МАТЕМАТИЧКИ ВЕСНИК
68, 3 (2016), 225–231
September 2016

оригинални научни рад
research paper

ON MONOTONICITY OF RATIOS OF SOME q-HYPERGEOMETRIC FUNCTIONS

Khaled Mehrez and Sergei M. Sitnik

Abstract. In this paper we prove monotonicity of some ratios of q-Kummer confluent hypergeometric and q-hypergeometric functions. The results are also closely connected with Turán type inequalities. In order to obtain main results we apply methods developed for the case of classical Kummer and Gauss hypergeometric functions in [S.M. Sitnik, Inequalities for the exponential remainder, preprint, Institute of Automation and Control Process, Far Eastern Branch of the Russian Academy of Sciences, Vladivostok 1993 (in Russian)] and [S.M. Sitnik, Conjectures on Monotonicity of Ratios of Kummer and Gauss Hypergeometric Functions, RGMIA Research Report Collection 17 (2014), Article 107].

1. Introduction

In this paper we prove results on monotonicity of ratios of some q-hypergeometric functions. These results are generalizations of our previous results on monotonicity of ratios of classical hypergeometric functions in [11, 12] and [8]. Also it is demonstrated that these results on monotonicity of ratios of hypergeometric functions are stronger than so-called Turán type inequalities for such functions. So it is a way to prove Turán type inequalities for different types of functions.

To start with formulations of our results on monotonicity of ratios of classical hypergeometric functions from [8, 11, 12] let us consider the simplest case of the series for the exponential function

$$\exp(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},\ x \geq 0,$$

its section $S_n(x)$ and series remainder $R_n(x)$ in the form

$$S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!},\quad R_n(x) = \exp(x) - S_n(x) = \sum_{k=n+1}^{\infty} \frac{x^k}{k!},\ x \geq 0. \tag{1}$$

2010 Mathematics Subject Classification: 33D15

Keywords and phrases: Kummer functions; Gauss hypergeometric functions; q-Kummer confluent hypergeometric functions; q-hypergeometric functions; Turán type inequalities.

Besides simplicity and elementary nature of these functions many mathematicians studied problems for them, including G. Szegö, S. Ramanujan, G. Hardy, W. Gautschi.

In the preprint [11] in 1993, inequalities of the form

$$m(n) \leq f_n(x) = \frac{R_{n-1}(x)R_{n+1}(x)}{[R_n(x)]^2} \leq M(n),\ x \geq 0. \tag{2}$$

were thoroughly studied. The search for the best constants $m(n) = m_{best}(n)$, $M(n) = M_{best}(n)$ has some history. The left-hand side of (2) was first proved by K. Menon with $m(n) = \frac{1}{2}$ (not best) and by H. Alzer with

$$m_{best}(n) = \frac{n+1}{n+2} = f_n(0); \tag{3}$$

cf. [11] for the more detailed history. It was also shown in [11] that the inequality (2) with the sharp lower constant (3) is a special case of a stronger inequality proved earlier in 1982 by W. Gautschi.

It seems that the right-hand side of (2) was first proved in [11] with $M_{best} = 1 = f_n(\infty)$. Several generalizations of inequality (2) and related results were also proved in [11]. Maybe it was the first example of so called Turan-type inequality (cf. [1, 4, 5, 9]) for special case of Kummer hypergeometric functions.

Obviously the above inequalities are consequences of the following conjecture, originally formulated in [11] in 1993 and recently revived in [12].

Conjecture. *The function $f_n(x)$ in (2) is increasing for $x \in [0;\infty)$, $n \in \mathbb{N}$.*

So the next inequality is valid

$$\frac{n+1}{n+2} = f_n(0) \leq f_n(x) < 1 = f_n(\infty). \tag{4}$$

The above conjecture may be reformulated in terms of Kummer hypergeometric functions. Only recently, in 2014, the above conjecture and its generalizations to Kummer, Gauss and generalized hypergeometric functions were proved in [8].

In this paper we prove q-versions as generalizations of these results. We also demonstrate that from the results on monotonicity of ratios of hypergeometric functions, the so-called Turán type inequalities (cf. [1, 4, 5, 9]) for such functions follow. So a way to prove monotonicity of ratios of hypergeometric functions is also a way to prove Turán type inequalities.

2. Notation and preliminaries

Throughout this paper we fix $q \in]0,1[$. We refer to [3] for the definitions, notation and properties of the q-shifted factorials and q-hypergeometric functions.

Next, let us recall the following results which will be used in the sequel.

Lemma 1. *Let (a_n) and (b_n) $(n = 0,1,2\dots)$ be real numbers such that $b_n > 0$, $n = 0,1,2,\dots$ and $\left(\frac{a_n}{b_n}\right)_{n\geq 0}$ is increasing (decreasing). Then $\left(\frac{a_0+\dots+a_n}{b_0+\dots+b_n}\right)_n$ is also increasing (decreasing).*

Lemma 2. (cf. [2, 10]) *Let* (a_n) *and* (b_n) $(n = 0, 1, 2 \dots)$ *be real numbers and let the power series* $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *be convergent for* $|x| < r$. *If* $b_n > 0$, $n = 0, 1, 2, \dots$ *and if the sequence* $\left(\frac{a_n}{b_n}\right)_{n\geq 0}$ *is (strictly) increasing (decreasing), then the function* $\frac{A(x)}{B(x)}$ *is also (strictly) increasing (decreasing) on* $[0, r[$.

2.1. Basic symbols. For $a \in \mathbb{R}$, let q-shifted factorials be defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k),$$

and write

$$(a_1, a_2, \dots, a_p; q) = (a_1;q)_n (a_2;q)_n \cdots (a_p;q)_n, \quad n = 0, 1, 2, \dots$$

Note that for $q \to 1$ the expression $\frac{(q^a;q)_n}{(1-q)^n}$ tends to $(a)_n = a(a+1)\cdots(a+n-1)$.

2.2. *q*-Kummer confluent hypergeometric functions. The q-Kummer confluent hypergeometric function is defined by

$$\phi(q^a, q^c; q, x) =_1 \phi_1(q^a, q^c; q, (1-q)x) = \sum_{n\geq 0} \frac{(q^a;q)_n (1-q)^n}{(q^c;q)(q;q)_n} x^n, \tag{5}$$

for all $a, c \in \mathbb{R}$ and $x > 0$, which for $q \to 1$ is reduced to the Kummer confluent hypergeometric function

$${}_1F_1(a;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} x^n.$$

2.3. *q*-hypergeometric functions. The q-hypergeometric series or basic hypergeometric series is defined by ([3])

$$\begin{aligned} &{}_p\Phi_r(a_1, \dots, a_p; b_1, \dots, b_r; q; x) \\ &\qquad = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \dots (a_p;q)_n}{(b_1;q)_n (b_2;q)_n \dots (b_r,q)_n (q;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+r-p} x^n, \end{aligned} \tag{6}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$, $a_k, b_k \in \mathbb{R} \in \mathbb{C}$, $b_k \neq q^{-n}$, $k = 1, \dots, r$, $n \in \mathbb{N}_0$, $0 < |q| < 1$. The left-hand side of (6) represents the q-hypergeometric function ${}_p\phi_r$ where the series converges. Assuming $0 < |q| < 1$, the following conditions are valid for the convergence of (6) (cf. [3]).

- $p < r + 1$: the series converges absolutely for $x \in \mathbb{C}$,
- $p = r + 1$: the series converges for $|x| < 1$,
- $p > r + 1$: the series converges only for $x = 0$, unless it terminates.

Since for $q \to 1$ the expression $\frac{(q^a;q)_n}{(1-q)^n}$ tends to $(a)_n = a(a+1)\cdots(a+n-1)$, we evaluate

$$\begin{aligned} \lim_{q\to 1} {}_p\Phi_r(q^{a_1}, \dots, q^{a_p}; q^{b_1}, \dots, q^{b_r}; q; x) &=_p F_r(a_1, \dots, a_p; b_1, \dots, b_r; x) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_r)_n n!} x^n, \end{aligned}$$

where ${}_pF_r$ stands for the generalized hypergeometric function.

3. Monotonicity of ratios of q-Kummer hypergeometric functions

In this section we consider the function

$$h(a,b,c,q,x)=\frac{\phi(q^a,q^{b-c},q,x)\phi(q^a,q^{b+c},q,x)}{\left[\phi(q^a,q^b,q,x)\right]^2}, \tag{7}$$

for all $a,b\in\mathbb{R}$ and $x>0$, The following theorem is the q-version of [8, Theorem 1].

Theorem 1. *Let $q\in]0,1[$, and a,b,c be real numbers. If $c>0$, then the function $x\mapsto h(a,b,c,q,x)$ is increasing on $[0,\infty[$. In particular, for $q\in]0,1[$ the following Turán type inequality*

$$\left[\phi(q^a,q^b,q,x)\right]^2\le\phi(q^a,q^{b-c},q,x)\phi(q^a,q^{b+c},q,x) \tag{8}$$

is valid for all $a,b,c\in\mathbb{R}$ such that $c>0$.

Proof. For convenience, let us write $\phi(q^a,q^b,q,x)$ as

$$\phi(q^a,q^b,q,x)=\sum_{n=0}^{\infty}u_n(a,b,q)x^n,$$

where $u_n(a,b,q)=\dfrac{(q^a;q)_n(1-q)^n}{(q^b;q)_n(q;q)_n}$. Then

$$h(a,b,c,q,x)=\frac{\left(\sum_{n=0}^{\infty}u_n(a,b-c,q)x^n\right)\left(\sum_{n=0}^{\infty}u_n(a,b-c,q)x^n\right)}{\left(\sum_{n=0}^{\infty}u_n(a,b,q)x^n\right)^2}=$$

$$=\frac{\sum_{n=0}^{\infty}v_n(a,b,c,q)x^n}{\sum_{n=0}^{\infty}w_n(a,b,q)x^n},$$

with $v_n(a,b,c,q)=\sum_{k=0}^{n}u_k(a,b-c,q)u_{n-k}(a,b+c,q)$ and
$w_n(a,b,q)=\sum_{k=0}^{n}u_k(a,b,q)u_{n-k}(a,b,q)$. Let us define a sequences $(A_{n,k})_{k\ge0}$ by

$$A_{n,k}(a,b,c,q)=\frac{u_k(a,b-c,q)u_{n-k}(a,b+c,q)}{u_k(a,b,q)u_{n-k}(a,b,q)}=\frac{(q^b;q)_k(q^b;q)_{n-k}}{(q^{b-c};q)_k(q^{b+c};q)_{n-k}}$$

and evaluate

$$\frac{A_{n,k+1}(a,b,c,q)}{A_{n,k}(a,b,c,q)}=\frac{(q^b;q)_{k+1}(q^b;q)_{n-k-1}(q^{b-c};q)_k(q^{b+c};q)_{n-k}}{(q^{b-c};q)_{k+1}(q^{b+c};q)_{n-k-1}(q^b;q)_k(q^b;q)_{n-k}}$$

$$=\left(\frac{(q^b;q)_{k+1}}{(q^b;q)_k}\right)\cdot\left(\frac{(q^{b-c};q)_k}{(q^{b-c};q)_{k+1}}\right)\cdot\left(\frac{(q^b;q)_{n-k-1}}{(q^b;q)_{n-k}}\right)\cdot\left(\frac{(q^{b+c};q)_{n-k}}{(q^{b+c};q)_{n-k}}\right)$$

$$=\left(\frac{1-q^{b+k}}{1-q^{b-c+k}}\right)\cdot\left(\frac{1-q^{b+c+n-k-1}}{1-q^{b+n-k-1}}\right).$$

Since $q\in]0,1[$ and $c>0$ it follows that $\frac{A_{n,k+1}(a,b,c,q)}{A_{n,k}(a,b,c,q)}\ge1$ and consequently the sequence $(A_{n,k}(a,b,c,q))_{k\ge0}$ is increasing. We conclude that C_n defined by $C_n=\frac{u_n}{v_n}$ is increasing by Lemma 1. Thus the function $x\mapsto h(a,b,c,q,x)$ is increasing on $[0,\infty[$ by Lemma 2. Furthermore,

$$\lim_{x\to0}h(a,b,q,x)=1,$$

and Turán type inequality (8) follows. So the proof of Theorem 1 is complete. ■

Remark 1. The inequality (8) is interesting as a consequence of monotonicity property we consider. This inequality itself is not new and may be found in [7].

4. Monotonicity of ratios of q-hypergeometric functions

In this section we consider the function $h_r(a,b,c,q)$ defined by

$$h_r(a,b,c,q) = \frac{\phi(q^{a_1},\dots,q^{a_{r+1}};q^{b_1-c_1},\dots,q^{b_r-c_r};q,x)\phi(q^{a_1},\dots,q^{a_{r+1}};q^{b_1-c_1},\dots,q^{b_r-c_r};q,x)}{[\phi(q^{a_1},\dots,q^{a_{r+1}};q^{b_1},\dots,q^{b_r};q,x)]^2} \quad (9)$$

where $a=(a_1,\dots,a_{r+1})$ $b=(b_1,\dots,b_r)$ and $c=(c_1,\dots,c_r)$ for all $a_k,b_k,c_k\in\mathbb{R}$, $b_k\neq q^{-n}$, $k=1,\dots,r$, $n\in\mathbb{N}_0$, $0<q<1$.

Theorem 2. *Let $r\in\mathbb{N}$, $q\in(0,1)$, $a=(a_0,\dots,a_r)$, $b=(b_1,\dots,b_r)$, $c=(c_1,\dots,c_r)$. If $c_i>0$ for $i=1,\dots,r$, then the function $h_r(a,b,c,q)$ is strictly increasing in x on $[0,1[$. Moreover, if $c_i>0$, and $q\in(0,1)$, then the next Turán type inequality holds*

$$\left[\phi(q^{a_1},\dots,q^{a_{r+1}};q^{b_1},\dots,q^{b_r};q,x)\right]^2 < \phi(q^{a_1},\dots,q^{a_{r+1}};q^{b_1-c_1},\dots,q^{b_p-c_p};q,x)\phi(q^{a_1},\dots,q^{a_{p+1}};q^{b_1-c_1},\dots,q^{b_r-c_r};q,x). \quad (10)$$

Proof. By using the equality (9), we can write h_r in the form

$$h_r(a,b,q,x) = \frac{\left(\sum_{n=0}^{\infty}\frac{(q^{a_1};q)_n\dots(q^{a_{r+1}};q)_n x^n}{(q^{b_1-c_1};q)_n\dots(q^{b_r-c_r};q)_n(q;q)_n}\right)}{\left(\sum\frac{(q^{a_1};q)_n\dots(q^{a_{r+1}};q)_n x^n}{(q^{b_1};q)_n\dots(q^{b_r};q)_n(q;q)_n}\right)^2}\cdot\left(\sum_{n=0}^{\infty}\frac{(q^{a_1};q)_n\dots(q^{a_{r+1}};q)_n x^n}{(q^{b_1+c_1};q)_n\dots(q^{b_r+c_r};q)_n(q;q)_n}\right) = \frac{\sum_{n=0}^{\infty}A_n(a,b,c,q)}{\sum_{n=0}^{\infty}B_n(a,b,c,q)}x^n,$$

with use of the following notation

$$A_n(a,b,c,q) = \sum_{k=0}^{n}U_k(a,b,c,q) = \sum_{k=0}^{n}\frac{\prod_{j=1}^{r+1}(q^{a_j};q)_{n-k}(q^{a_j};q)_k}{(q;q)_k(q;q)_{n-k}\prod_{j=1}^{r}(q^{b_j-c_j};q)_k(q^{b_j+c_j};q)_{n-k}}$$

and

$$B_n(a,b,c,q) = \sum_{k=0}^{n}V_k(a,b,c,q) = \sum_{k=0}^{n}\frac{\prod_{j=1}^{r+1}(q^{a_j};q)_{n-k}(q^{a_j};q)_k}{(q;q)_k(q;q)_{n-k}\prod_{j=1}^{r}(q^{b_j};q)_k(q^{b_j};q)_{n-k}}.$$

For fixed $n\in\mathbb{N}$ we define a sequence $(W_{n,k}(a,b,c,q))_{k\geq 0}$ by

$$W_{n,k}(a,b,c,q) = \frac{U_k(a,b,c,q)}{V_k(a,b,c,q)} = \prod_{j=1}^{r}\frac{(q^{b_j};q)_k(q^{b_j};q)_{n-k}}{(q^{b_j-c_j};q)_k(q^{b_j+c_j};q)_{n-k}}.$$

For $n, k \in \mathbb{N}$ we evaluate

$$\begin{aligned}
&\frac{W_{n,k+1}(a,b,c,q)}{W_{n,k}(a,b,c,q)} \\
&= \prod_{j=1}^{r} \left[\frac{(q^{b_i};q)_{k+1}}{(q^{b_j};q)_k}\right] \cdot \left[\frac{(q^{b_j};q)_{n-k-1}}{(q^{b_j};q)_{n-k}}\right] \cdot \left[\frac{(q^{b_j-c_j};q)_k}{(q^{b_j};q)_{k+1}}\right] \cdot \left[\frac{(q^{b_j+c_j};q)_{n-k}}{(q^{b_j+c_j};q)_{n-k-1}}\right] \\
&= \prod_{j=1}^{r} \left[\frac{1-q^{b_j+k}}{1-q^{b_j-c_j+k}}\right] \cdot \left[\frac{1-q^{b_j+c_j+n-k-1}}{1-q^{b_j+n-k-1}}\right].
\end{aligned}$$

Since $0 < q < 1$ and $c_j > 0$ for $j = 1, \ldots, r$ we conclude that $(W_{n,k})_k$ is increasing and consequently $\left(C_n = \frac{A_n}{B_n}\right)_{n\geq 0}$ is increasing too, by Lemma 1. Thus the function $x \mapsto h_r(a,b,c,q)$ is increasing on $[0,1[$ by Lemma 2. Therefore the inequality (10) follows immediately from the monotonicity of the function $h_r(a,b,c,q)$. ■

There are applications of considered inequalities in the theory of transmutation operators for estimating transmutation kernels and norms [6, 13, 14] and for problems of function expansions by systems of integer translations of Gaussians [7, 15].

REFERENCES

[**1**] A. Baricz, K. Raghavendarb, A. Swaminathan, *Turán type inequalities for q-hypergeometric functions*, J. Approx. Theory **168** (2013) 69–79.

[**2**] M. Biernacki, J. Krzyz, *On the monotonicity of certain functionals in the theory of analytic functions*, Ann. Univ. M. Curie-Sklodowska **2** (1995) 134–145.

[**3**] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge Univ. Press, 2nd ed., 2004.

[**4**] D.B. Karp, S.M. Sitnik, *Log-convexity and log-concavity of hypergeometric-like functions*, J. Math. Anal. Appl. **364**:2 (2010) 384–394.

[**5**] D.B. Karp, S.M. Sitnik, *Inequalities and monotonicity of ratios for generalized hypergeometric function*, J. Approx. Theory **161** (2009) 337–352.

[**6**] V.V. Katrakhov, S.M. Sitnik, *A boundary-value problem for the steady-state Schrödinger equation with a singular potential*, Soviet Math. Doklady **30**:2 (1984), 468–470.

[**7**] E.A. Kiselev, L.A. Minin, I.Ya. Novikov, S.M. Sitnik, *On the Riesz constants for systems of integer translates*, Math. Notes **96**:2 (2014) 228–238.

[**8**] K. Mehrez, S.M. Sitnik, *Inequalities for Sections of Exponential Function Series and Proofs of Some Conjectures on Monotonicity of Ratios of Kummer, Gauss and Generalized Hypergeometric Functions*, RGMIA Research Report Collection 17 (2014), Article 132.

[**9**] J. Pintz, A. Biró, K. Györy, G. Harcos, M. Simonovits, J. Szabados (Eds.), *Number Theory, Analysis, and Combinatorics*, Proceedings of the Paul Turán Memorial Conference held on August 22-26, 2011 in Budapest. De Gruyter, 2014.

[**10**] S. Ponnusamy, M. Vuorinen, *Asymptotic expansions and inequalities for hypergeometric functions*, Mathematika **44** (1997) 278–301.

[**11**] S.M. Sitnik, *Inequalities for the exponential remainder*, preprint, Institute of Automation and Control Process, Far Eastern Branch of the Russian Academy of Sciences, Vladivostok 1993 (in Russian).

[**12**] S.M. Sitnik, *Conjectures on Monotonicity of Ratios of Kummer and Gauss Hypergeometric Functions*, RGMIA Research Report Collection 17 (2014), Article 107.

[**13**] S.M. Sitnik, *Factorization and estimates of the norms of Buschman-Erdelyi operators in weighted Lebesgue spaces*, Soviet Math. Doklady **44**:2 (1992) 641–646.

[**14**] S.M. Sitnik, *Buschman-Erdelyi transmutations, classification and applications*, In: Analytic Methods Of Analysis and Differential Equations: Amade 2012. (M.V. Dubatovskaya, S.V. Rogosin, eds), Cambridge Scientific Publishers, Cottenham, Cambridge (2013), pp. 171–201.

[**15**] M.V. Zhuravlev, E.A. Kiselev, L.A. Minin, S.M. Sitnik, *Jacobi theta-functions and systems of integral shifts of Gaussian functions*, J. Math. Sciences **173**:2 (2011) 231–241.

(received 06.12.2015; in revised form 10.03.2016; available online 21.03.2016)

Kh.M.: University of Kairouan, Department of Mathematics, ISSAT Kasserine 3100, Tunisia.

E-mail: `k.mehrez@yahoo.fr`

S.M.S.: Voronezh Institute of the Ministry of Internal Affairs of Russia, Patriotov pr. 53, Voronezh, 394065, Russia

and

Peoples' Friendship University of Russia, M. - Maklaya str., 6, Moscow, 117198, Russia

E-mail: `pochtaname@gmail.com`