TRIGONOMETRIC POLYNOMIAL RINGS AND THEIR FACTORIZATION PROPERTIES

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Abstract. Consider the rings S and S', of real and complex trigonometric polynomials over the field Q and its algebraic extension $Q(i)$ respectively. Then S is an FFD, whereas S' is a Euclidean domain. We discuss irreducible elements of S and S', and prove a few results on the trigonometric polynomial rings T and T' introduced by G. Picavet and M. Picavet in [Trigonometric polynomial rings, Commutative ring theory, Lecture notes on Pure Appl. Math., Marcel Dekker, Vol. 231 (2003), 419–433]. We consider several examples and discuss the trigonometric polynomials in terms of irreducibles (atoms), to study the construction of these polynomials from irreducibles, which gives a geometric view of this study.

1. Introduction

Trigonometric polynomials are widely used in different fields of engineering and science, like trigonometric interpolation applied to the interpolation of periodic functions, approximation theory, discrete Fourier transform, and real and complex analysis, etc. We are developing this study by keeping in mind the possibility that studying factorization properties of these polynomials could help studying the above fields and especially Fourier series, that is, the study of big waves (a trigonometric polynomial) in terms of small wavelets (irreducibles). The study of Fourier series is a vast field of study by itself and this study will help to understand a big Fourier series in terms of smaller Fourier series.

We refer to [10, 11] and reference therein, for a short review of some of the recent interesting results on nonnegative trigonometric polynomials and their applications in Fourier series, signal processing, approximation theory, function theory and number theory. Many applications, especially in mechanical engineering and in numerical analysis lead to quantifier elimination problems with trigonometric functions involved (see [22]). Decompositions of trigonometric polynomials with applications to multivariate subdivision schemes is studied in [12], random almost periodic trigonometric polynomials and applications to ergodic theory can be found in [7], a detailed treatment of trigonometric series can be found in [31], and a new

proof of a theorem of Littlewood concerning flatness of unimodular trigonometric polynomials is given in [4], this proof is shorter and simpler than Littlewood's. Inspired by the above stated applications and a lot more, we investigate trigonometric polynomials using an algebraic approach. Throughout this article we follow the notation and definitions introduced in [23, 30] unless mentioned otherwise.

By a trigonometric polynomial we mean a finite linear combination of $\sin(nx)$ and $\cos(nx)$ with n a natural number. The coefficients may be taken as real or complex. For complex coefficients, there is no difference between such a finite linear combination and a finite Fourier series. Any function of the form

$$a_0 + \sum_{k=1}^{n} (a_k \cos kx + ib_k \sin kx) \ : \ x \in \mathbb{R}, \ a_k, \ b_k \in \mathbb{C}$$

is called a complex trigonometric polynomial of degree n. The familiar Fourier coefficient formulas

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \text{ and } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

for $n > 0$, show that the coefficients are uniquely determined. Using Euler's formula the above polynomial can be rewritten as

$$\sum_{k=-n}^{n} c_k e^{inx} \ : \ x \in \mathbb{R}, \ c_k \in \mathbb{C}$$

Analogously, we can define a real trigonometric polynomial and its degree. The set of all trigonometric polynomials form a ring, in fact an integral domain. In particular, if we restrict the coefficients to be real even then we get an integral domain. The degree of a non-zero trigonometric polynomial is defined as the largest value of n for which a_n and b_n are not both zero. The degrees behave just like ordinary polynomials, that is, the product of two trigonometric polynomials of degree m and n respectively, is a trigonometric polynomial of degree $m + n$.

In polynomial rings, factorization properties of integral domains have been a frequent topic of recent mathematical literature. Following Cohn [6], an integral domain, say D, is atomic if each non-zero non-unit of D is a product of irreducible elements (*atoms*) of D, and it is well known that UFDs, PIDs and Noetherian domains are atomic domains. An integral domain D satisfies the *ascending chain condition on principal ideals (ACCP)* if there does not exist any infinite strictly ascending chain of principal integral ideals of D. Every PID, UFD and Noetherian domain satisfies ACCP and a domain satisfying ACCP is atomic. Grams [15] and Zaks [29] provided examples of atomic domains which do not satisfy ACCP. An integral domain D is a *bounded factorization domain (BFD)* if it is atomic and for each non-zero non-unit of D, there is a bound on the length of factorization into products of irreducible elements (cf. [1]). For examples of BFDs are UFDs and Noetherian or Krull domains, cf. [1, Proposition 2.2].

As we know, an integral domain D is said to be a *half-factorial domain (HFD)* if D is atomic and whenever $x_1, \ldots, x_m = y_1, \ldots, y_n$, where $x_1, x_2, \ldots, x_m$, $y_1, y_2, \ldots, y_n$ are irreducibles in D, then $m = n$. A UFD is obviously an HFD, but

the converse fails, since any Krull domain D with $CI(D) \cong \mathbb{Z}_2$ is an HFD [28], but not a UFD. Moreover, a polynomial extension of an HFD is not an HFD, for example $\mathbb{Z}[\sqrt{-3}][X]$ is not an HFD, as $\mathbb{Z}[\sqrt{-3}]$ is an HFD but not integrally closed [9]. Following [1], an integral domain D is an *idf-domain* if each non-zero element of D has at most a finite number of non-associate irreducible divisors. Also an integral domain is a *finite factorization domain (FFD)* if each non-zero non-unit of D has only a finite number of non-associate divisors and hence, only a finite number of factorizations up to order and associates. Moreover an integral domain D is an FFD if and only if D is atomic and an idf-domain [1, Theorem 5.1].

In general,

$$\textbf{UFD} \Longrightarrow \textbf{idf} - \textbf{domain} \Longleftarrow \textbf{FFD} \Longrightarrow \textbf{BFD} \Longrightarrow \textbf{ACCP} \Longrightarrow \textbf{Atomic}.$$

But none of the above implications is reversible. In this study we investigate the factorization properties of trigonometric polynomial rings. The basic concepts, notions and terminology are as standard in [23, 26, 27], unless mentioned otherwise. In [24, Theorem], J. F. Ritt deduced the following: "If $1 + a_1 e^{\alpha_1 x} + \cdots + a_n e^{\alpha_n x}$ is divisible by $1 + b_1 e^{\beta_1 x} + \cdots + b_r e^{\beta_r x}$ with no $b = 0$, then every β is a linear combination of $\alpha_1, \ldots, \alpha_n$ with rational coefficients". Recently G. Picavet and M. Picavet [23] investigated some factorization properties in trigonometric polynomial rings. Actually, when we replace all α_k above by im, with $m \in \mathbb{Z}$, we obtain trigonometric polynomials. Whereas

$$T' = \left\{ \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) \ : \ n \in \mathbb{N}, \ a_k, b_k \in \mathbb{C} \right\} \text{ and}$$

$$T = \left\{ \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) \ : \ n \in \mathbb{N}, \ a_k, b_k \in \mathbb{R} \right\}$$

are the trigonometric polynomial rings over $\mathbb{C}$ and $\mathbb{R}$, respectively. Here T' is a Euclidean domain and T is a Dedekind half factorial domain (see [23, Theorem 2.1 & Theorem 3.1].

In our previous papers [26] and [27], this study is extended to factorization properties of subrings in T' and T, where we introduce subrings S_0' and S_1' of T'. We have also proved that S_0' is a Noetherian HFD and S_1' is a Euclidean domain. In this paper, we continue the above investigations to find the factorization properties in trigonometric polynomial rings, begun in [23] and extended in [26, 27]. Moreover, we establish the study of factorization properties of trigonometric polynomials with coefficients from the field $\mathbb{Q}$ and its algebraic extension $\mathbb{Q}(i)$, instead of $\mathbb{R}$ and $\mathbb{C}$, that is, we set

$$S' = \left\{ \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) \ : \ n \in \mathbb{N}, \ a_k, b_k \in \mathbb{Q}(i) \right\} \text{ and}$$

$$S = \left\{ \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) \ : \ n \in \mathbb{N}, \ a_k, b_k \in \mathbb{Q} \right\}.$$

Note that while studying the factorization properties of trigonometric polynomials over the field $\mathbb{Q}(i)$, instead of $\mathbb{C}$, we are left with less results, as $\mathbb{Q}(i)$ is not algebraically closed. This paper is a continuation of [26], where we have already studied

the factorization properties of the subrings S_1' (Euclidean domain$\simeq (\mathbb{Q}[X])_X$) and S_0' (Noetherian HFD$\simeq (\mathbb{Q}+X\mathbb{Q}(i)[X])_X$) of S', where $S_1' \subseteq S_0' \subseteq S'$.

Indeed, $\sin^2 x = (1 - \cos x)(1 + \cos x)$ shows that two different non-associated irreducible factorizations of the same element may appear. Throughout we denote by $\cos kx$ and $\sin kx$ the two functions $x \mapsto \cos kx$ and $x \mapsto \sin kx$ (defined over $\mathbb{R}$). Also from basic trigonometric identities, it is obvious that for each $n \in \mathbb{N} \setminus \{1\}$, $\cos nx$ represents a polynomial in $\cos x$ with degree n and $\sin nx$ represents the product of $\sin x$ and a polynomial in $\cos x$ with degree $n - 1$. Conversely, by trigonometric linearization formulas, it follows that any product $\cos^n x \sin^p x$ can be written as:

$$\sum_{k=0}^{q} (a_k \cos kx + b_k \sin kx), \text{ where } q \in \mathbb{N} \text{ and } a_k, \ b_k \in \mathbb{Q}.$$

Hence $S = \mathbb{Q}[\cos x, \sin x] \subseteq \mathbb{R}[\cos x, \sin x] = T$ and $S' = \mathbb{Q}(i)[\cos x, \sin x] \subseteq \mathbb{C}[\cos x, \sin x] = T'$.

Let us describe what is ahead of us. In Section 2 we prove that the ring S is an FFD being a Dedekind domain and in Section 3 we prove that the ring S' is a Euclidean domain (isomorphic to $\mathbb{Q}(i)[X]_{\alpha X}$). In Section 4 we establish few results on localization of T. In Section 5 we construct several interesting examples. Some of these examples also verify the results of [23], on T. Finally, in Section 6, we discuss the trigonometric polynomials in terms of irreducibles (atoms), which turns the direction of this study towards geometry. In some sense, in this section our aim is to study the geometrical behavior of trigonometric polynomials. In this section we have used computer package `Mathematica` [19] to draw trigonometric polynomials.

2. The ring $\mathbb{Q}[\cos x, \sin x]$

In what follows, we consider the ring S of real trigonometric polynomials over the field $\mathbb{Q}$. We prove that S is a finite factorization Dedekind domain. We also describe irreducible elements and units in S.

The ring of real trigonometric polynomials T possesses some interesting features, for instance the identity $\sin^2 x = (1 - \cos x)(1 + \cos x)$ provides an example of non-unique factorization. But this identity breaks down over the ring of complex trigonometric polynomials T'. Now consider the ring S of real trigonometric polynomials over the field $\mathbb{Q}$. The identity $\sin^2 x = (1 - \cos x)(1 + \cos x)$ again provides an example of non-unique factorization in the ring S. The degree of a non-zero trigonometric polynomial of S, is defined as the largest value of n for which a_n and b_n are not both zero. The degrees behave just like ordinary polynomials, that is, the product of two trigonometric polynomials of degree m and n respectively, is a trigonometric polynomial of degree $m + n$. Actually, for each $n \in N \setminus \{1\}$, we get $\cos nx$ as a polynomial in $\cos x$ of degree n and $\sin nx$ as the product of $\sin x$ by a polynomial in $\cos x$ of degree $n - 1$. So each polynomial $P = \sum_{k=0}^{n}(a_k \cos kx + b_k \sin kx) \in S$ can be put in the form $P = A(\cos x) + \sin x B(\cos x)$, where $A(\cos x)$ and $B(\cos x)$

are polynomials in $\cos x$ of degree k and $k - 1$ respectively. This can be seen more clearly from the following definition.

DEFINITION 1. Let $P = \sum_{k=0}^{n} a_k \cos^k x + \left(\sum_{j=0}^{p} b_j \cos^j x \right) \sin x$, $a_k, b_j \in \mathbb{Q}$. The degree of P is defined as

$$\delta(P) = \begin{cases} \sup\{k, \ j + 1/a_k, \ b_j \neq 0\}, & \text{if } P \neq 0, \\ -\infty, & \text{if } P = 0. \end{cases}$$

This definition is due to the fact that the family $\{\cos^k x, \sin x \cos^k x\}$ is a basis of S over $\mathbb{Q}$ and gives $\delta(PQ) = \delta(P) + \delta(Q)$, for any $P, Q \in S$. In particular, a trigonometric polynomial of degree one is irreducible and $\mathbb{Q} \setminus \{0\}$ is the set of unit elements of S. This gives us the following proposition.

PROPOSITION 1. *The trigonometric polynomial ring S forms an integral domain. Furthermore,*

 (i) *the units (invertible elements) in this domain are the elements of degree zero, that is, the constant functions,*

(ii) *the elements of degree one are irreducible.*

Proof. The proof is straightforward, just as with ordinary polynomials, and we leave the details to the reader. ∎

THEOREM 1. *The integral domain $\mathbb{Q}[\cos x, \sin x]$ is an FFD.*

Proof. Consider the substitution morphism $g \colon \mathbb{Q}[X, Y] \to \mathbb{Q}[\cos x, \sin x]$, defined by $g(X) = \cos x$ and $g(Y) = \sin x$, such that $g(X^2 + Y^2 - 1) = g(X^2) + g(Y^2) - 1 = \cos^2 x + \sin^2 x - 1 = 0$, this implies $(X^2 + Y^2 - 1) = \mathrm{Ker}(g)$, therefore $\mathbb{Q}[\cos x, \sin x] \simeq \mathbb{Q}[X, Y]/(X^2 + Y^2 - 1)$. Since $\mathbb{Q}[X, Y]/(X^2 + Y^2 - 1) \simeq \mathbb{Q}[X][Y]/(X^2 + Y^2 - 1)$, with $\mathbb{Q}[X]$ factorial and $1 - X^2$ is square free in $\mathbb{Q}[X]$, $\mathbb{Q}[X, Y]/(X^2 + Y^2 - 1)$ is integrally closed [13, Lemma 11.1]. Also $\mathbb{Q}[X, Y]/(X^2 + Y^2 - 1)$ is one dimensional and Noetherian. Therefore, $\mathbb{Q}[\cos x, \sin x]$ is a Dedekind domain and hence a Krull domain. Since a Krull domain is both atomic and an idf-domain, so it is an FFD [1, Theorem 5.1]. ∎

REMARK 1. We have the following interesting remarks for the real trigonometric polynomial rings.

(1) $\mathbb{R}[\cos x, \sin x]$ is an HFD [23, Theorem 3.1], whereas $\mathbb{Q}[\cos x, \sin x]$ is an FFD.

(2) S is a free $\mathbb{Q}[\cos x]$-module and has a basis $\{1, \sin x\}$.

(3) $\mathbb{Q}[\cos x]$ is a Euclidean domain because $\mathbb{Q}[\cos x] \simeq \mathbb{Q}[X]$, therefore the FFD S lies between the two Euclidean domains $\mathbb{Q}[\cos x]$ and $\mathbb{Q}(i)[\cos x, \sin x]$ (see Theorem 2).

(4) S' is a free S-module and has a basis $\{1, i\}$.

(5) T is an S-module, also T' is an S-module.

(6) T' is an S'-module.

(7) The quotient field of $\mathbb{Q}[\cos x, \sin x]$ is $\mathbb{Q}(\cos x)[\sin x]$ and the quotient field of $\mathbb{Q}(i)[\cos x, \sin x]$ is $\mathbb{Q}(i)(\cos x)[\sin x]$.

3. The ring $\mathbb{Q}(i)[\cos x, \sin x]$

Consider the ring S' of complex trigonometric polynomials over the field $\mathbb{Q}(i)$. We will prove that S' is a Euclidean domain and describe the irreducible elements of S'. The complex exponential forms of sine and cosine shows that the ring of trigonometric polynomials with coefficients from $\mathbb{Q}(i)$ is the same as the ring of polynomials in positive and negative powers of $z = e^{ix}$ with coefficients from the field $\mathbb{Q}(i)$. To see that this is a unique factorization ring, we define the degree of a polynomial in z and z^{-1} as the difference between the largest and smallest exponents appearing in non-zero terms. According to this definition, the elements of degree zero are the monomials, which are exactly the invertible elements in this ring. The usual proof that ordinary polynomials over a field form a Euclidean domain then goes through with no essential change. To prove this we proceed as follows.

Consider the isomorphism $f \colon \mathbb{Q}(i)[X]_X \to S'$ defined through the substitution morphism $X \to e^{ix}$. Note that this isomorphism exists for more general case, that is, if we replace X by αX with $\alpha \in \mathbb{Q}(i)$, then again we have the isomorphism $f \colon \mathbb{Q}(i)[X]_{\alpha X} \to S'$ through the same substitution morphism $X \to e^{ix}$.

An arbitrary element $z \in \mathbb{Q}(i)[\cos x, \sin x]$ has the form $e^{-inx}P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{Q}(i)[X]$ and $deg(P) = 2n$, which is obtained by the complex exponential forms of sine and cosine, i.e. the relations $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$.

Conversely, as $e^{ix} = \cos x + i \sin x$, so it follows $e^{-inx}P(e^{ix}) \in S'$, $n \in \mathbb{N}$, $P(X) \in \mathbb{Q}(i)[X]$. Hence, we can find an isomorphism $f \colon \mathbb{Q}(i)[X]_{\alpha X} \to S'$ through the substitution morphism $X \to e^{ix}$, where $\alpha \in \mathbb{Q}(i)$ such that $\alpha X \in \mathbb{Q}(i)[X]$. As each element $z \in \mathbb{Q}(i)[X]_{\alpha X}$ can be written uniquely as $(\alpha X)^k P(X)$, $k \in \mathbb{Z}$, $P(X) \in \mathbb{Q}(i)[X]$ and $P(0) \neq 0$.

The mapping ϕ defining the Euclidean domain $\mathbb{Q}(i)[X]_{\alpha X}$ is given by $\phi(z) = deg(P)$ [25, Proposition 7]. A consequence of these observations is the following theorem.

THEOREM 2. *The integral domain $\mathbb{Q}(i)[\cos x, \sin x]$ is a Euclidean domain with quotient field $\mathbb{Q}(i)(\cos x)[\sin x]$. The irreducible elements of S' are, up to units, trigonometric polynomials of the form $\cos x + i \sin x - a$, $a \in \mathbb{Q}(i) \setminus \{0\}$.*

As a generalization of Ritt's factorization theorem the following corollary tells us about the factorization of elements in $T' = \mathbb{C}[\cos x, \sin x]$, which first appeared in [23, Corollary 1].

COROLLARY 1. *Let*

$$z = \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx), \ n \in \mathbb{N} \setminus \{1\}, \ a_k, b_k \in \mathbb{C} \ with \ (a_n, b_n) \neq (0, 0).$$

Let d be a common divisor of the integers k such that $(a_k, b_k) \neq (0,0)$. Then z has a unique factorization

$$z = \lambda(\cos nx - i \sin nx) \prod_{j=1}^{\frac{2n}{d}} (\cos dx + i \sin dx - \alpha_j), \;\; where \; \lambda, \alpha_j \in \mathbb{C} \setminus \{0\}.$$

REMARK 2. The factorization in [23, Corollary 1] is possible due to the fact that $\mathbb{C}$ is algebraically closed. Now there is a natural question whether there exists a similar type of factorization for the elements of $S' = \mathbb{Q}(i)[\cos x, \sin x]$. The answer is negative because $\mathbb{Q}(i)$ is not algebraically closed. So we are not able to find the same kind of irreducible decomposition for the elements of S'. But this would be quite interesting to find out such a irreducible factorization.

REMARK 3. As we showed that the mapping ϕ which defines the Euclidean domain $S' = \mathbb{Q}(i)[\cos x, \sin x]$ is given by $\phi(z) = deg(P)$, where $P(X) \in \mathbb{Q}(i)[X]$ is such that $z = f[X^k P(X)]$ with $k \in \mathbb{Z}$ and $P(0) \neq 0$, for any $z \in S' \setminus \{0\}$. The restriction of ϕ to $\mathbb{Q}[\cos x]$ provides that $\mathbb{Q}[\cos x]$ is a Euclidean domain if we define $\phi(P(\cos x)) = 2 \deg(P)$, where $P(\cos x) \in \mathbb{Q}[\cos x]$ and $P \in \mathbb{Q}[X]$, whereas

$$P(\cos x) = \sum_{k=0}^{n} a_k \cos^k x, \; n \in \mathbb{N}, \; a_k \in \mathbb{R}, \; a_n \neq 0, \; \text{then}$$

$$P(\cos x) = \sum_{k=0}^{n} a_k \left(\frac{e^{ix} + e^{-ix}}{2} \right)^k = \sum_{k=0}^{n} a_k \left(\frac{e^{i2x} + 1}{2e^{ix}} \right)^k.$$

By using the substitution morphism $X \to e^{ix}$, we have

$$P(\cos x) = f\left(\sum_{k=0}^{n} a_k \left(\frac{X^2 + 1}{2X} \right)^k \right) = f\left(\frac{2^n X^n}{2^n X^n} \sum_{k=0}^{n} a_k \left(\frac{X^2 + 1}{2X} \right)^k \right)$$

$$= f(X^{-n} 2^{-n} \sum_{k=0}^{n} a_k (X^2 + 1)^k (2X)^{n-k} = f(X^{-n} h(X)),$$

where

$$h(X) = 2^{-n} \sum_{k=0}^{n} a_k (X^2 + 1)^k (2X)^{n-k}$$

$$= 2^{-n} a_n (X^2 + 1)^n + \sum_{k=1}^{n-1} a_k (X^2 + 1)^k (2X)^{n-k}.$$

Therefore $h(0) = 2^{-n} a_n \neq 0$ and hence $\phi[P(\cos x)] = 2n = 2deg(P)$.

4. Ideals generated by irreducibles

Consider the rings T and T' of real and complex trigonometric polynomials over the field $\mathbb{R}$ and its algebraic extension $\mathbb{C}$ respectively. Then T is a Dedekind Half factorial domain, whereas T' is a Euclidean domain (see [23, Theorem 2.1 & Theorem 3.1]). We prove a few results on localization of T with respect to the ideals generated by irreducibles. Furthermore we give an example of fraction ring in T.

PROPOSITION 2. *Let $z = a\cos x + b\sin x + c \in T = \mathbb{R}[\cos x, \sin x]$, with $(a,b) \neq (0,0)$ and $c^2 > a^2 + b^2$ such that $m_o = (z)$. Then T_{m_o} is a principal ideal domain.*

Proof. The irreducible element $z = a\cos x + b\sin x + c \in T$, with $(a,b) \neq (0,0)$ and $c^2 > a^2 + b^2$ generates a maximal ideal [23, Theorem 3.8]. Since $T = \mathbb{R}[\cos x, \sin x]$ is a Dedekind domain, therefore by [3, p. 494], T_{m_o} is a PID. ∎

PROPOSITION 3. *For each $z = a\cos x + b\sin x + c \in T = \mathbb{R}[\cos x, \sin x]$ with $(a,b) \neq (0,0)$ and $c^2 = a^2 + b^2$, $T/(z)$ is local with all its elements nilpotent.*

Proof. Let $z = a\cos x + b\sin x + c \in T = \mathbb{R}[\cos x, \sin x]$ and $(a,b) \neq (0,0)$. Then (z) is the square of a maximal ideal if and only if $c^2 = a^2 + b^2$ [23, Theorem 3.8]. Since (z) is an ideal of T which is a square of a maximal ideal, so $T/(z)$ has a unique prime ideal and therefore is local [16, Exercise 14, p. 148], and every element of $T/(z)$ is nilpotent [16, Exercise 15, p. 148]. ∎

REMARK 4. The ring $T/(z)$ has a minimal prime ideal which contains all zero devisors, and all non units of $T/(z)$ are zero divisors [16, Exercise 15, p. 148]. Thus the non units of $T/(z)$ are contained in a minimal prime ideal. Also every zero divisor of $T/(z)$ is nilpotent, therefore (z) is a primary ideal [30, p. 152].

PROPOSITION 5. *Let $z = a\cos x + b\sin x + c \in T = \mathbb{R}[\cos x, \sin x]$, with $(a,b) \neq (0,0)$ and $c^2 > a^2 + b^2$ such that $mT_m \neq (0)$ where $m = (z)$. Then mT_m-adic completion $\hat{T}_m$ of T_m is a discrete valuation ring.*

Proof. The irreducible element $z \in T$, where $z = a\cos x + b\sin x + c$ with $(a,b) \neq (0,0)$ and $c^2 > a^2 + b^2$ generates a maximal ideal [23, Theorem 3.8]. Since $T = \mathbb{R}[\cos x, \sin x]$ is a Dedekind domain, therefore for every maximal ideal $m = (z)$, T_m is either a field or a discrete valuation ring [3, Theorem 1, p. 494]. As $mT_m \neq (0)$, so T_m is not a field, which implies that T_m is a discrete valuation ring having only one maximal ideal mT_m, hence the mT_m-adic completion $\hat{T}_m$ of T_m is again a discrete valuation ring [20, Exercise, p. 85]. ∎

REMARK 5. By [23, p. 422], we observe that the elements of T are classes, so there must be an equivalence relation on T. Also the ideal generated by $\cos^2 x + \sin^2 x - 1$ gives $\mathbb{R}[\cos x, \sin x] \simeq \mathbb{R}[\cos x, \sin x]/(\cos^2 x + \sin^2 x - 1)$.

5. Some examples

Trigonometric polynomial rings give rise to some interesting examples. A very much familiar example of non-unique factorization in an integral domain is $(2)(3) = (1+\sqrt{-5})(1-\sqrt{-5})$. If we consider real trigonometric polynomials, another obvious example of a non-unique factorization is $\sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x)$ as $\sin x$, $1 - \cos x$ and $1 + \cos x$ are irreducibles in T. This identity asserts that two different looking pairs of factors have the same product. Actually it is a valid example of non-unique factorization in the integral domain T. In this section we discuss some examples one by one.

EXAMPLE 1. A ring of algebraic integers can sometimes be enlarged to another one in such a way that it restores unique factorization, although the question of how and when it can be done is not elementary. As an example, consider the ring $\mathbb{Z}[-3]$ consisting of numbers of the form $a + b\sqrt{-3}$, where a and b are integers. This ring does not have the unique factorization, as the equation $2 \cdot 2 = (1 + \sqrt{-3})(1 - \sqrt{-3})$ shows. We can restore unique factorization by enlarging the ring $\mathbb{Z}[-3]$ to the ring of all algebraic integers in the field $\mathbb{Q}(\sqrt{-3})$, which is $\mathbb{Z}[w]$, where $w = \frac{(1+\sqrt{-3})}{2}$ is a complex cube root of unity. But this does not work for the ring $\mathbb{Z}[-5]$ used above, because that is already the ring of all algebraic integers in the field $\mathbb{Q}(\sqrt{-5})$. The ring of all algebraic integers in the enlarged field $\mathbb{Q}(\sqrt{-5}, i)$, however, which can be shown to be the ring $\mathbb{Z}[\theta]$, where $\theta = \frac{(i+\sqrt{-5})}{2}$ is a root of $x^4 + 3x^2 + 1$, is an enlargement of $\mathbb{Z}[\sqrt{-5}]$ that does have unique factorization. The proof of last assertion can be easily established by standard arguments based on Minkowski's estimate, as described in [2, Chapter 12], [5, Chapter 13], or [18, Chapter 5].

EXAMPLE 2. Consider the ring of real trigonometric polynomials which provides the non-unique factorization $\sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x)$, where $\sin x$, $1 - \cos x$ and $1 + \cos x$ are irreducibles in T. This example breaks down if we consider the ring of complex trigonometric polynomials T'. So the change of coefficients alters the nature of factorization in the ring. Introducing complex coefficients produces more units. All the non-zero constant multiples of powers of $z = \cos x + i \sin x$ and $z^{-1} = \cos x - i \sin x$ are units in T'. Our example breaks down because the factors involved cease to be irreducible. We have

$$\sin x = \frac{z - z^{-1}}{2i} = \frac{z^{-1}(z-1)(z+1)}{2i},$$
$$1 - \cos x = \frac{-z + 2 - z^{-1}}{2} = \frac{-z^{-1}(z-1)^2}{2},$$
$$1 + \cos x = \frac{z^{-1}(z+1)^2}{2},$$

so both sides become

$$\frac{-z^{-2}(z-1)^2(z+1)^2}{4},$$

when expressed as a product of irreducible factors.

EXAMPLE 3. Apparently the polynomial $P = \cos^2 x + \sin^2 x$ has degree 2 in $\mathbb{R}[\cos x, \sin x]$, but this is not true because $\phi(P) = \phi(\cos^2 x + \sin^2 x) = \cos^2 x + 1 - \cos^2 x = 1$, which implies that $\delta(P) = \delta(1) = 0$.

EXAMPLE 4. $R_\circ = \mathbb{R}[\cos^2 x, \cos^3 x]$ is an FFD which is not an HFD. If $M_\circ = (\cos^2 x, \cos^3 x)$, then $M_\circ$ is a maximal ideal of $R_\circ$. Also, $A = (R_\circ)_{M_\circ}$ is a 1-dimensional local domain, hence a G-domain. This example is a consequence of the fact that $\mathbb{R}[X] \simeq \mathbb{R}[\cos x]$.

EXAMPLE 5. $R = \mathbb{Z} + \cos x \mathbb{Q}[\cos x]$ is a GCD domain [8, Corollary 1.3]. Also, R is 2-dimensional and the complete integral closure of R is $\mathbb{Q}[\cos x]$ [14, Exercise 2, p. 144].

Following [23, p. 425], if $z \in T$ is irreducible, then we have three possibilities: (z) is a maximal ideal, (z) is a square of a maximal ideal or (z) is a product of two distinct maximal ideals. The following examples demonstrate this fact.

EXAMPLE 6. Consider $\cos^2 x = (1 - \sin x)(1 + \sin x)$. The two irreducible elements $1 - \sin x$ and $1 + \sin x$ generate the ideals $(1 - \sin x)$ and $(1 + \sin x)$ respectively, which are the squares of maximal ideals [23, Theorem 3.8]. Therefore we have two maximal ideals $M_1 = (1 - \sin x, \cos x)$ and $M_2 = (1 + \sin x, \cos x)$. So three irreducible elements $\cos x$, $1 - \sin x$ and $1 + \sin x$ generate the following products $M_1 M_2$, M_1^2 and M_2^2, where M_1^2, M_2^2 are principal ideals [23, Corollary 3.9]. Also $M_1 M_2$ is a principal ideal [23, Proposition 3.10].

6. The geometrical behavior of trigonometric polynomials

6.1. Real trigonometric polynomials. Consider the ring of real trigonometric polynomials $T = \mathbb{R}[\cos x, \sin x]$. T is an HFD, therefore it is an atomic domain, that is, every element can be expressed as a product of irreducibles (atoms). So we can study $P \in T$ as a product of atoms. For this, first we discuss a general irreducible element of the form $a \cos x + b \sin x + c$, $(a, b, c) \in \mathbb{R}^3$ with $(a, b) \neq (0, 0)$.

6.1.1. *The behavior of irreducible elements.* The irreducible element $a \cos x + b \sin x + c$, $(a, b, c) \in \mathbb{R}^3$ with $(a, b) \neq (0, 0)$ represents a wave, where constant c has a direct relation with the translation of the wave. An increase in constant c translates the wave upward and vice versa. It follows that, if we change the sign of constant c, we get the same wave with a downward translation, whereas the shape of the wave remains the same.

The coefficient a of $\cos x$ in $a \cos x + b \sin x + c$, $(a, b, c) \in \mathbb{R}^3$ plays a double role. Firstly it represents the properties of $\cos x$ in the wave, the greater be the value of a, the greater is the resemblance of wave with $\cos x$ and vice versa. Secondly, it has a direct relation with the amplitude of wave, that is, the greater be the value of a, the greater is the amplitude of the wave and vice versa, whereas the shape of the wave remains the same. Similarly the coefficient b of $\sin x$ in $a \cos x + b \sin x + c$, $(a, b, c) \in \mathbb{R}^3$ also plays a double role in the same manner as $\cos x$.

OBSERVATION. For each $P \in T$, all irreducibles in the factorization of P have the same wavelength, that is, 2π.

6.1.2. *Behavior of a trigonometric polynomial in terms of its atoms.* Consider the three non-associated irreducible factorizations of $\cos 3x$ given in example 5:

$$\cos 3x = (2 \cos x - \sqrt{3})(2 \cos x + \sqrt{3}) \cos x$$
$$= (1 - 2 \sin x)(1 + 2 \sin x) \cos x$$
$$= (\cos x - \sqrt{3} \sin x)(\cos x + \sqrt{3} \sin x) \cos x.$$

In the following three figures, we have shown the waves for above three different factorizations respectively. Whereas the wave with minimum amplitude represents $\cos 3x$. We have used `Mathematica` to draw trigonometric polynomials.
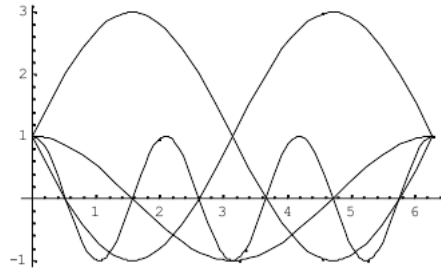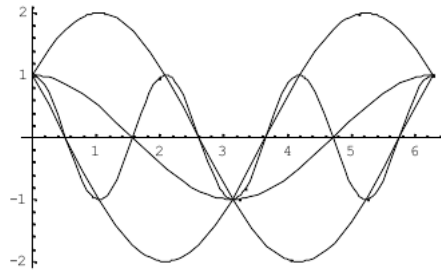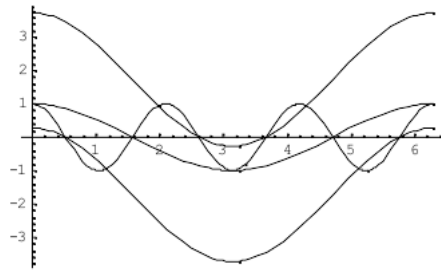
Fig. 1

Fig. 2

Fig. 3

Let $P \in T$. As T is an atomic domain so there exist finite number of irreducible elements of the form $a\cos x + b\sin x + c$, $(a, b, c) \in \mathbb{R}^3$ with $(a, b) \neq (0, 0)$ such that $P = P_1 \ldots, P_n$ where $P_i = a\cos x + b\sin x + c$, $(a, b, c) \in \mathbb{R}^3$ with $(a, b) \neq (0, 0)$.

Note that, each one of $P, P_1, \ldots, P_n$ constitutes a wave, that is why we are studding a trigonometric polynomial as a wave.

OBSERVATION. Let k and t represent the number of crests and troughs in the wave P, whereas $k_1, \ldots, k_n$ and $t_1, \ldots, t_n$ represent the number of crests and troughs in the waves $P_1, \ldots, P_n$ respectively. Then we have

$$k = \sum_{i=1}^{n} k_i \text{ and } t = \sum_{i=1}^{n} t_i.$$

REMARK 6. The repetition of P_i in the factorization of P compels P to behave like P_i, greater be the exponent of P_i, greater is the resemblance between P and P_i.

Also note the presence of symmetry in the wave $P \in T$, especially the irreducible factors (atoms), which appear in pairs in the factorization of P observe symmetry with each other. One factor in this pair occurs with positive sign and the other with a negative sign. In some cases this pair of factors behave very similar to each other, do not intersect even at a single point, distance between them remains constant at each point and in some cases it behave quite opposite, and intersect at so many points. For example, consider the following two factorizations of $\cos 3x$,

$$\cos 3x = (2\cos x - \sqrt{3})(2\cos x + \sqrt{3})\cos x$$
$$= (1 - 2\sin x)(1 + 2\sin x)\cos x.$$

Fig. 1 and Fig. 2 depicts that $(1 + 2\sin x)$ and $(1 - 2\sin x)$ behave oppositely, where $(2\cos x + \sqrt{3})$ and $(2\cos x - \sqrt{3})$ behave in a symmetric way and make no intersection.

REMARK 7. Three types of irreducible elements can occur in the factorization of a trigonometric polynomial, and they are $a\cos x + c$, $(a, c) \in \mathbb{R}^2$, $a \neq 0$, $b\sin x + c$, $(b, c) \in \mathbb{R}^2$, $b \neq 0$ and $a\cos x + b\sin x + c$, $(a, b, c) \in \mathbb{R}^3$, $a \neq 0$, $b \neq 0$. These three types of irreducible elements give three different types of waves. The first two, $a\cos x + c$, $(a, c) \in \mathbb{R}^2$, $a \neq 0$ and $b\sin x + c$, $(b, c) \in \mathbb{R}^2$, $b \neq 0$ produce waves which resemble with the waves $\cos x$ and $\sin x$ respectively, and the third one produces a wave that have a mixed behavior.

OBSERVATION. For each $P \in T$, wavelength of $P = \frac{2\pi}{n}$, where 2π is the wavelength of P_i and n is the total number of irreducible factors in the factorization of P.

Another question which is still unanswered is, can we identify the relation between the amplitude of wave P and its irreducibles P_i. We do not know the answer but we have the following observations.

REMARK 8. When we talk about the amplitude of a single factor, we come to know that amplitude of the factors $a\cos x + c$, $(a, c) \in \mathbb{R}^2$, $a \neq 0$ and $b\sin x + c$, $(b, c) \in \mathbb{R}^2$, $b \neq 0$ is a and b, respectively. Also the amplitude of $a\cos x + b\sin x + c$, $(a, b, c) \in \mathbb{R}^3$, $a \neq 0$, $b \neq 0$ depends upon both a and b, the greater be the values of a and b, the greater is the amplitude.

6.2. Complex trigonometric polynomials Consider the ring of complex trigonometric polynomials $T' = \mathbb{C}[\cos x, \sin x]$. Since T' is a UFD, therefore we can study $P \in T'$ as a product of irreducibles with irreducible elements of the form $\cos x + i\sin x - a$, $a \in \mathbb{C} \setminus \{0\}$. Geometrically these irreducible elements represent circles in plane, where constant a denotes the translation along one of the two axis. Now question is how we can study the geometrical behavior of trigonometric polynomials in T' by considering its factorization into irreducible elements. To answer this, we make the following discussion.

As above, the irreducible element $\cos x + i\sin x - a$, $a \in \mathbb{C} \setminus \{0\}$ is a circle. What happens to this circle when we form a polynomial by multiplying these irreducible elements? The answer is that each polynomial $P \in T'$ is a geometric figure, whose starting point coincides with its end point. Actually, there is a 1-1 correspondence

between polynomials in T' and the number of figures that can be formed by a circle, by translating it or molding it in any shape, without braking it at any point. The most known figures, which can be formed are ellipse and flowers with different number of leaves etc. So we conclude that each $P \in T'$ is a circle molded into some shape.

6.3. Future work and applications. It is exciting that at the end of this paper, there are still some directions for future research work. In polynomial rings, factorization properties of integral domains have been a frequent topic of recent mathematical literature but the study of factorization properties in trigonometric polynomials has not been addressed that much. So it seems to be really interesting to investigate factorization properties of trigonometric polynomial rings and this can open a new challenge for the researchers.

In addition to the applications mentioned in the introduction, we would like to highlight an application of trigonometric polynomials in symbolic computation. In [21], J. Mulholland and M. Monagan presented algorithms for simplifying ratios of trigonometric polynomials and algorithms for dividing, factoring and computing greatest common divisors of trigonometric polynomials. The provided algorithms do not always lead to the simplest form. A possible direction of study could be to provide enough general algorithms for finding a simplest form.

REFERENCES

[1] D. D. Anderson, D. F. Anderson and M. Zafrullah, *Factorization in integral domains*, J. Pure Appl. Algebra **69** (1990), 1–19.

[2] E. Artin, *Theory of Algebraic Number*s, Math. Inst. Göttingen, 1959.

[3] N. Bourbaki, *Commutative Algebra*, Addison-Wesley Pub. Co. Paris, 1972.

[4] P. Borwein and T. Erdélyi, *Trigonometric polynomials with many real zeros and Littlewood-type problem*, Proc. Amer. Math. Soc. **129** (2001), 725–730.

[5] H. Cohn, *A Classical Invitation to Algebraic Numbers and Class Fields*, Springer, New York, 1978.

[6] P. M. Cohn, *Bezout rings and their subrings*, Proc. Camb. Phil. Soc. **64** (1968), 251–264.

[7] G. Cohen and C. Cuny, *On random almost periodic trigonometric polynomials and applications to ergodic theory*, Annals Probab., **34** (2006), 39–79.

[8] D. Costa, J. Mott and M. Zafrullah, *The construction $D + XD_S[X]$*, J. Algebra, **53** (1978), 423–439.

[9] J. Coykendall, *A characterization of polynomial rings with the Half-Factorial Property. Factorization in integral domains*, Lecture notes in pure and applied mathematics, Marcel Dekker (New York), **189** (1997) 291–294.

[10] D. K. Dimitrov, *Extremal positive trigonometric polynomials*, Approx. Theory, A volume dedicated to Blagovest Sendov (2002), 1–24.

[11] B. Dumitrescu, *Positive Trigonometric Polynomials and Signal Processing Applications*, Springer-Verlag, Dordrecht, 2007.

[12] N. Dyn and M. Skopina, *Decompositions of trigonometric polynomials with applications to multivariate subdivision schemes*, preprint.

[**13**] R. F. Fossum, *The Divisor Class Group of a Krull Domain*, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[**14**] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

[**15**] A. Grams, *Atomic domains and the ascending chain condition for principal ideals*, Proc. Camb. Phil. Soc. **75** (1974), 321–329.

[**16**] T. W. Hungerford, *Algebra*, Holt, Rinehart and Winston, 1974.

[**17**] I. Kaplansky, *Commutative Rings*, The University of Chicago Press, 1974.

[**18**] S. Lang, *Algebraic Number Theory*, Addison-Wesley, Reading, 1970.

[**19**] `Mathematica Edition: Version 8.0`, Wolfram Research, Inc., Champaign, 2010, avaialble at: http://www.wolfram.com/mathematica/

[**20**] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, London, New York.

[**21**] J. Mulholland and M. Monagan, *Algorithms for trigonometric polynomials*, Proceedings of ISSAC 2001, ACM Press, pp. 245–252, 2001.

[**22**] P. Pau and J. Schicho, *Quantifier elimination for trigonometric polynomials by cylindrical trigonometric decomposition*, J. Symbolic Computation, **29** (2000), 971–983.

[**23**] G. Picavet and M. Picavet, *Trigonometric polynomial rings. Commutative ring theory*, Lecture notes on Pure Appl. Math. Marcel Dekker, **231** (2003), 419–433.

[**24**] J. F. Ritt, *A factorization theory for functions* $\sum_{i=1}^{n} a_i e^{\alpha_i x}$, Trans. Amer. Math. Soc. **29** (1927), 584–596.

[**25**] P. Samuel, *About Eucleadean rings*, J. Algebra **19** (1971), 282–301.

[**26**] T. Shah and E. Ullah, *Factorization properties of subrings in trigonometric polynomial rings*, Publ. Inst. Math., Nouvelle Sér., **86** (2009), 123–131.

[**27**] T. Shah and E. Ullah, *Subrings in trigonometric polynomial rings*, Acta Math. Acad. Paedag.e Nyíregyháziensis, **26** (2010), 35–43.

[**28**] A. Zaks, *Half-factorial domains*, Israel J. Math. 37 (1980), 281–302.

[**29**] A. Zaks, *Atomic rings without a.c.c. on principal ideals*, J. Algebra **74** (1982), 223–231.

[**30**] O. Zariski and P. Sammuel, *Commutative Algebra*, Vol. 1, Springer-Verlag, New York, Heidelberg, Berlin, 1958.

[**31**] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, Cambridge, 1968.

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