the existence of fixed points requires the exclusion of periodic points of prime periods 2 and 3. Furthermore, these mappings can have at most three fixed points, which enables us to derive a necessary condition for the uniqueness of a fixed point.

Consequently, we demonstrate that a subclass of Kannan type mappings and Chatterjea type mappings is naturally contained in the broader classes of their respective perimetric contractions on quadrilaterals. To support our theoretical results, we provide non-trivial examples that illustrate the distinctions between these newly introduced classes and several well-known classes of mappings.

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Department of Mathematics, National Institute of Technology, Durgapur, West Bengal, India

E-mail: anish22.mathematics@gmail.com

ORCID iD: <https://orcid.org/0009-0009-3399-1596>

Department of Mathematics, A.B.N. Seal College, Cooch Behar, West Bengal, India

E-mail: real.analysis77@gmail.com

ORCID iD: <https://orcid.org/0000-0002-9678-9610>

Department of Mathematics, National Institute of Technology, Durgapur, West Bengal, India

E-mail: lakshmikdey@yahoo.co.in

ORCID iD: <https://orcid.org/0000-0001-5389-6048>