

ULAM STABILITY FOR MULTI-TERM FRACTIONAL DELAY DIFFERENTIAL EQUATIONS EQUIPPED WITH NONLOCAL MULTI-POINT AND MULTI-STRIP BOUNDARY CONDITIONS

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Abstract. In this paper we focus on Ulam type stability for a class of nonlocal boundary value problems containing multi-term delay fractional differential equations and nonlocal multi-point and multi-strip boundary conditions. The application of Ulam stability to any type of boundary conditions causes some misunderstandings which are mainly connected with the solutions of the applied inequality and the corresponding boundary condition. If the inequality is not defined in an appropriate way, then its solutions could not satisfy the boundary condition and the corresponding integral presentation is not valid. In this paper we suggest some ways to avoid these misunderstandings. In one of them we consider a fractional integral inequality. In the second one we suggest a fractional differential inequality with a boundary condition. In both cases we slightly change the classical definition of Ulam stability. In the last way we consider a parameter in the boundary condition and a fractional differential inequality. This parameter depends significantly on the chosen solution of the fractional differential inequality. We illustrate the three suggested approaches only on Ulam-Hyers stability and generalized Ulam-Hyers stability with appropriate changes of the classical definitions. The same approaches could be applied similarly to other types of stability such as Ulam-Hyers-Rassias stability, generalized Ulam-Hyers-Rassias stability.

1. Introduction

Boundary value problems for the fractional differential equations in which the unknown function is included both under the sign of fractional derivative and the boundary conditions, were studied by many authors. The interest is connected with their applications in problems of physics, mechanics and other applied sciences (see, for example, [5, 12, 14]). Several qualitative problems such as existence, uniqueness, stability, are studied for various types of fractional derivatives and linear and nonlinear boundary conditions. One of these problems is Ulam type stability, such as Ulam-Hyers

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stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, generalized Ulam-Hyers-Rassias stability. Note that when Ulam type of stability is studied for initial value problems, the base of the study is to be defined in appropriate way a differential inequality and then to be compared the closeness of an arbitrary chosen solution of this inequality and a solution of the studied problem. The main point in this case is that the initial value of the solution of given problem is taken to be equal to the initial value of the chosen solution of the inequality. By this way there is a deep connection between both, the chosen solution of the inequality and the solution of the studied problem. When Ulam type of stability is applied to a boundary value problem the same idea has to be kept (see, for example, [15]). Otherwise there are some misunderstandings in the application and proof of the Ulam type of stability.

The main purpose of this paper is to consider a nonlinear multi-term delay fractional differential equation with nonlocal multi-point and multi-substrip boundary conditions and to apply Ulam type of stability (BVP). The importance of the multi-strip type boundary conditions corresponds to the values of the unknown function given on the segments of arbitrary lengths of the given domain (see, for example, the applications studied in [3, 7, 9–11, 13]).

We focus on Ulam-Hyers stability and generalized Ulam-Hyers stability and suggest three different approaches to study Ulam type of stability and to avoid some misunderstandings in the literature. In the first approach we consider a fractional integral inequality. It is based on the integral presentation of the studied boundary value problem. In the second approach we consider a fractional differential inequality combined with the boundary conditions. In both approaches the solution of the studied BVP is given and it does not depend on the chosen solution of the corresponding inequality. In connection with this we define a modified Ulam-Hyers stability and modified generalized Ulam-Hyers stability. To keep the mentioned above idea of Ulam stability applied to initial value problems and to have a deep connection between the chosen solution of the inequality and the solution of BVP we suggest the last approach- we introduce a parameter in the boundary conditions and later we choose this parameter to depend on the values of the chosen solution of the inequality. Note this idea was suggested in [1, 2], and applied to linear boundary value problems for Caputo fractional differential equations. In our paper in all three approaches we obtain sufficient conditions for Ulam type stability.

The same approaches could be applied similarly to other types of stability such as Ulam-Hyers-Rassias stability, generalized Ulam-Hyers-Rassias stability.

2. Statement of the problem and some known results

We consider the nonlinear multi-term delay fractional differential equation

$$\sum_{i=1}^q \alpha_i {}^c\mathcal{D}^{\gamma_i} y(t) = f(t, y(t), y(\omega t)), \quad t \in (0, 1], \quad (1)$$

with nonlocal multi-point and multi-substrip boundary data (BVP)

$$\begin{aligned} y(0) &= y_0, \quad \frac{d^\zeta y(0)}{dt^\zeta} = 0, \quad \zeta = 1, 2, \dots, q-2, \\ y(1) &= \sum_{j=1}^n b_j \int_{\xi_j}^{\eta_j} y(s) ds + \sum_{k=1}^m a_k y(\sigma_k), \end{aligned} \quad (2)$$

where q is a given natural number, ${}^c\mathcal{D}^{\gamma_i}$ denotes the Caputo fractional derivative of order γ_i with $\gamma_\rho \in (q-1, q) : \rho \in [1, r]$ is a natural number, $\gamma_i \in (0, 1)$ for $i = 1, 2, \dots, r$, $i \neq \rho$, $\omega \in (0, 1)$, $\alpha_i, b_j, a_k \in \mathbb{R}$, $i \in \{1, 2, \dots, q\}$, $j \in \{1, 2, \dots, n\}$, $k \in \{1, 2, \dots, m\}$, $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $\sigma_k \in (0, 1)$, $\xi_j, \eta_j \in (0, 1)$, $\xi_j < \eta_j$, $k = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.

Consider the space $C^q([0, 1], \mathbb{R})$ with a norm $\|u\| = \sup_{t \in [0, 1]} |u(t)|$ for $u \in C^q([0, 1], \mathbb{R})$.

We will introduce the following assumption :

Assumption A1. Let $\beta_1 = 1 - \frac{1}{q} \sum_{j=1}^n b_j (\eta_j^q - \xi_j^q) - \sum_{k=1}^m a_k \sigma_k^{q-1} \neq 0$.

We represent some basic definitions given in the general case in [8] and adopted to our studied problem.

DEFINITION 2.1. The Riemann-Liouville fractional integral of order $\omega > 0$ for the function $v : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$I_0^\omega v(t) = \frac{1}{\Gamma(\omega)} \int_0^t \frac{v(\vartheta)}{(t-\vartheta)^{1-\omega}} d\vartheta, \quad t \in (0, 1],$$

where Γ denotes the Euler gamma function.

DEFINITION 2.2. Let $\omega > 0 : p-1 < \omega < p$, $p = [\omega] + 1$ with $[\omega]$ equals to the integer part of ω . The Caputo fractional derivative of fractional order $\omega > 0$ for the function $v : [0, 1] \rightarrow \mathbb{R}$, is defined by

$${}^cD^\omega v(t) = \frac{1}{\Gamma(p-\omega)} \int_0^t \frac{v^{(p)}(\vartheta)}{(t-\vartheta)^{1+\omega-p}} d\vartheta, \quad t \in (0, 1].$$

Note BVP (1) is studied in [4] and the existence is investigated. The following result will be applied in this study:

LEMMA 2.3 ([4]). Let **Assumption A1.** be satisfied and $y \in C([0, 1], \mathbb{R})$. Then the solution of the linear problem:

$$\begin{cases} \sum_{i=1}^q \alpha_i {}^c\mathcal{D}^{\gamma_i} v(t) = y(t), & t \in [0, 1], \\ v(0) = v_0, \quad v^{(\zeta)}(0) = 0, & \zeta = 1, 2, 3, \dots, q-2, \\ v(1) = \sum_{j=1}^n b_j \int_{\xi_j}^{\eta_j} v(s) ds + \sum_{k=1}^m a_k v(\sigma_k), \end{cases} \quad (3)$$

is given by

$$v(t) = v_0 + \frac{t^{q-1}}{\beta_1} \left[v_0 \left(\sum_{j=1}^n b_j (\eta_j^q - \xi_j^q) + \sum_{k=1}^m a_k - 1 \right) \right]$$

$$\begin{aligned}
& + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \sum_{j=1}^n b_j \int_{\xi_j}^{\eta_j} \int_0^s (s-z)^{\gamma_\rho-1} y(z) dz ds \\
& - \sum_{j=1}^n b_j \sum_{i=1, i \neq \rho}^q \frac{\alpha_i}{\alpha_\rho} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_{\xi_j}^{\eta_j} \int_0^s (s-z)^{\gamma_\rho-\gamma_i-1} v(z) dz ds \\
& + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \sum_{k=1}^m a_k \int_0^{\sigma_k} (\sigma_k - z)^{\gamma_\rho-1} y(z) dz \\
& - \sum_{k=1}^m a_k \sum_{i=1, i \neq \rho}^q \frac{\alpha_i}{\alpha_\rho} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_0^{\sigma_k} (\sigma_k - z)^{\gamma_\rho-\gamma_i-1} v(z) dz \\
& - \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \int_0^1 (1-z)^{\gamma_\rho-1} y(z) dz \\
& + \sum_{i=1, i \neq \rho}^q \frac{\alpha_i}{\alpha_\rho} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_0^1 (1-z)^{\gamma_\rho-\gamma_i-1} v(z) dz \Big] \\
& + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \int_0^t (t-z)^{\gamma_\rho-1} y(z) dz \\
& - \sum_{i=1, i \neq \rho}^q \frac{\alpha_i}{\alpha_\rho} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_0^t (t-z)^{\gamma_\rho-\gamma_i-1} v(z) dz.
\end{aligned} \tag{4}$$

Based on Lemma 2.3 and equality (4) we define the operator $\mathcal{F} : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ by (see [4])

$$\begin{aligned}
\mathcal{F}(v) = & v_0 + \frac{t^{q-1}}{\beta_1} \left[v_0 \left(\sum_{j=1}^n b_j (\eta_j^q - \xi_j^q) + \sum_{k=1}^m a_k - 1 \right) \right. \\
& + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \sum_{j=1}^n b_j \int_{\xi_j}^{\eta_j} \int_0^s (s-z)^{\gamma_\rho-1} f(z, v(z), v(\omega z)) dz ds \\
& - \sum_{j=1}^n b_j \sum_{i=1, i \neq \rho}^q \frac{\alpha_i}{\alpha_\rho} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_{\xi_j}^{\eta_j} \int_0^s (s-z)^{\gamma_\rho-\gamma_i-1} v(z) dz ds \\
& + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \sum_{k=1}^m a_k \int_0^{\sigma_k} (\sigma_k - z)^{\gamma_\rho-1} f(z, v(z), v(\omega z)) dz \\
& - \sum_{k=1}^m a_k \sum_{i=1, i \neq \rho}^q \frac{\alpha_i}{\alpha_\rho} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_0^{\sigma_k} (\sigma_k - z)^{\gamma_\rho-\gamma_i-1} v(z) dz \\
& - \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \int_0^1 (1-z)^{\gamma_\rho-1} f(z, v(z), v(\omega z)) dz \\
& \left. + \sum_{i=1, i \neq \rho}^q \frac{\alpha_i}{\alpha_\rho} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_0^1 (1-z)^{\gamma_\rho-\gamma_i-1} v(z) dz \right]
\end{aligned} \tag{5}$$

$$\begin{aligned}
& + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \int_0^t (t-z)^{\gamma_\rho-1} f(z, v(z), v(\omega z)) dz \\
& - \sum_{i=1, i \neq \rho}^q \frac{\alpha_i}{\alpha_\rho} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_0^t (t-z)^{\gamma_\rho - \gamma_i - 1} v(z) dz.
\end{aligned}$$

In [4] it is written without a proof that the existence of a fixed point of the operator $\mathcal{F}$ in the set $C([0, 1], \mathbb{R})$ implies the existence of a solution of the problem (1), (2). It would be correct if the fractional orders of the applied fractional derivatives are from the interval $(0, 1)$, but here $\gamma_\rho \in (q-1, q)$.

We will use the following known in the literature results.

PROPOSITION 2.4. – *If $v \in C[0, 1]$ and $\gamma > 0$ then ${}^c D^\gamma(I_0^\gamma v(t)) = v(t)$ (see [8, Lemma 2.21]).*

– *If $v \in C^q[0, 1]$ and $\gamma_\rho \in (q-1, q)$ then $I_0^{\gamma_\rho}({}^c D^{\gamma_\rho} v(t)) = v(t) - \sum_{k=0}^{q-1} \frac{v^{(k)}(0)}{k!} t^k$ (see [8, Lemma 2.22]).*

– *If $\gamma \in (0, 1)$ and $v \in C[0, 1]$ then $I_0^\gamma({}^c D^\gamma v(t)) = v(t) - v(0)$ (see [8, Lemma 2.22]).*

Based on Proposition 2.4 we obtain the following lemma.

LEMMA 2.5. *Let $\gamma_\rho \in (q-1, q)$, $\rho \leq r$ is a natural number, $\gamma_i \in (0, 1)$ for $i = 1, 2, \dots, r$, $i \neq \rho$.*

If the function $v \in C([0, 1], \mathbb{R})$ is a fixed point of the operator $\mathcal{F}$ defined by (5), then the function v is a solution of BVP (1), (2).

If the function $v \in C^q([0, 1], \mathbb{R})$ is a solution of BVP (1), (2), then it is a fixed point of the operator $\mathcal{F}$ defined by (5).

COROLLARY 2.6. *Any solution $v \in C^q([0, 1], \mathbb{R})$ of BVP (1), (2) satisfies the integral equations $v(t) = \mathcal{F}v(t)$, $t \in [0, 1]$, where the operator $\mathcal{F}$ is defined by (5).*

In the forthcoming analysis, we need the following assumption:

Assumption A2. The function $f \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exist $L_1, L_2 > 0$, such that

$$|f(t, v_1, v_2) - f(t, y_1, y_2)| \leq L_1 \|v_1 - y_1\| + L_2 \|v_2 - y_2\| \text{ for } t \in [0, 1], v_1, v_2, y_1, y_2 \in \mathbb{R}.$$

We will apply Banach contraction principle to prove the existence and uniqueness of a solution of BVP (1), (2). Note, this principle is applied in [4] but since there are several mistakes in the proof of [4, Theorem 3.5] we will prove the result in our paper.

THEOREM 2.7. *Let the **Assumption A1.** and **Assumption A2.** be satisfied and*

$\lambda < 1$, where

$$\begin{aligned} \lambda = \frac{1}{|\beta_1|} & \left[\frac{(L_1 + L_2)}{|\alpha_\rho| \Gamma(2 + \gamma_\rho)} \sum_{j=1}^n |b_j| (\xi_j^{\gamma_\rho+1} - \eta_j^{\gamma_\rho}) \right. \\ & + \sum_{j=1}^n |b_j| \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(2 + \gamma_\rho - \gamma_i)} (\xi_j^{\gamma_\rho - \gamma_i + 1} - \eta_j^{\gamma_\rho - \gamma_i + 1}) \\ & + \frac{L_1 + L_2}{\alpha_\rho \Gamma(1 + \gamma_\rho)} \left(1 + |\beta_1| + \sum_{k=1}^m |a_k| \sigma_k^{\gamma_\rho} \right) \\ & + (1 + |\beta_1|) \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(1 + \gamma_\rho - \gamma_i)} \\ & \left. + \sum_{k=1}^m |a_k| \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(1 + \gamma_\rho - \gamma_i)} \sigma_k^{\gamma_\rho - \gamma_i} \right]. \end{aligned} \quad (6)$$

Then BVP (1), (2) has a unique solution on $C^q([0, 1], \mathbb{R})$.

Proof. First we will prove the operator $\mathcal{F}$ is a contraction in $C^q([0, 1], \mathbb{R})$.

Apply the equalities $\frac{1}{\Gamma(\mu)} \int_0^A (A-s)^\mu ds = \frac{A^\mu}{\Gamma(1+\mu)}$ and $\frac{1}{\Gamma(\mu)} \int_\beta^\xi \int_0^z (z-s)^\mu ds dz = \frac{\beta^{1+\mu} - \xi^{\mu+1}}{\Gamma(2+\mu)}$ and obtain for any $v_1, v_2 \in C^q([0, 1], \mathbb{R})$ and $t \in [0, 1]$, the inequalities

$$\begin{aligned} |\mathcal{F}v_1(t) - \mathcal{F}v_2(t)| & \leq \frac{t^{q-1}}{|\beta_1|} \left[\frac{(L_1 + L_2)}{|\alpha_\rho| \Gamma(\gamma_\rho)} \sum_{j=1}^n |b_j| \int_{\xi_j}^{\eta_j} \int_0^s (s-z)^{\gamma_\rho-1} dz ds \right. \\ & + \sum_{j=1}^n |b_j| \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_{\xi_j}^{\eta_j} \int_0^s (s-z)^{\gamma_\rho - \gamma_i - 1} dz ds \|v_1 - v_2\| \\ & + \frac{L_1 + L_2}{\alpha_\rho \Gamma(\gamma_\rho)} \sum_{k=1}^m |a_k| \int_0^{\sigma_k} (\sigma_k - z)^{\gamma_\rho-1} dz \|v_1 - v_2\| \\ & + \sum_{k=1}^m |a_k| \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_0^{\sigma_k} (\sigma_k - z)^{\gamma_\rho - \gamma_i - 1} dz \|v_1 - v_2\| \\ & + \frac{L_1 + L_2}{\alpha_\rho \Gamma(\gamma_\rho)} \int_0^1 (1-z)^{\gamma_\rho-1} dz \|v_1 - v_2\| \\ & + \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_0^1 (1-z)^{\gamma_\rho - \gamma_i - 1} dz \|v_1 - v_2\| \Big] \\ & + \frac{L_1 + L_2}{|\alpha_\rho| \Gamma(\gamma_\rho)} \int_0^t (t-z)^{\gamma_\rho-1} dz \|v_1 - v_2\| \\ & + \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_0^t (t-z)^{\gamma_\rho - \gamma_i - 1} dz \|v_1 - v_2\| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{|\beta_1|} \left[\frac{(L_1 + L_2)}{|\alpha_\rho| \Gamma(2 + \gamma_\rho)} \sum_{j=1}^n |b_j| (\xi_j^{\gamma_\rho+1} - \eta_j^{\gamma_\rho}) \right. \\
&+ \sum_{j=1}^n |b_j| \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(2 + \gamma_\rho - \gamma_i)} (\xi_j^{\gamma_\rho - \gamma_i + 1} - \eta_j^{\gamma_\rho - \gamma_i + 1}) \\
&+ \frac{L_1 + L_2}{\alpha_\rho \Gamma(1 + \gamma_\rho)} \left(1 + |\beta_1| + \sum_{k=1}^m |a_k| \sigma_k^{\gamma_\rho} \right) \\
&+ (1 + |\beta_1|) \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(1 + \gamma_\rho - \gamma_i)} \\
&\left. + \sum_{k=1}^m |a_k| \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(1 + \gamma_\rho - \gamma_i)} \sigma_k^{\gamma_\rho - \gamma_i} \right] \|v_1 - v_2\|
\end{aligned}$$

or $\|\mathcal{F}v_1 - \mathcal{F}v_2\| \leq \lambda \|v_1 - v_2\|$, where λ is defined by (6).

The above inequality proves $\mathcal{F}$ is a contraction since $\lambda < 1$.

We will choose $\Delta > 0$ such that $\mathcal{F}B_\Delta \subset B_\Delta$, where $B_\Delta = \{v \in C^q([0, 1], \mathbb{R}) : \|v\| \leq \Delta\}$. Let $v \in B_\Delta$. Then $\|\mathcal{F}(v)\| \leq \|\mathcal{F}(v) - \mathcal{F}(0)\| + \|\mathcal{F}(0)\|$. According to the proved above we have $\|\mathcal{F}(v)\| \leq \lambda \|v\| + \|\mathcal{F}(0)\| \leq \lambda \Delta + \|\mathcal{F}(0)\|$.

Choose Δ such that $\|\mathcal{F}(0)\| + \lambda \Delta < \Delta$ or $\|\mathcal{F}(0)\| < (1 - \lambda)\Delta$ or $\frac{\|\mathcal{F}(0)\|}{(1 - \lambda)} < \Delta$, where

$$\begin{aligned}
\mathcal{F}(0) = \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} &\left(\sum_{j=1}^n b_j \int_{\xi_j}^{\eta_j} \int_0^s (s - z)^{\gamma_\rho - 1} f(z, 0, 0) dz ds \right. \\
&+ \sum_{k=1}^m a_k \int_0^{\sigma_k} (\sigma_k - z)^{\gamma_\rho - 1} f(z, 0, 0) dz \\
&\left. - \int_0^1 (1 - z)^{\gamma_\rho - 1} f(z, 0, 0) dz + \int_0^t (t - z)^{\gamma_\rho - 1} f(z, 0, 0) dz \right).
\end{aligned}$$

Thus, by Banach fixed point theorem there exists a unique fixed point of the operator $\mathcal{F}$ in $C^q([0, 1], \mathbb{R})$. According to Lemma 2.5 the fixed point is the unique solution of BVP (1), (2) on $[0, 1]$. $\square$

3. Ulam-Hyers stability results

Note Ulam type stability for BVP (1) is studied in [4]. Unfortunately, there are some misunderstandings.

Let us discuss [4, Definition 4.3]. The authors define the inequality (4.16) depending on $\varepsilon > 0$. Note the inequality (4.16) has many solution. Therefore, the written in Remark 4.1 is not correct. Actually, if a function $v(t)$, $t \in [0, 1]$, is a solution of the

inequality (4.16) then there exists a function $\vartheta \in C([0, 1], \mathbb{R})$ such that $|\vartheta(t)| \leq \epsilon$, $t \in [0, 1]$, and

$$\sum_{i=1}^p \alpha_i^c \mathcal{D}^{\gamma_i} v(t) - f(t, v(t), v(wt)) - \vartheta(t) = 0. \quad (7)$$

The solution of (7) could not satisfy the boundary conditions (2) and therefore Lemma 4.3 is not applicable for it.

To illustrate the above written we consider a very simple scalar differential equation $x' = x + e^t$, $t \in [0, 1]$ with a boundary condition $x(1) = 2$ (compare with (1.1)). Its solution is $x(t) = e^t(2e^{-1} + t - 1)$. Now let $\epsilon > 0$ and consider the differential inequality $|y' - y - e^t| \leq \epsilon$ (compare with (4.16)). Then $y(t) = e^t(t + 0.5\epsilon) - 0.5\epsilon$ is one of the possible solutions of the differential inequality. It is obvious $y(1) = e(1 + 0.5\epsilon) - 0.5\epsilon \neq 2$, i.e. the solution of the differential inequality could not satisfy the given boundary condition.

Now look at the proof of [4, Theorem 4.6]. According to [4, Definition 4.6] the authors consider a solution $v(t)$ of inequality (4.16). This solution could not satisfy the boundary condition, i.e. $v(1) \neq \sum_{j=1}^n b_j \int_{\xi_j}^{\eta_j} v(s) ds + \sum_k + k = 1^m a_k v(\sigma_k)$ and [4, Lemma 4.3] is not applicable for the chosen function $v(t)$ and the proof of [4, Theorem 4.3] is not correct.

Similar is the situation about [4, Definition 4.4, Definition 4.5 and Theorem 4.7].

By the way, in the proof of [4, Lemma 4.3] there is a typo, the function $v \in C([0, 1], \mathbb{R})$ is not a solution of (1.1) but a solution of the boundary value problem written in the statement of the lemma.

We will suggest three different ways to avoid the mentioned above misunderstandings.

3.1 Fractional integral inequality

Let $\epsilon > 0$ be an arbitrary number. Consider the following fractional integral inequality

$$|v(t) - \mathcal{F}v(t)| \leq \epsilon, \quad \text{for } t \in [0, 1], \in \mathbb{R} \quad (8)$$

where the integral operator $\mathcal{F}$ is defined by (5).

REMARK 3.1. If the function $v \in C([0, 1], \mathbb{R})$ satisfy the inequality (8), then there exists a function $\vartheta \in C([0, 1], \mathbb{R})$: $|\vartheta(t)| \leq \epsilon$, $t \in [0, 1]$, such that the function v satisfies the fractional integral equation

$$v(t) - \mathcal{F}v(t) - \vartheta(t) = 0. \quad (9)$$

In this case we will change slightly the classical definition for Ulam-Hyers stability in the following way:

DEFINITION 3.2. The BVP (1), (2) is called *modified Ulam-Hyers* (MUH) stable, if for the unique solution $v^*(t)$ of BVP (1), (2) there exists a constant $K_1 \geq 0$ such that, for any $\epsilon > 0$ and any solution $v \in C([0, 1], \mathbb{R})$ of the fractional integral inequality (8) the inequality $\|v - v^*\| \leq K_1 \epsilon$ holds.

DEFINITION 3.3. The BVP (1), (2) is called *modified generalized Ulam-Hyers* (MGUH) stable, if for the unique solution $v^*(t)$ of BVP (1), (2) there exists a positive function $\kappa : [0, \infty) \rightarrow [0, \infty)$ with $\kappa(0) = 0$ such that for any $\varepsilon > 0$ and any solution $v \in C([0, 1], \mathbb{R})$ of the fractional integral inequality (8) the inequality $\|v - v^*\| \leq \kappa(\varepsilon)$ holds.

Now we will obtain sufficient conditions for the defined above Ulam type stability.

THEOREM 3.4. Let **Assumption A1.** and **Assumption A2.** be satisfied and $\lambda < 1$, where λ is given in (6). Then BVP (1), (2) is MUH stable and MGUH stable.

Proof. According to Theorem 2.7 there exists unique solution v^* of BVP (1), (2) and according to Corollary 2.6 the equality $v^*(t) = \mathcal{F}v^*(t)$, $t \in [0, 1]$, holds.

Let $v \in C([0, 1], \mathbb{R})$ be a solution of (8). Then apply **Assumption A2.**, equalities $\frac{1}{\Gamma(\mu)} \int_0^A (A-s)^\mu ds = \frac{A^\mu}{\Gamma(1+\mu)}$ and $\frac{1}{\Gamma(\mu)} \int_\beta^\xi \int_0^z (z-s)^\mu ds dz = \frac{\beta^{1+\mu} - \xi^{\mu+1}}{\Gamma(2+\mu)}$ and obtain

$$\begin{aligned}
|v(t) - v^*(t)| &= |v - \mathcal{F}v^*(t)| \leq |v - \mathcal{F}v(t)| + |\mathcal{F}v(t) - \mathcal{F}v^*(t)| \\
&\leq \varepsilon + \frac{1}{|\beta_1|} \left[\frac{L_1 + L_2}{|\alpha_\rho| \Gamma(2 + \gamma_\rho)} \sum_{j=1}^n |b_j| (\eta_j^{1+\gamma_\rho} - \xi_j^{1+\gamma_\rho}) \right. \\
&\quad + \sum_{j=1}^n b_j \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(2 + \gamma_\rho - \gamma_i)} (\eta_j^{1+\gamma_\rho - \gamma_i} - \xi_j^{1+\gamma_\rho - \gamma_i}) \\
&\quad + \frac{L_1 + L_2}{|\alpha_\rho| \Gamma(\gamma_\rho)} \sum_{k=1}^m |a_k| \frac{\sigma_k^{\gamma_\rho}}{\gamma_\rho} + \sum_{k=1}^m |a_k| \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \frac{\sigma^{\gamma_\rho - \gamma_i}}{\gamma_\rho - \gamma_i} \\
&\quad + \frac{L_1 + L_2}{|\alpha_\rho| \Gamma(\gamma_\rho)} \frac{1}{\gamma_\rho} + \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \frac{1}{\gamma_\rho - \gamma_i} \Big] \|v^* - v\| \\
&\quad + \left(\frac{L_1 + L_2}{|\alpha_\rho| \Gamma(\gamma_\rho)} \frac{t^{\gamma_\rho}}{\gamma_\rho} + \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \frac{t^{\gamma_\rho - \gamma_i}}{\gamma_\rho - \gamma_i} \right) \|v^* - v\| \quad (10) \\
&\leq \varepsilon + \frac{1}{|\beta_1|} \left[\frac{L_1 + L_2}{|\alpha_\rho| \Gamma(2 + \gamma_\rho)} \sum_{j=1}^n |b_j| (\eta_j^{1+\gamma_\rho} - \xi_j^{1+\gamma_\rho}) \right. \\
&\quad + \sum_{j=1}^n |b_j| \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(2 + \gamma_\rho - \gamma_i)} (\eta_j^{1+\gamma_\rho - \gamma_i} - \xi_j^{1+\gamma_\rho - \gamma_i}) \\
&\quad + \frac{L_1 + L_2}{|\alpha_\rho| \Gamma(1 + \gamma_\rho)} (1 + |\beta_1| + \sum_{k=1}^m |a_k| \sigma_k^{\gamma_\rho}) \\
&\quad + \sum_{k=1}^m |a_k| \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(1 + \gamma_\rho - \gamma_i)} \sigma^{\gamma_\rho - \gamma_i} \\
&\quad \left. + \sum_{i=1, i \neq \rho}^q \frac{|\alpha_i|}{|\alpha_\rho|} \frac{1}{\Gamma(1 + \gamma_\rho - \gamma_i)} (1 + \beta_1) \right] \|v^* - v\| = \varepsilon + \lambda \|v^* - v\|,
\end{aligned}$$

or $\|v - v^*\| \leq \frac{1}{1-\lambda} \varepsilon$.

It follows the MUH stability of BVP (1), (2). If we take $\kappa(\varepsilon) = \frac{1}{1-\lambda} \varepsilon$, then we obtain the MGUH stability of BVP (1), (2). $\square$

3.2 Fractional differential inequality with a boundary condition

Let $\varepsilon > 0$ be an arbitrary number. Consider the following fractional differential inequality

$$\left| \sum_{i=1}^p \alpha_i^c \mathcal{D}^{\gamma_i} v(t) - f(t, v(t), v(wt)) \right| \leq \varepsilon, \quad \text{where } t \in [0, 1] \quad (11)$$

with a boundary condition (2).

REMARK 3.5. If the function $v \in C^q([0, 1], \mathbb{R})$ satisfy the inequality (11) and the boundary condition (2), then there exists a function $\vartheta \in C([0, 1], \mathbb{R}) : |\vartheta(t)| \leq \varepsilon$, $t \in [0, 1]$, such that the function v satisfies the fractional differential equation

$$\sum_{i=1}^p \alpha_i^c \mathcal{D}^{\gamma_i} v(t) - f(t, v(t), v(wt)) - \vartheta(t) = 0. \quad (12)$$

with the boundary condition (2). Then according to Corollary 2.6 the integral equality

$$\begin{aligned} v(t) = & \mathcal{F}v(t) + \frac{t^{q-1}}{\beta_1} \left[\frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \sum_{j=1}^n b_j \int_{\xi_j}^{\eta_j} \int_0^s (s-z)^{\gamma_\rho-1} \vartheta(z) dz ds \right. \\ & + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \sum_{k=1}^m a_k \int_0^{\sigma_k} (\sigma_k - z)^{\gamma_\rho-1} \vartheta(z) dz \\ & \left. - \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \int_0^1 (1-z)^{\gamma_\rho-1} \vartheta(z) dz \right] + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \int_0^t (t-z)^{\gamma_\rho-1} \vartheta(z) dz. \end{aligned} \quad (13)$$

holds where the operator $\mathcal{F}$ is defined by (5).

REMARK 3.6. Remark 3.5 and integral presentation (13) are very important for the Ulam type stability. Note that (13) is true for a function satisfying the fractional inequality (11) if this function also satisfies the boundary condition (2). If a function is satisfying only the inequality (13), it is not sure that it will satisfy (2) and the integral presentation (13) could not be true for it.

The written in Remark 3.6 explains the reason for studying the suggested approach, i.e., to consider both, (13) with (2), not only (13) as it is done in [4].

Now, for a function $v \in C^q([0, 1], \mathbb{R})$, satisfying (13), (2) we could apply the following result which is similar to [4, Lemma 4.3] with slight changes:

LEMMA 3.7. Let $v \in C^q([0, 1], \mathbb{R})$ be a solution of the problem (13), (2) with function $\vartheta \in C([0, 1], \mathbb{R}) : |\vartheta(t)| \leq \varepsilon$, $t \in [0, 1]$ (see Remark 3.5).

Then the inequality $|v(t) - \mathcal{F}v(t)| \leq \mathcal{S}_1 \varepsilon$, holds where

$$\mathcal{S}_1 = \frac{1}{|\alpha_\rho| \Gamma(\gamma_\rho + 1)} \left(\frac{1}{|\beta_1|} \left[(\gamma_\rho + 1) \sum_{j=1}^n |b_j| (\eta_j^{\gamma_\rho+1} - \xi_j^{\gamma_\rho+1}) + \sum_{k=1}^m |a_k| \sigma_k^{\gamma_\rho+1} \right] + 1 \right). \quad (14)$$

In this case we will changes slightly the classical definition for Ulam-Hyers stability in the following way.

DEFINITION 3.8. The BVP (1), (2) is called *modified Ulam-Hyers* (MUH) stable, if for the unique solution $v^*(t)$ of BVP (1), (2) there exists a constant $K_1 \geq 0$ such that, for any $\varepsilon > 0$ and any solution $v \in C^q([0, 1], \mathbb{R})$ of the fractional differential inequality (13) with a boundary condition (2) the inequality $\|v - v^*\| \leq K_1 \varepsilon$ holds.

DEFINITION 3.9. The BVP (1), (2) is called *modified generalized Ulam-Hyers* (MGUH) stable, if for the unique solution $v^*(t)$ of BVP (1), (2) there exists a positive function $\kappa : [0, \infty) \rightarrow [0, \infty)$ with $\kappa(0) = 0$ such that for any $\varepsilon > 0$ and any solution $v \in C^q([0, 1], \mathbb{R})$ of the fractional differential inequality (13) with a boundary condition (2) the inequality $\|v - v^*\| \leq \kappa(\varepsilon)$ holds.

Now we will obtain sufficient conditions for the defined above Ulam type stability.

THEOREM 3.10. Let **Assumption A1.** and **Assumption A2.** be satisfied and $\lambda < 1$, where λ is given by (6). Then BVP (1), (2) is MUH stable and MGUH stable.

Proof. According to Theorem 2.7 there exists unique solution v^* of BVP (1), (2).

Let $v \in C^q([0, 1], \mathbb{R})$ be a solution of (13), (2). According to Remark 3.5 there exists a function $\vartheta \in C([0, 1], \mathbb{R}) : |\vartheta(t)| \leq \varepsilon, \quad t \in [0, 1]$, such that the function v satisfies the fractional differential equation (12). According to Lemma 3.7 the inequality $|v(t) - \mathcal{F}v(t)| \leq \mathcal{S}_1 \varepsilon, \quad t \in [0, 1]$, where the constant $\mathcal{S}_1$ is defined by (14). Then we have

$$\begin{aligned} |v(t) - v^*(t)| &= |v - \mathcal{F}v^*(t)| \leq |v - \mathcal{F}v(t)| + |\mathcal{F}v(t) - \mathcal{F}v^*(t)| \\ &\leq \mathcal{S}_1 \varepsilon + \lambda |v(t) - v^*(t)|, \end{aligned}$$

$$\text{or} \quad \|v - v^*\| \leq \frac{\mathcal{S}_1}{1 - \lambda} \varepsilon.$$

It follows the MUH stability of BVP (1), (2). If we take $\kappa(\varepsilon) = \frac{\mathcal{S}_1}{1 - \lambda} \varepsilon$, then we obtain the MGUH stability of BVP (1), (2). $\square$

3.3 Fractional differential inequality

In both above approaches given in Sections 3.1 and 3.2 the solution of the given problem is one and the same for any chosen solution of the corresponding inequality. Therefore, it changes the meaning of the classical Ulam stability.

Now we will give an approach which keeps the idea of Ulam stability for initial values, i.e. the solution of the given problem will depend on the particular chosen solution of the corresponding inequality. For this purpose we will introduce a parameter in the boundary conditions (2) and later the parameter will be defined by the help of the values of the chosen solution of the corresponding inequality.

Consider the following boundary conditions.

$$\begin{aligned} y(0) &= y_0, \quad \frac{d^\zeta y(0)}{dt^\zeta} = 0, \quad \zeta = 1, 2, \dots, q-2, \\ y(1) &= \sum_{j=1}^n b_j \int_{\xi_j}^{\eta_j} y(s) ds + \sum_{k=1}^m a_k y(\sigma_k) + p, \end{aligned} \quad (15)$$

where p is a parameter.

Then the operator $\mathcal{F} : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ defined by (5) will be slightly changed. Define the operator $\mathcal{G} : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$

$$\begin{aligned} \mathcal{G}(v) &= v_0 + \frac{t^{q-1}}{\beta_1} \left[v_0 \left(\sum_{j=1}^n b_j (\eta_j^q - \xi_j^q) + \sum_{k=1}^m a_k - 1 \right) \right. \\ &\quad + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \sum_{j=1}^n b_j \int_{\xi_j}^{\eta_j} \int_0^s (s-z)^{\gamma_\rho-1} f(z, v(z), v(\omega z)) dz ds \\ &\quad - \sum_{j=1}^n b_j \sum_{i=1, i \neq \rho}^q \frac{\alpha_i}{\alpha_\rho} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_{\xi_j}^{\eta_j} \int_0^s (s-z)^{\gamma_\rho-\gamma_i-1} v(z) dz ds \\ &\quad + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \sum_{k=1}^m a_k \int_0^{\sigma_k} (\sigma_k - z)^{\gamma_\rho-1} f(z, v(z), v(\omega z)) dz \\ &\quad - \sum_{k=1}^m a_k \sum_{i=1, i \neq \rho}^q \frac{\alpha_i}{\alpha_\rho} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_0^{\sigma_k} (\sigma_k - z)^{\gamma_\rho-\gamma_i-1} v(z) dz \\ &\quad - \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \int_0^1 (1-z)^{\gamma_\rho-1} f(z, v(z), v(\omega z)) dz \\ &\quad + \sum_{i=1, i \neq \rho}^q \frac{\alpha_i}{\alpha_\rho} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_0^1 (1-z)^{\gamma_\rho-\gamma_i-1} v(z) dz + p \left. \right] \\ &\quad + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \int_0^t (t-z)^{\gamma_\rho-1} f(z, v(z), v(\omega z)) dz \\ &\quad - \sum_{i=1, i \neq \rho}^q \frac{\alpha_i}{\alpha_\rho} \frac{1}{\Gamma(\gamma_\rho - \gamma_i)} \int_0^t (t-z)^{\gamma_\rho-\gamma_i-1} v(z) dz. \end{aligned} \quad (16)$$

Similar to Corollary 2.6 we have the following.

COROLLARY 3.11. *Any solution $v \in C^q([0, 1], \mathbb{R})$ of BVP (1), (15) satisfies the integral equations $v(t) = \mathcal{G}v(t)$, $t \in [0, 1]$, where the operator $\mathcal{G}$ is defined by (16).*

Note Theorem 2.7 is true for BVP (1), (15).

Let $\varepsilon > 0$ be an arbitrary number. Consider the following fractional differential

inequality

$$\left| \sum_{i=1}^p \alpha_i^c \mathcal{D}^{\gamma_i} v(t) - f(t, v(t), v(wt)) \right| \leq \varepsilon, \quad \text{for } t \in [0, 1] \quad (17)$$

$$\frac{d^\zeta v(0)}{dt^\zeta} = 0, \quad \zeta = 1, 2, \dots, q-2.$$

REMARK 3.12. If the function $v \in C^q([0, 1], \mathbb{R})$ satisfy the inequality (17) then there exists a function $\vartheta \in C([0, 1], \mathbb{R}) : |\vartheta(t)| \leq \varepsilon, \quad t \in [0, 1]$, such that the function v satisfies the fractional differential equation

$$\sum_{i=1}^p \alpha_i^c \mathcal{D}^{\gamma_i} v(t) - f(t, v(t), v(wt)) - \vartheta(t) = 0. \quad (18)$$

If we choose $y_0 = v(0)$ and $p = v(1) - \sum_{j=1}^n b_j \int_{\xi_j}^{\eta_j} v(s) ds + \sum_{k=1}^m a_k v(\sigma_k)$, then the function v satisfies boundary conditions (15) and according to Corollary 3.11 the integral equality

$$\begin{aligned} v(t) = & \mathcal{G}v(t) + \frac{t^{q-1}}{\beta_1} \left[\frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \sum_{j=1}^n b_j \int_{\xi_j}^{\eta_j} \int_0^s (s-z)^{\gamma_\rho-1} \vartheta(z) dz ds \right. \\ & + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \sum_{k=1}^m a_k \int_0^{\sigma_k} (\sigma_k - z)^{\gamma_\rho-1} \vartheta(z) dz \\ & \left. - \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \int_0^1 (1-z)^{\gamma_\rho-1} \vartheta(z) dz \right] + \frac{1}{\alpha_\rho \Gamma(\gamma_\rho)} \int_0^t (t-z)^{\gamma_\rho-1} \vartheta(z) dz. \end{aligned} \quad (19)$$

holds where the operator $\mathcal{G}$ is defined by (16).

REMARK 3.13. Remark 3.12 and integral presentation (19) are very important for the Ulam type stability.

Now, for a function $v \in C^q([0, 1], \mathbb{R})$, satisfying (13), (2) we could apply the following result which is similar to Lemma 3.7.

LEMMA 3.14. Let $v \in C^q([0, 1], \mathbb{R})$ be a solution of the problem (13), (15) with function $\vartheta \in C([0, 1], \mathbb{R}) : |\vartheta(t)| \leq \varepsilon, \quad t \in [0, 1]$ (see Remark 3.12).

Then the inequality $|v(t) - \mathcal{G}v(t)| \leq \mathcal{S}_1 \varepsilon$, holds where the constant $\mathcal{S}_1$ is defined by equality (14).

In this case we will use the classical definition for Ulam-Hyers stability.

DEFINITION 3.15. The BVP (1), (2) is called *Ulam-Hyers* (UH) stable, if there exists a constant $K_1 \geq 0$ such that, for any $\varepsilon > 0$ and any solution $v \in C^q([0, 1], \mathbb{R})$ of the fractional differential inequality (13) with a boundary condition (15) the inequality $\|v - v^*\| \leq K_1 \varepsilon$ holds where $v^*(t)$ is a solution of BVP (1), (15).

DEFINITION 3.16. The BVP (1), (2) is called *generalized Ulam-Hyers* (GUH) stable, if there exists a positive function $\kappa : [0, \infty) \rightarrow [0, \infty)$ with $\kappa(0) = 0$ such that for any

$\varepsilon > 0$ and any solution $v \in C^q([0, 1], \mathbb{R})$ of the fractional differential inequality (19) with a boundary condition (15) the inequality $\|v - v^*\| \leq \kappa(\varepsilon)$ holds where $v^*(t)$ is a solution of BVP (1), (15).

Now we will obtain sufficient conditions for the defined above Ulam type stability.

THEOREM 3.17. *Let **Assumption A1.** and **Assumption A2.** be satisfied and $\lambda < 1$, where λ is given in (6). Then BVP (1), (2) is UH stable and GUH stable.*

Proof. Let $v \in C^q([0, 1], \mathbb{R})$ be a solution of inequality (19). According to Remark 3.12 there exists a function $\vartheta \in C([0, 1], \mathbb{R}) : |\vartheta(t)| \leq \varepsilon, \quad t \in [0, 1]$, such that the function v satisfies the fractional differential equation (12).

We choose $y_0 = v(0)$ and $p = v(1) - \sum_{j=1}^n b_j \int_{\xi_j}^{\eta_j} v(s) ds + \sum_{k=1}^m a_k v(\sigma_k)$ and substitute them in boundary condition (15). Then

- According to Theorem 2.7 for the operator $\mathcal{G}$ and Corollary 3.11 there exists unique solution v^* of BVP (1), (15) with the chosen y_0 and p and $v^*(t) = \mathcal{G}v^*(t), \quad t \in [0, 1]$;
- the function v satisfies boundary conditions (15) and according to Corollary 3.11 the integral equality (19) holds.

According to Lemma 3.14 the inequality $|v(t) - \mathcal{G}v(t)| \leq \mathcal{S}_1 \varepsilon, \quad t \in [0, 1]$, where the constant $\mathcal{S}_1$ is defined by (14). Then we have

$$\begin{aligned} |v(t) - v^*(t)| &= |v - \mathcal{G}v^*(t)| \leq |v - \mathcal{G}v(t)| + |\mathcal{G}v(t) - \mathcal{G}v^*(t)| \\ &\leq \mathcal{S}_1 \varepsilon + \lambda |v(t) - v^*(t)|, \end{aligned}$$

or
$$\|v - v^*\| \leq \frac{\mathcal{S}_1}{1 - \lambda} \varepsilon.$$

It follows the UH stability of BVP (1), (15). If we take $\kappa(\varepsilon) = \frac{\mathcal{S}_1}{1 - \lambda} \varepsilon$, then we obtain the GUH stability of BVP (1), (2). $\square$

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REFERENCES

- [1] R. Agarwal, S. Hristova, D. O'Regan, *Boundary value problems for fractional differential equations of Caputo type and Ulam type stability: basic concepts and study*, Axioms, **12** (2023), 226.
- [2] R. Agarwal, S. Hristova, D. O'Regan, *Ulam stability for boundary value problems of differential equations. Main misunderstandings and how to avoid them*, Mathematics, **12**(11), (2024), 1626.
- [3] B. Ahmad, T. Hayat, S. Asghar, *Diffraction of a plane wave by an elastic knife-edge adjacent to a strip*, Can. Appl. Math. Q., **9** (2002), 303–316.
- [4] N. Alghamdi, R. P. Agarwal, B. Ahmad, E. A. Alharbi, W. Shammakh, *Existence and stability results for multi-term fractional delay differential equations equipped with nonlocal multi-point and multi-strip boundary conditions*, Carpathian J. Math., **41**(3), (2025), 651–671.
- [5] R. Gorenflo, F. Mainardi, *Fractional Calculus: integral and differential equations of fractional order*, In: Fractional Calculus in Continuum and Statistical Mechanics (Udine, 1996; CISM Courses and Lectures, 1997, Springer, Vienna, 223–276.

- [6] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.
- [7] M. Kawasaki, *Exploration of electronic functionalities in metal oxides by combinatorial lattice engineering*, Bull. Chem. Soc. Jpn., **86** (2013), 1341–1358.
- [8] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.
- [9] D.-J. Lee, S. J. Lee, W.-S. Lee, J.-W. Yu, *Diffraction by dielectric-loaded multiple slits in a conducting plane: TM case*, Prog. Electromagn. Res., **131** (2012), 409–424.
- [10] S. P. Lipshitz, T. C. Scott, B. Salvy, *On the acoustic impedance of baffled strip radiators*, J. Audio Eng. Soc., **43** (1995), 573–580.
- [11] M. Lundqvist, *Silicon Strip Detectors for Scanned Multi-Slit X-Ray Imaging*, Kungl Tekniska Hogskolan, Stockholm, 2003.
- [12] K. B. Oldham, J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.
- [13] M. N. Ozisik, *Boundary Value Problems of Heat Conduction*, Dover, New York, 2002.
- [14] I. Podlubny, *Fractional Differential Equations. Mathematics in Sciences and Engineering*, 198, Academic Press, San-Diego, 1999.
- [15] H. A. Wahash, M. S. Abdo1, S. K. Panchal, *Existence and Ulam-Hyers stability of the implicit fractional boundary value problem with ψ -Caputo fractional derivative*, J. Appl. Math. Comput. Mech., **19(1)** (2020), 89–101.

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