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RADIUS ESTIMATES OF A SUBCLASS OF UNIVALENT FUNCTIONS

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Abstract. For analytic functions f normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk U, a class $P_\alpha(\lambda)$ of f defined by $|D_z^\alpha(\frac{z}{f(z)})| \le \lambda$, where D_z^α denotes the fractional derivative of order α, $m \le \alpha < m+1$, $m \in \mathbf{N}_0$, is introduced. In this article, we study the problem when $\frac{1}{r} f(rz) \in P_\alpha(\lambda)$, $3 \le \alpha < 4$.

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in $U := \{z \in \mathbf{C} : |z| < 1\}$ and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \tag{1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in U.

In [1], Srivastava and Owa, gave definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbf{C}$ as follows:

Definition 1.1. The fractional derivative D_z^α of order α is defined, for a function $f(z)$, by

$$D_z^\alpha f(z) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha}\, d\zeta, & 0 \le \alpha < 1, \\ (\frac{d}{dz})^{m+1} D_z^{\alpha-m} f(z), & m \le \alpha < m+1,\ m \in \mathbf{N}_0. \end{cases}$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbf{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

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For $f(z) \in \mathcal{A}$, we define the class $P_\alpha(\lambda)$ of $f(z)$ if $f(z)$ satisfies $\frac{f(z)}{z} \neq 0$, $(z \in U)$ and

$$\Big| D_z^\alpha \Big(\frac{z}{f(z)}\Big) \Big| \leq \lambda, \quad z \in U, \tag{2}$$

for some real $\lambda > 0$ and $m \leq \alpha < m+1$, $m \in \mathbf{N}_0$.

Obradović and Ponnusamy [2] have studied the subclass $P_2(\lambda)$ for $f(z) \in \mathcal{A}$ satisfying $\frac{f(z)}{z} \neq 0$, $(z \in U)$ and $\big|\big(\frac{z}{f(z)}\big)''\big| \leq \lambda$, $(z \in U)$ for some real $\lambda > 0$.

Recently, Kuroki et al. studied the subclass $P_3(\lambda)$ for $f(z) \in \mathcal{A}$ satisfying $\frac{f(z)}{z} \neq 0$, $(z \in U)$ and $\big|\big(\frac{z}{f(z)}\big)'''\big| \leq \lambda$, $(z \in U)$ for some real $\lambda > 0$ (see [3]).

In this work, we study the problem when $\frac{1}{r} f(rz) \in P_\alpha(\lambda)$, $3 \leq \alpha < 4$. For this purpose, we need the following result.

Lemma 1.1. [4] *If $f(z) \in \mathcal{S}$ and*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \tag{3}$$

then $\sum\limits_{n=1}^{\infty} (n-1)|b_n|^2 \leq 1$.

2. Results

First we derive the following result.

Theorem 2.1. *Let $f \in \mathcal{A}$ and $\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \neq 0$, $(z \in U)$. If $f(z)$ satisfies*

$$\sum_{n=m+1}^{\infty} n(n-1)(n-2)\cdots(n-m)|b_n| \leq \lambda,$$

$(\lambda > 0$, $m \leq \alpha < m+1$, $m \in \mathbf{N}_0)$, then $f(z) \in P_\alpha(\lambda)$.

Proof. By Definition 1.1, we observe

$$\begin{aligned}
\Big| D_z^\alpha \Big(\frac{z}{f(z)}\Big) \Big| &\leq \frac{\sum_{n=m+1}^{\infty} n(n-1)(n-2)\cdots(n-m)|b_n|}{\Gamma(m+1-\alpha)} \int_0^z |(z-\zeta)|^{m-\alpha}\, d\zeta \\
&\leq \frac{\sum_{n=m+1}^{\infty} n(n-1)(n-2)\cdots(n-m)|b_n|}{\Gamma(m+2-\alpha)} \\
&< \sum_{n=m+1}^{\infty} n(n-1)(n-2)\cdots(n-m)|b_n| \leq \lambda.
\end{aligned}$$

Hence, $f(z) \in P_\alpha(\lambda)$. ∎

Corollary 2.2. *Let $f \in \mathcal{A}$ and $\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \neq 0$, $(z \in U)$. If $f(z)$ satisfies*

$$\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)|b_n| \leq \lambda,$$

$(\lambda > 0$, $3 \leq \alpha < 4)$, then $f(z) \in P_\alpha(\lambda)$.

Proof. By letting $m = 3$ in Theorem 2.1. ■

THEOREM 2.3. *Let $f \in \mathcal{S}$ and $\lambda > 0$. Then the function $\frac{1}{r} f(rz)$, $(r > 0,$ $z \in U)$ belongs to the class $P_\alpha(\lambda)$ for $3 \le \alpha < 4$ and $0 < r \le r_0(\lambda)$, where $r_0(\lambda)$ is the smallest root of the equation*

$$\begin{aligned} F(r) := r^2\big(&A_1(r) - 11(1-r^2)A_2(r) + 47(1-r^2)^2 A_3(r) - 97(1-r^2)^3 A_4(r) \\ &+ 96(1-r^2)^4 A_5(r) - 36(1-r^2)^5 A_6(r)\big) - \lambda^2(1-r^2)^8 = 0, \end{aligned} \tag{4}$$

where

$$\begin{aligned} A_1(r) &:= 5 + 424r^2 + 2989r^4 + 3544r^6 + 989r^8 + 88r^{10}, \\ A_2(r) &:= 5 + 197r^2 + 668r^4 + 268r^6 + 14r^8 + 9r^{10}, \\ A_3(r) &:= 5 + 86r^2 + 108r^4 - 14r^6 + 9r^8, \\ A_4(r) &:= 5 + 23r^2 - r^4 - 3r^6, \\ A_5(r) &:= 1 + 4r^2 + r^4, \\ A_6(r) &:= 1 + r^2, \end{aligned}$$

in the interval $(0,1)$.

Proof. Let $f \in \mathcal{S}$. Since $\frac{z}{f(z)} \neq 0$, $(z \in U)$, if we write $\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n$, then we have

$$\frac{z}{\frac{1}{r} f(rz)} = 1 + \sum_{n=1}^{\infty} (r^n b_n) z^n,$$

for $0 < r < 1$. It follows from Lemma 1.1 that

$$\sum_{n=4}^{\infty} (n-1)|b_n|^2 \le \sum_{n=1}^{\infty} (n-1)|b_n|^2 \le 1.$$

To verify that $\frac{1}{r} f(rz) \in P_\alpha(\lambda)$ for $3 \le \alpha < 4$, we have to show that

$$\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)|r^n b_n| \le \lambda, \qquad (\lambda > 0, \quad 3 \le \alpha < 4) \tag{5}$$

by mean of Corollary 2.2. Now by the Cauchy-Schwarz inequality for the left-hand side of (5), we have

$$\begin{aligned} &\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)|r^n b_n| \\ &\quad = \sum_{n=4}^{\infty} \Big(n^2(n-1)(n-2)^2(n-3)^2|r^n|^2\Big)^{\frac{1}{2}} \Big((n-1)|b_n|^2\Big)^{\frac{1}{2}} \\ &\quad \le \Big(\sum_{n=4}^{\infty} n^2(n-1)(n-2)^2(n-3)^2|r^n|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=4}^{\infty} (n-1)|b_n|^2\Big)^{\frac{1}{2}} \\ &\quad \le \Big(\sum_{n=4}^{\infty} n^2(n-1)(n-2)^2(n-3)^2 r^{2n}\Big)^{\frac{1}{2}} \end{aligned}$$

$$= \frac{1}{(1-r^2)^4}\Big[r^2\Big(A_1(r) - 11(1-r^2)A_2(r) + 47(1-r^2)^2A_3(r)$$
$$- 97(1-r^2)^3A_4(r) + 96(1-r^2)^4A_5(r) - 36(1-r^2)^5A_6(r)\Big)\Big]^{\frac{1}{2}}.$$

Consequently, $\frac{1}{r}f(rz) \in P_\alpha(\lambda)$ for $3 \le \alpha < 4$ and $0 < r \le r_0(\lambda)$, where $r_0(\lambda)$ is the positive solution for the equation (4), which satisfies that $F(0) = -\lambda^2 < 0$ and $F(1) = A_1(1) > 0$. Hence (4) has a solution $r_0(\lambda)$ in the interval $(0,1)$. This completes the proof of the theorem. ■

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