MATEMATIČKI VESNIK
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APPLICATIONS OF HARMONIC AND GEOMETRIC MEANS IN THE THEORY OF SUBORDINATION

Stanislawa Kanas

Abstract. The concept of subordination is attributed to Lindelöf (1909), but Littlewood and Rogosinski introduced the terms and discovered the basic relations. Two of the earliest first-order differential subordinations were given in 1935 by Goluzin and in 1947 by Robinson, but it was Miller and Mocanu who worked out the problem in detail. Different types of differential subordination were considered in the early articles of the originators of this theory, and then by other mathematicians. In this paper we introduce and explore differential subordinations related to generalized Pythagorean means i.e. arithmetic, geometric and harmonic means. By transferring the obtained results to the ground of analytic functions we obtain new concepts in geometric functions theory.

1. Introduction

Following Lindelöf, Littlewood and Rogosinski, we say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$ (where f and g are analytic in $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$), if there exists a Schwarz function ω (i.e. analytic in $\mathbb{D}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{D}$) such that $f(z) = g(\omega(z)),\ z \in \mathbb{D}$. If, additionally g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $[f(0) = g(0),\ f(\mathbb{D}) \subset g(\mathbb{D})]$.

Let $\mathcal{H}$ denote a set of all function holomorphic in $\mathbb{D}$, and $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions f with a normalization $f(0) = f'(0) - 1 = 0$, and let $\mathcal{CV} \subset \mathcal{H}$ denote the subclass of convex functions h (i.e. such that $h(\mathbb{D})$ is convex set). Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ of all univalent functions.

Definition 1.1. Let $\psi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ and $h \in \mathcal{H}$ be univalent. We say that a nonconstant function $p \in \mathcal{H}[\psi]$ if p satisfies the first-order differential subordination and is called its solution if

$$\psi(p(z), zp'(z)) \prec h(z), \quad z \in \mathbb{D}. \tag{1}$$

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If $h \in \mathcal{H}$ is an univalent function such that $p \prec h$ for all functions $p \in \mathcal{H}[\psi]$ satisfying (1), then h is called a dominant of (1).

The condition (1) constitutes the most general form of the first-order differential subordination. However, we can specify it by the introduction of an additional function $\varphi(p(z))$ thanks to which it is possible to introduce various modifications, as follows: $\psi(p(z), p(z) + zp'(z)\varphi(p(z))) \prec h(z)$, $z \in \mathbb{D}$.

The beginnings of the first-order subordination studies are due to Goluzin [1] who considered the simple first-order differential subordination $zp'(z) \prec h(z)$, that was generalized in 1947 by Robinson [9] to the form $p(z) + zp'(z) \prec h(z)$. For both cases, under some assumption on h, the best dominants of subordination have been found.

The classical Pythagorean means i.e. an arithmetic (A), a geometric (G), and a harmonic mean (H) of the positive real two numbers a, b have beautiful geometric interpretation and satisfy $H \leq G \leq A$. All means were generalized to the weighted cases or to their special version - convex weighted means. For the arithmetic mean we have then $A_\alpha(a, b) = (1-\alpha)a + \alpha\, b$, for the geometric $G_\alpha(a, b) = a^\alpha b^{1-\alpha}$, while for the harmonic

$$H_\alpha(a, b) = \frac{1}{\alpha\frac{1}{a} + (1-\alpha)\frac{1}{b}} = \frac{ab}{\alpha(b-a) + a},$$

where $0 \leq \alpha \leq 1$, that reduce to the classical case when $\alpha = 1/2$. All three means, both in a classical and weighted sense found numerous applications in each fields of mathematical sciences. Since in this paper we will deal only with generalized means, in the sequel for simplicity, we will omit the parameter α, writing A, G, H for generalized means instead of $A_\alpha, G_\alpha, H_\alpha$.

The fundamental example of the first-order differential subordination (1) related to the arithmetic mean has been considered by Miller, Mocanu and Reade and was of the form $p(z) + \alpha\, zp'(z) \prec h(z)$, $z \in \mathbb{D}$, which is the case of arithmetic means of the form $A(p(z), p(z) + zp'(z)) = (1-\alpha)p(z) + \alpha(p(z) + zp'(z))$. If we put $zp'(z)\varphi(p(z))$ instead of $zp'(z)$ we obtain $p(z) + \alpha\, zp'(z)\varphi(p(z)) \prec h(z)$, $z \in \mathbb{D}$, that reduces to the arithmetic case $A(p(z), p(z) + zp'(z)\varphi(p(z))) = (1-\alpha)p(z) + \alpha(p(z) + zp'(z)\varphi(p(z)))$ which contains many special cases, for example $p(z) + \frac{zp'(z)}{p(z)} \prec h(z)$, $z \in \mathbb{D}$, or, more general, Briot-Bouquet differential subordination [8, pp. 80-98]

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z),\ z \in \mathbb{D},\ p(z) \neq -\gamma/\beta.$$

The first-order differential subordination related to the geometric mean has grown on the basis of the work of Lewandowski et al. [7], who introduced γ-starlike functions consisting of $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma} > 0,$$

that with $p(z) = \frac{zf'(z)}{f(z)}$ comes down to the geometric mean in (1)

$$G(p(z), p(z) + zp'(z)/p(z)) = p(z)^{1-\gamma}\left(p(z) + \frac{zp'(z)}{p(z)}\right)^{\gamma} = p(z)\left(1 + \frac{zp'(z)}{p(z)}\varphi(p(z))\right)^{\gamma},$$

where $\varphi(p(z)) = 1/p(z)$, see [6].

The study of (1) related to the harmonic mean was started by the author in [5], where the classical harmonic means of the form $\psi(p(z), zp'(z)) = \frac{2p(z)\left[p(z)+zp'(z)\right]}{2p(z)+zp'(z)}$, have been considered. Similar problems were continued by author in [4] where the generalized harmonic differential subordination

$$H(p(z), p(z) + zp'(z)) = \frac{p(z)\,[p(z) + zp'(z)]}{p(z) + \alpha zp'(z)},$$

or

$$H(p(z), p(z) + zp'(z)) = \frac{p^2(z) + zp'(z)}{p(z) + \alpha zp'(z)/p(z)}$$

have been studied.

The combined Pythagorean means appeared in [3], where the authors have investigated differential subordinations involving arithmetic and geometric means of the form

$$\psi(p(z), zp'(z)) = \alpha[p(z)]^{\delta} + (1-\alpha)\left[p(z) + \frac{zp'(z)}{p(z)}\right]^{\mu}.$$

The main aim of a paper is to involve all types of Pythagorean means by considering the first-order differential subordination of the form

$$\psi(p(z), p(z) + zp'(z)\varphi(p(z))) \prec h(z),\ z \in \mathbb{D},$$

where h is convex in $\mathbb{D}$.

The following definition and lemma are crucial to prove our results.

Definition 1.2 ([8, p.21]). Denote by Q the set of functions q that are analytic and injective on $\bar{\mathbb{D}} \setminus E(q)$, where

$$E(q) = \left\{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty\right\},$$

such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(q)$.

If $q \in Q$ then $\Delta = q(\mathbb{D})$ is a simply connected domain and its boundary consists of either a simple closed curve or the union of pairwise disjoint simple regular curves, each of which converges to ∞ in both directions.

Lemma 1.3 ([8, p. 24]). *Let q be an analytic and univalent function on $\mathbb{D}$, and $q \in Q$. Let p be a non-constant analytic function in $\mathbb{D}$ with $p(0) = q(0)$. If p is not subordinate to q, then there exist points $z_0 \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D} \setminus E(q)$, and $m \geq 1$ for which $p(\mathbb{D}_{|z_0|}) \subset q(\mathbb{D}_{|z_0|})$, $p(z_0) = q(\zeta_0)$, $z_0p'(z_0) = m\zeta_0 q'(\zeta_0)$. Here $\mathbb{D}_{|z_0|} = \{z \in \mathbb{C} :\ |z| < |z_0| < 1\}$.*

It is commonly known that $h \in \mathcal{CV}$ with $h'(0) \neq 0$ is convex if and only if

$$\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

We note that if h is a convex function and $h(\mathbb{D})$ is bounded, then $\partial h(\mathbb{D})$ is a Jordan curve and h has a continuous injective extension to $\mathbb{D}$. If $h(\mathbb{D})$ is unbounded, then for every $\zeta \in \partial\mathbb{D} \setminus E(h)$ the unrestricted limit $\lim\limits_{\mathbb{D}\ni z\to\zeta} h(z) =: h(\zeta) \in \mathbb{C}$ exists.

2. Combined means

Since arithmetic means have been the most frequently considered means, we focus on harmonic and geometric means and their combinations. First, we consider harmonic means of the expressions $p(z)$ and $p(z)+zp'(z)\varphi(p(z))$ with $\mathrm{Re},\varphi(p(z))>0$. Some special cases of combined means have been studied by the author in [5].

THEOREM 2.1. *Let $\alpha\in\langle 0,1\rangle$, $h\in Q$ be a convex function and let φ be analytic in a domain containing $h(\mathbb{D})$ and satisfying $\operatorname{Re}\varphi(h(\zeta))\geq 0$ for $\zeta\in\partial\mathbb{D}\setminus E(h)$. If $p\in\mathcal{H}[\psi]$, where $\psi(p(z),p(z)+\alpha zp'(z)\varphi(p(z)))=H(p(z),p(z)+\alpha zp'(z)\varphi(p(z)))$ and*

$$H(p(z),p(z)+\alpha zp'(z)\varphi(p(z)))=\frac{p(z)(p(z)+zp'(z)\varphi(p(z)))}{p(z)+\alpha zp'(z)\varphi(p(z))}\prec h(z),\ \ z\in\mathbb{D},$$

then $p(z)\prec h(z)$ in $\mathbb{D}$.

Proof. We note that $h(0)=p(0)$. Suppose, on the contrary, that $p(z)\not\prec h(z)$. Then, by Lemma 1.3, there exist $z_0\in\mathbb{D}$, $\zeta_0\in\partial\mathbb{D}\setminus E(h)$ and $m\geq 1$ such that $p(z_0)=h(\zeta_0)$ and $z_0p'(z_0)=m\zeta_0h'(\zeta_0)$.

Applying the above relations we obtain

$$\frac{p(z_0)(p(z_0)+z_0p'(z_0)\varphi(p(z_0)))}{p(z_0)+\alpha z_0p'(z_0)\varphi(p(z_0))}=\frac{h(\zeta_0)(h(\zeta_0)+m\zeta_0h'(\zeta_0)\varphi(h(\zeta_0)))}{h(\zeta_0)+\alpha m\zeta_0h'(\zeta_0)\varphi(h(\zeta_0))}. \tag{2}$$

Because the function h is convex, $h(\mathbb{D})$ is the convex set and $h(\zeta_0)\in\partial h(\mathbb{D})$ for $\zeta_0\in\partial\mathbb{D}\setminus E(h)$. Thus, there exists an open half-plane $\mathcal{P}$, supporting the domain $h(\mathbb{D})$ at $h(\zeta_0)$. Thus $h(\zeta_0)\in\partial\mathcal{P}$. Since $\zeta_0h'(\zeta_0)$ is the outward normal to the boundary of the convex domain $h(\mathbb{D})$ at the point $h(\zeta_0)$, $\operatorname{Re}\varphi(h(\zeta))\geq 0$ for $\zeta\in\partial\mathbb{D}\setminus E(h)$ hence a vector $h(\zeta_0)+m\alpha\zeta_0h'(\zeta_0)\varphi(h(\zeta_0)\in\mathcal{P}$ (cf. the proof of [8, Theorem 3.4a]).

The key function in our next considerations is the inversion $T(z)=1/z$, for $z\neq 0$, which has the involutory property $T(T(z))=z$. As a special case of the Möbius transformation, T maps generalized circles to generalized circles and open sets to open sets. First, we note that the harmonic mean is a composition of the arithmetic mean and the inversion, i.e.,

$$H(u,v)=\frac{1}{\alpha\frac{1}{u}+(1-\alpha)\frac{1}{v}}=\frac{1}{\alpha T(u)+(1-\alpha)T(v)}=T(A(T(u),T(v)).$$

Applying the inversion T to the halfplane $\mathcal{P}$, we obtain a halfplane (when $0\in\partial\mathcal{P}$) or some disk (when $0\notin\partial\mathcal{P}$), and both domains $T(\mathcal{P})$ are convex. Hence $A(T(h(\zeta_0)),T(h(\zeta_0)+m\alpha\zeta_0h'(\zeta_0)\varphi(h(\zeta_0))\in T(\mathcal{P})$. Using the involutory property of T we get

$$T\big(A(T(h(\zeta_0)),T(h(\zeta_0)+m\alpha\zeta_0h'(\zeta_0)\varphi(h(\zeta_0))\big)\in\mathcal{P},$$

that is

$$B(\zeta_0):=\frac{h(\zeta_0)[h(\zeta_0)+m\zeta_0h'(\zeta_0)\varphi(h(\zeta_0))]}{h(\zeta_0)+\alpha m\zeta_0h'(\zeta_0)\varphi(h(\zeta_0))}\in\mathcal{P},$$

or equivalently $B(\zeta_0)\notin h(\mathbb{D})$, which contradicts to the assumption. □

Next, we consider the geometric mean of a particular form below.

THEOREM 2.2. *Let $\alpha, \gamma \in \langle 0, 1\rangle$, $h \in Q$ be a convex function and let φ be analytic in a domain containing $h(\mathbb{D})$ and satisfying* $\operatorname{Re}\varphi(h(\zeta)) \geq 0$ *for* $\zeta \in \partial\mathbb{D} \setminus E(h)$. *If* $p \in \mathcal{H}[\psi]$, *where* $\psi(p(z), p(z) + \alpha z p'(z)\varphi(p(z))) = G(p(z), p(z) + \alpha z p'(z)\varphi(p(z)))$ *and*

$$G(p(z), p(z) + \alpha z p'(z)\phi(p(z))) = p(z)\left[\frac{p(z) + zp'(z)\varphi(p(z))}{p(z) + \alpha z p'(z)\varphi(p(z))}\right]^{\gamma} \prec h(z), \ z \in \mathbb{D},$$

then $p(z) \prec h(z)$ *in* $\mathbb{D}$.

Proof. We first note that

$$\begin{aligned} G((p(z), p(z) + \alpha z p'(z)\phi(p(z))) &= G(p(z), H(p(z), p(z) + \alpha z p'(z)\phi(p(z)) \\ &= p(z)^{1-\gamma}\left[\frac{p(z)(p(z) + zp'(z)\varphi(p(z)))}{p(z) + \alpha z p'(z)\varphi(p(z))}\right]^{\gamma}. \end{aligned}$$

The case $\gamma = 0$ is obvious, then we will consider only $\gamma \in (0, 1\rangle$. We note that $h(0) = p(0)$. As in the proof of Theorem 2.1 suppose, on the contrary, that $p(z) \not\prec h(z)$. Then, by Lemma 1.3, there exist $z_0 \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D} \setminus E(h)$ and $m \geq 1$ such that $p(z_0) = h(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 h'(\zeta_0)$.

Applying the above relations we obtain

$$p(z_0)^{1-\gamma}\left[\frac{p(z_0)(p(z_0)+z_0p'(z_0)\varphi(p(z_0)))}{p(z_0)+\alpha z_0 p'(z_0)\varphi(p(z_0))}\right]^{\gamma} = h(\zeta_0)^{1-\gamma}\left[\frac{h(\zeta_0)(h(\zeta_0)+m\zeta_0 h'(\zeta_0)\varphi(h(\zeta_0)))}{h(\zeta_0)+\alpha m\zeta_0 h'(\zeta_0)\varphi(h(\zeta_0))}\right]^{\gamma}.$$

In Theorem 2.1 we prove that the expression $B(\zeta_0)$ in a square brackets of a right hand side (2) is the element of a supporting half-plane $\mathcal{P}$. Now, we prove that also $h(\zeta_0)[B(\zeta_0)]^{\gamma} \in \mathcal{P}$. In the connected open set $\mathbb{C} \setminus L$, where L is the half-line from the origin to infinity, such that $L \cap \mathcal{P} = \emptyset$, the principal value of a branch of $\log w$ is well defined. In the case, when $\mathcal{P}$ is the halfplane $\log(\mathcal{P}) = S$ which is the horizontal strip of width π, so is the convex domain. Note also that the boundary points are mapped onto boundary points. Then the arithmetic mean of $\log(h(\zeta_0))$ and $\log\big(B(\zeta_0)\big)$ is also the element of $\log(\mathcal{P})$ that is equivalent to

$$\begin{aligned} &(1-\gamma)\log(h(\zeta_0)) + \gamma\log\left\{\frac{h(\zeta_0)(h(\zeta_0) + m\zeta_0 h'(\zeta_0)\varphi(h(\zeta_0)))}{h(\zeta_0) + \alpha m\zeta_0 h'(\zeta_0)\varphi(h(\zeta_0))}\right\} \\ &= \log\left\{h^{1-\gamma}(\zeta_0)\left[\frac{h(\zeta_0)(h(\zeta_0) + m\zeta_0 h'(\zeta_0)\varphi(h(\zeta_0)))}{h(\zeta_0) + \alpha m\zeta_0 h'(\zeta_0)\varphi(h(\zeta_0))}\right]^{\gamma}\right\} \\ &= \log\left\{h(\zeta_0)^{1-\gamma}[B(\zeta_0)]^{\gamma}\right\} \in \log(\mathcal{P}). \end{aligned}$$

This and the univalence of $\log^{-1} w$ in the strip S yields that $h(\zeta_0)^{1-\gamma}[B(\zeta_0)]^{\gamma} \in \mathcal{P}$ or equivalently $h(\zeta_0)^{1-\gamma}[B(\zeta_0)]^{\gamma} \notin h(\mathbb{D})$, which contradicts to the assumption. □

3. Special cases

REMARK 3.1. Let us note, that the result given in Theorem 2.1, in an abbreviated form, are the following

$$H(u, v) = T(A(T(u), T(v)) \prec h \Rightarrow u \prec h, \tag{3}$$

and in the Theorem 2.2:

$$G(u, H(u,v)) = G\left(u, T(A(T(u), T(v))\right) \prec h \Rightarrow u \prec h. \tag{4}$$

Hence, it has a much more general meaning, due to the fact that the arithmetic, harmonic or geometric means can be replaced by special cases of those appearing in Theorems 2.1 and 2.2. Examples of such applications are listed below.

Setting $\alpha = 1/2, u = p(z)$ and $v = p(z) + zp'(z)\varphi(p(z))$ we have from (3) the following corollary.

Corollary 3.2. *Let $h \in Q$ be a convex function. Also let φ be analytic in a domain containing $h(\mathbb{D})$ and satisfying $\operatorname{Re}\varphi\left(h(\zeta)\right) \geq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(h)$. If $p(0) = h(0)$, $p \in \mathcal{H}[\psi]$, where $\psi(u,v) = H(u,v)$ and*

$$H(u,v) = \frac{2\,p(z)(p(z) + zp'(z)\varphi(p(z)))}{2\,p(z) + zp'(z)\varphi(p(z))} \prec h(z),\ \ z \in \mathbb{D}, \tag{5}$$

then $p(z) \prec h(z)$ in $\mathbb{D}$.

Putting $u = p(z), v = p(z) + \frac{zp'(z)}{\beta p(z)+\delta}$ we obtain from (3).

Corollary 3.3. *Let $\alpha \in \langle 0, 1\rangle$ and $\beta > 0$, $\operatorname{Re}\delta \geq 0$. Also, let $h \in Q$ be a convex function satisfying $\operatorname{Re}(\beta h(z) + \delta) \geq 0, h(z) \neq -\delta/\beta$. If $p(0) = h(0)$, $p \in \mathcal{H}[\psi]$, where $\psi(u,v) = H(u,v)$ and*

$$H(u,v) = \frac{p(z)\left(p(z) + \frac{zp'(z)}{\beta p(z)+\delta}\right)}{p(z) + \alpha\,\frac{zp'(z)}{\beta p(z)+\delta}} \prec h(z),\ \ z \in \mathbb{D},$$

then $p(z) \prec h(z)$ in $\mathbb{D}$.

Applying (3) to $u = 1/p(z), v = 1/(p(z) + zp'(z)\varphi(p(z)))$ we have.

Corollary 3.4. *Let $\alpha \in \langle 0, 1\rangle$ and let $h \in Q$ be a convex function. Also let φ be analytic in a domain containing $h(\mathbb{D})$ and fulfilling $\operatorname{Re}\varphi\left(h(\zeta)\right) \geq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(h)$. If $p(0) = h(0) = 1$, $p \in \mathcal{H}[\psi]$, where $\psi(u,v) = H(u,v)$ and*

$$H(u,v) = \frac{1}{p(z) + \alpha\, zp'(z)\varphi(p(z))} \prec h(z),\ \ z \in \mathbb{D},$$

then $p(z) \prec h(z)$ in $\mathbb{D}$.

Similarly, for the geometric means we obtain from (4) the following.

For $\alpha = 1/2, u = p(z), v = p(z) + zp'(z)\varphi(p(z))$, we have from (4) the next corollary.

Corollary 3.5. *Let $\gamma \in \langle 0, 1\rangle$ and let $h \in Q$ is a convex function. Also let φ be analytic in a domain containing $h(\mathbb{D})$ and satisfying $\operatorname{Re}\varphi\left(h(\zeta)\right) \geq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(h)$. If $p(0) = h(0)$, $p \in \mathcal{H}[\psi]$, where $\psi(u,v) = G(u,v)$ and*

$$G(u,v) = p(z)\left\{\frac{2(p(z) + zp'(z)\varphi(p(z)))}{2\,p(z) + zp'(z)\varphi(p(z))}\right\}^{\gamma} \prec h(z),\ \ z \in \mathbb{D},$$

then $p(z) \prec h(z)$ in $\mathbb{D}$.

Setting $u = p(z), v = p(z) + zp'(z)\varphi(p(z))$, we obtain from (4) the following corollary.

Corollary 3.6. *Let* $\alpha, \gamma \in \langle 0,1\rangle$, $\beta > 0$, $\operatorname{Re}\delta \geq 0$. *Also, let* $h \in Q$ *be a convex function such that* $\operatorname{Re}(\beta h(z) + \delta) \geq 0$, $h(z) \neq -\delta/\beta$. *If* $p(0) = h(0)$, $p \in \mathcal{H}[\psi]$, *where* $\psi(u,v) = G(u,v)$ *and*

$$G(u,v) = p(z)\left\{\frac{p(z)\left(p(z) + \frac{zp'(z)}{\beta p(z)+\delta}\right)}{p(z) + \alpha\frac{zp'(z)}{\beta p(z)+\delta}}\right\}^{\gamma} \prec h(z),\ z \in \mathbb{D},$$

then $p(z) \prec h(z)$ *in* $\mathbb{D}$.

Let $u = 1/p(z), v = 1/(p(z)+zp'(z)\varphi(p(z)))$ then by (4), we arrive at the following corollary.

Corollary 3.7. *Let* $\alpha, \gamma \in \langle 0,1\rangle$ *and let* $h \in Q$ *be a convex function. Also let* φ *be analytic in a domain containing* $h(\mathbb{D})$ *and satisfying* $\operatorname{Re}\varphi\,(h(\zeta)) \geq 0$ *for* $\zeta \in \partial\mathbb{D}\setminus E(h)$. *If* $p(0) = h(0) = 1$, $p \in \mathcal{H}[\psi]$, *where* $\psi(u,v) = G(u,v)$ *and*

$$G(u,v) = p(z)\left\{\frac{1}{p(z) + \alpha\, zp'(z)\varphi(p(z))}\right\}^{\gamma} \prec h(z),\ z \in \mathbb{D}, \tag{6}$$

then $p(z) \prec h(z)$ *in* $\mathbb{D}$.

The preceding results yield a number of interesting special cases through suitable choices of the convex function h. In particular, we consider the classical choices $h(z) = (1+z)/(1-z)$ and $h(z) = [(1+z)/(1-z)]^{\lambda}$.

Corollary 3.8.

$$\operatorname{Re} H(u,v) = \operatorname{Re} T(A(T(u),T(v)) > 0 \Rightarrow \operatorname{Re} u > 0,$$

and

$$\operatorname{Re} G(u,H(u,v)) = \operatorname{Re} G\,(u, T(A(T(u),T(v))) > 0 \Rightarrow \operatorname{Re} u > 0.$$

Corollary 3.9. *Let* $\lambda \in (0,1\rangle$. *Then*

$$|\arg\, H(u,v)| = |\arg\, T(A(T(u),T(v))| < \frac{\lambda\pi}{2} \Rightarrow |\arg\, u| < \frac{\lambda\pi}{2},$$

and

$$|\arg\, G(u,H(u,v))| = |\arg\, G\,(u, T(A(T(u),T(v)))\,| < \frac{\lambda\pi}{2} \Rightarrow |\arg\, u| < \frac{\lambda\pi}{2}.$$

4. Aplication

The above theorems and corollaries give completely new results in Geometric Function Theory depending on the choice of functions p and h, below.

Choosing $\varphi(p(z)) = 1/p(z)$, with $p(z) = z/f(z)$, in Corollary 3.2 and 3.5, we obtain the following theorem.

THEOREM 4.1. *Let $\alpha \in \langle 0,1\rangle$, and let a convex function $h \in Q$ be such that $h(0) = 1$. If $f \in \mathcal{A}$ and*

$$\frac{\frac{z}{f(z)}\left(\frac{z}{f(z)}+1-\frac{zf'(z)}{f(z)}\right)}{\frac{z}{f(z)}+\alpha\left(1-\frac{zf'(z)}{f(z)}\right)} \prec h(z),\ z \in \mathbb{D}, \quad \text{then} \quad \frac{z}{f(z)} \prec h(z) \text{ in } \mathbb{D}.$$

THEOREM 4.2. *Let $\alpha, \gamma \in \langle 0,1\rangle$, and let a convex function $h \in Q$ be such that $h(0)$=1. If $f \in \mathcal{A}$ and*

$$\frac{z}{f(z)}\left[\frac{\frac{z}{f(z)}\left(\frac{z}{f(z)}+1-\frac{zf'(z)}{f(z)}\right)}{\frac{z}{f(z)}+\alpha\left(1-\frac{zf'(z)}{f(z)}\right)}\right]^{\gamma} \prec h(z),\ z \in \mathbb{D}, \quad \text{then} \quad z/f(z) \prec h(z) \text{ in } \mathbb{D}.$$

For the same choice of φ and $p(z)$, Corollaries 3.4 and 3.7 yield the next theorem.

THEOREM 4.3. *Let $\alpha \in \langle 0,1\rangle$, and let a convex function $h \in Q$ be such that $h(0) = 1$. If $f \in \mathcal{A}$ and*

$$\frac{\frac{f(z)}{z}}{1+\alpha\left(\frac{f(z)}{z}-f'(z)\right)} \prec h(z),\ z \in \mathbb{D}, \quad \text{then} \quad \text{in } \mathbb{D}.$$

then $\frac{z}{f(z)} \prec h(z)$ *in* $\mathbb{D}$.

THEOREM 4.4. *Let $\alpha, \gamma \in \langle 0,1\rangle$, and let a convex function $h \in Q$ be such that $h(0)$=1. If $f \in \mathcal{A}$ and*

$$\frac{z}{f(z)}\left[\frac{\frac{f(z)}{z}}{1+\alpha\left(\frac{f(z)}{z}-f'(z)\right)}\right]^{\gamma} \prec h(z),\ z \in \mathbb{D}, \quad \text{then} \quad z/f(z) \prec h(z) \text{ in } \mathbb{D}.$$

Applying Corollaries 3.4 and 3.7 to $\varphi(p(z)) \equiv 1$, and $p(z) = f'(z)$ we obtain the following.

THEOREM 4.5. *Let $\alpha \in \langle 0,1\rangle$, and let a convex function $h \in Q$ be such that $h(0) = 1$. If $f \in \mathcal{A}$ and*

$$\frac{f'(z)\left(1+\frac{zf''(z)}{f'(z)}\right)}{1+\alpha\frac{zf''(z)}{f'(z)}} \prec h(z),\ z \in \mathbb{D}, \quad \text{then} \quad f'(z) \prec h(z) \text{ in } \mathbb{D}.$$

THEOREM 4.6. *Let $\alpha, \gamma \in \langle 0,1\rangle$, and let a convex function $h \in Q$ be such that $h(0) = 1$. If $f \in \mathcal{A}$ and*

$$f'(z)\left[\frac{1+\frac{zf''(z)}{f'(z)}}{f'(z)+\alpha\frac{zf''(z)}{f'(z)}}\right]^{\gamma} \prec h(z),\ z \in \mathbb{D}, \quad \text{then} \quad f'(z) \prec h(z) \text{ in } \mathbb{D}.$$

5. Discussion

The solution of the problem of generalized three known Pythagorean means can be found in the literature mainly in relation to the arithmetic mean, see e.g. [8].

Research on differential subordination of harmonic and geometric means has begun with the works of [2–5]. In these papers, general theorems of differential subordination covering harmonic, geometric, and combined (arithmetic and geometric) means were discussed. In [4] and [2], the authors found the best dominants for special cases of harmonic means.

However, it should be noted the importance and relevance of the considered problems. The obtained relations can be used to determine the properties of the solution of a special differential equation as well as to determine the properties of inclusion between classes of functions defined by subordination.

The most important, and still open, problem is to find the best dominant of such subordination for an arbitrarily chosen convex function h, hence the direction of future research may be to find the best dominants.

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Institute of Mathematics and Information Technology, The State University of Applied Sciences in Chełm, Pocztowa 54, 22-100 Chełm, Poland

E-mail: stankanas@gmail.com

ORCID iD: https://orcid.org/0000-0001-7135-6504