STRONG LINEAR PRESERVERS OF UT-TOEPLITZ WEAK MAJORIZATION ON $\mathbb{R}^n$

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Abstract. Let $x,y \in \mathbb{R}^n$, we say $x$ is ut-Toeplitz weak majorized by $y$ (written as $x \prec_{uT} y$) if there exists an upper triangular substochastic Toeplitz matrix $A$ such that $x = Ay$. In this paper, we characterize all linear functions that strongly preserve $\prec_{uT}$ on $\mathbb{R}^n$.

1. Introduction

Majorization is one of the interesting concepts in matrix analysis and there are special researches on it and its linear preservers in recent years. Considering $M_n(\mathbb{R})$ as the space of all real $n \times n$ matrices, $D \in M_n(\mathbb{R})$ is called doubly (sub)stochastic if its entries are all nonnegative and the sum of its entries in each row and column is (less than or) equal to 1. Let $\mathbb{R}^n$ be the vector space of all real $n \times 1$ vectors. For $x,y \in \mathbb{R}^n$, it is said that $x$ is (weak) majorized by $y$ and denoted by $(x \prec_w y) x \prec y$ if there is a doubly (sub)stochastic matrix $D$ such that $x = Dy$. It is well known that $x \prec y$ if and only if $\sum_{j=1}^{n} x[j] \leq \sum_{j=1}^{n} y[j]$, for $k = 1,2,\ldots,n - 1$, and $\sum_{j=1}^{n} x[j] = \sum_{j=1}^{n} y[j]$, and $x \prec_w y$ if and only if $\sum_{j=1}^{k} x[j] \leq \sum_{j=1}^{k} y[j]$, for $k = 1,2,\ldots,n$, where $x[j]$ is the $j^{th}$ largest element of vector $x$. For more study see [8].

Definition 1.1. A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a linear preserver of a relation $\sim$ on $\mathbb{R}^n$ if for all $x,y \in \mathbb{R}^n$ $x \sim y \Rightarrow Tx \sim Ty$, and it is called a strongly linear preserver of the relation if $x \sim y \iff Tx \sim Ty$.

There are some researches on characterization of linear or nonlinear preservers of special kinds of (weak) majorization. For example, in [1, 3] authors have characterized strong linear preservers and linear preservers of g-tridiagonal majorization.

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respectively. In [10] authors have characterized strong linear preservers and linear pre-
servers of circulant majorization. In [9] authors have characterized nonlinear preserver
of some special weak majorization, and also in [2, 4, 5, 7] authors have characterized
linear preservers of some other special majorizations.

In this paper we introduce ut-Toeplitz weak majorization and characterize all
linear maps that strongly preserve upper triangular Toeplitz weak majorization. Ac-
tually this kind of majorization is a particular case of that introduced by Ilkhanizadeh
Manesh in [6].

2. Preliminaries and notations

The $k^{th}$ diagonal of a matrix $A = [a_{i,j}]$ is the collection of entries $a_{i,j}$ where $j - i = k$. The $0^{th}$ diagonal of a matrix is known as the main diagonal. A matrix $A$ is called
Toeplitz if all entries of each diagonal are equal. We denote a Toeplitz matrix by
$A = [a_{-n,-n-1}] \cdots [a_0 \backslash a_1 \cdots a_{n-1}]$ where $a_i$ is the amount of the $i^{th}$ diagonal, and
if the Toeplitz matrix is upper triangular we use the notation $A = [a_0 \backslash a_1 \cdots a_{n-1}]$.

**Definition 2.1.** Let $x, y \in \mathbb{R}^n$. We say that $x$ is ut-Toeplitz weak majorized by
$y$ (written as $x \preceq \text{uT} y$) if there exists an upper triangular substochastic Toeplitz matrix
$D \in M_n(\mathbb{R})$ such that $x = Dy$.

For $x \in \mathbb{R}^n$ we use the notation $x \geq 0$ if all entries of $x$ are nonnegative. Obviously
if $x$ is weak majorized by $y$ and $y \geq 0$, then $x \geq 0$. Also if $x \preceq \text{uT} y$, then $x = 0$.

We use $\phi(x)$ for the vector space generated by \{ $y \in \mathbb{R}^n : y \preceq \text{uT} x$ \}. Also the linear
operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is identified with its matrix representation under the canonical
basis, $e_1, \ldots, e_n$, in $\mathbb{R}^n$.

In this paper we also use the following special upper triangular substochastic
Toeplitz matrices.

$U_0 = I, \quad U_1 = \begin{pmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}, \ldots, U_{n-1} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}$

Actually every upper triangular substochastic Toeplitz matrix is the form of $\sum_{i=0}^{n-1} c_i U_i$, where $0 \leq c_i \leq 1$ and $\sum_{i=0}^{n-1} c_i \leq 1$.

3. Linear preservers of ut-Toeplitz majorization

We start this section by stating some preliminaries and properties of ut-Toeplitz weak majorization on $\mathbb{R}^n$. We will use these properties to prove our main theorem. The
following lemma describes vectors that are ut-Toeplitz weak majorized by some special vectors in $\mathbb{R}^n$.

**Lemma 3.1.** Let $x, y \in \mathbb{R}^n$ and $x \prec_{uT} y$. If $y \geq 0$ and $k$ is the largest index such that $y_k \neq 0$, then:

(i) $x_i = 0$, $\forall i > k$;
(ii) $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, $\forall 1 \leq i \leq k$.

**Proof.** Since $x \prec_{uT} y$ there is a substochastic upper triangular Toeplitz matrix $T = [t_0 \ t_1 \ \cdots \ t_{n-1}]$ such that $x = Ty$.

Obviously $x_i = 0$ for each $i \geq k$ and $x_j = \sum_{i=1}^{k-j+1} t_{i-1} y_{i+j-1}$. Considering $\sum_{i=1}^{n} t_{i-1} \leq 1$, we have

$$\sum_{i=1}^{k} x_i = t_0 y_1 + \cdots + t_{k-1} y_k + \cdots + t_0 y_{k-1} + t_1 y_k + t_0 y_k$$

$$= t_0 y_1 + (t_0 + t_1) y_{k+1} + \cdots + (\sum_{i=1}^{k-l+1} t_{i-1}) y_k \leq \sum_{i=1}^{k} y_i. \qed$$

**Lemma 3.2.** Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then $k$ is the largest index that $x_k \neq 0$ if and only if $\phi(x) = (e_1, \ldots, e_k)$.

**Proof.** Let $k$ be the largest index such that $x_k \neq 0$. We know $U_i x \prec_{uT} x$. Since $x_j = 0$ for each $j > k$, $U_0 x = x_1 e_1 + \cdots + x_k e_k, \ldots, U_{k-2} x = x_{k-1} e_1 + x_k e_2, U_{k-1} x = x_k e_1$ and $U_i x = 0, \forall j \geq k$. Hence $\phi(x)$ contains $e_1, \ldots, e_k$, which means $(e_1, \ldots, e_k) \subseteq \phi(x)$. On the other hand by part (i) of Lemma 3.1 if $y \prec_{uT} x$, then $y_i = 0$ for each $i > k$, which means that each $y \in \phi(x)$ is a linear combination of $e_1, \ldots, e_k$. Hence $\phi(x) = (e_1, \ldots, e_k)$. Proof of the converse is obvious. \qed

**Lemma 3.3.** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map strongly preserves $\prec_{uT}$. Then $T$ is an invertible upper triangular matrix.

**Proof.** First we prove that $T$ is invertible. Let $Tx = 0$. Since $T$ is a linear operator $T(0) = 0 = T(x)$. Considering that $T$ strongly preserves $\prec_{uT}$, implies $x \prec_{uT} 0$. Hence $x = 0$.

To prove that $T$ is an upper triangular matrix we apply the induction principle. By Lemma 3.2 we know that $\phi(e_1) = (e_1)$. Since $T$ is invertible, $dimT\phi(e_1) = dimT(e_1) = 1$. Since $T$ strongly preserves $\prec_{uT}$, we have

$$T\phi(e_1) = \{\{Tx : x \prec_{uT} e_1\}\} = \{\{Tx : Tx \prec_{uT} Te_1\}\} = \phi(T(e_1)).$$

Hence considering $dimT\phi(e_1) = 1$ and Lemma 3.2, we have $Te_1 = (a_{11}, 0, \ldots, 0)^t$.

Suppose that $Te_i = (a_{i1}, \ldots, a_{ii}, 0, \ldots, 0)^t$, for each $i < k$. Now we prove for $k$. By Lemma 3.2 we have $\phi(e_k) = (e_1, \ldots, e_k)$. Since $T$ is invertible, $dimT\phi(e_k) = dimT(e_1, \ldots, e_k) = k$. Obviously $e_i \prec_{uT} e_k$, for each $i < k$, hence $e_1, \ldots, e_{k-1} \in \phi(e_k)$, that means $T(e_1, \ldots, e_{k-1}) \subseteq T^\phi(e_k)$.

Considering the hypothesis of induction, we have $Te_i = (a_{i1}, \ldots, a_{ii}, 0, \ldots, 0)^t$, for each $i < k$, which means $T(e_1, \ldots, e_{k-1}) = (e_1, \ldots, e_{k-1})$. Now if the index of the largest nonzero entry of $Te_k$ is less than $k$, then $Te_k \notin T(e_1, \ldots, e_{k-1})$ and
consequently \( \dim T\phi(e_k) < k \) that is not true. On the other hand let the index of the largest nonzero entry of \( T e_k \) be greater than \( k \). Since \( T \) strongly preserves \( \prec_{uT} \), \( T\phi(e_k) = \{(Tx : x \prec_{uT} e_k)\} = \{(Tx : Tx \prec_{uT} Te_k)\} = \phi(T(e_k)) \) which implies \( \dim T\phi(e_k) > k \) that is impossible.

Hence \( T e_k = (a_{1,k}, \ldots, a_{k,k}, 0, \ldots, 0)^t \) for each \( 1 \leq k \leq n \), that means \( T \) is an upper triangular matrix.

\[ \text{Theorem 3.4. Let } T : \mathbb{R}^n \to \mathbb{R}^n \text{ be a linear operator. If } T \text{ is an upper triangular Toeplitz matrix then } T \text{ preserves } \prec_{uT}. \text{ Moreover } T \text{ strongly preserves } \prec_{uT} \text{ if and only if } T \text{ is an invertible upper triangular Toeplitz matrix.} \]

\[ \text{Proof. Let } T \text{ be an upper triangular Toeplitz matrix and } T_n \text{ be the set of all nonsingular, upper triangular Toeplitz matrices of size } n. \text{ It is well known that } T_n \text{ is an Abelian group. Let } T \in T_n \text{ and } x, y \in \mathbb{R}^n \text{ be such that } x \prec_{uT} y. \text{ Then } x = Dy \text{ for some substochastic matrix } D \in T_n. \text{ We obtain } Tx = TDy = DTy \text{ so that } Tx \prec_{uT} Ty, \text{ that is } T \text{ is a linear preserver of } \prec_{uT}. \]

Now let \( T \) be an invertible upper triangular Toeplitz matrix. To prove \( T \) strongly preserves \( \prec_{uT} \), it suffices to show that if \( Tx \prec_{uT} Ty \), then \( x \prec_{uT} y \). \( Tx \prec_{uT} Ty \) implies \( Tx = DTy \) for some substochastic matrix \( D \in T_n \), hence \( Tx = TDy \). Since \( T \) is invertible we have \( x = Dy \), hence \( x \prec_{uT} y \), and the proof is complete.

To prove the converse of the theorem, let \( T \) strongly preserves \( \prec_{uT} \). Then by Theorem 3.3, \( T \) is an invertible upper triangular matrix. To show \( T \) is Toeplitz, first we show that all entries on the main diagonal are equal. Since \( T \) is an invertible upper triangular matrix \( a_{i,i} \neq 0 \), for each \( 1 \leq i \leq n \). We assume that \( a_{n,n} > 0 \) (proof for the case \( a_{n,n} < 0 \) is similar). Consider an arbitrary natural number \( 1 \leq k \leq n \). Obviously \( e_k \prec_{uT} e_n \), hence \( T e_k \prec_{uT} T e_n \) that means there is an upper triangular substochastic Toeplitz matrix \( W = [w_0, \ldots, w_{n-1}] \) such that \( T e_k = W T e_n \).

\[ T e_k = (a_{1,k}, a_{2,k}, \ldots, a_{k,k}, 0, \ldots, 0)^t \]

\[ = (\sum_{j=1}^{n} w_{j-1} a_{j,n}, \ldots, \sum_{j=1}^{n-k+1} w_{j-1} a_{j+k-1,n}, \ldots, w_{0} a_{n,n})^t \]

We have \( w_0 a_{n,n} = 0 \). Since \( a_{n,n} \neq 0 \), we obtain \( w_0 = 0 \). Considering the \((n-1)th\) entry of \( W T e_n \), i.e. \( w_0 a_{n-1,n} + w_1 a_{n,n} = w_1 a_{n,n} = 0 \), implies \( w_1 = 0 \). Continuing this process we have \( w_{i-1} = 0 \) for each \( 1 \leq i \leq n - k \). Consequently the \( k \)th entry of \( T W e_n \) is equal to \( w_{n-k} a_{n,n} \). Hence by the equation (1) we have \( a_{k,k} = w_{n-k} a_{n,n} \), which implies that \( a_{k,k} \leq a_{n,n} \).

Since \( T \) is onto, there is \( y \in \mathbb{R}^n \) such that \( Ty = U_k T e_n \). Also since \( T \) strongly preserves \( \prec_{uT} \) and \( Ty \prec_{uT} T e_n \), we have \( y \prec_{uT} e_n \). Hence there is an upper triangular substochastic Toeplitz matrix \( W = [w_0, \ldots, w_{n-1}] \) such that \( y = W e_n \) which implies that \( U_k T e_n = Ty = T W e_n \). We have the following equation:

\[ (a_{k,n}, \ldots, a_{n,n}, 0, \ldots, 0)^t = (\sum_{j=1}^{n} a_{1,j} w_{n-j}, \ldots, \sum_{j=k}^{n} a_{k,j} w_{n-j}, \ldots, a_{n,n} w_0) \]

Like the above argument we have \( w_{i-1} = 0 \) for each \( 1 \leq i \leq n - k \) and hence
Since $a_{n,n} = w_{n-k}a_{kk}$ which implies that $a_{n,n} \leq a_{k,k}$. Hence we proved $a_{k,k} = a_{n,n}$ for each $1 \leq k \leq n$.

Suppose that the entries of $ith$ diagonal for each $1 \leq i \leq k$ are all equal to a constant number $a_i$. We show that the entries of $(k+1)th$ diagonal are equal. To reach this aim we show that $a_{n-k,n} = a_{j-k,j}$ for each $k + 1 \leq j \leq n-1$. We know $Te_j \prec_u T T e_n$, hence $(a_{1,j}, \ldots, a_{j-k,j}, a_{k,k}, \ldots, a_{1,n}, 0, \ldots, 0)^T \prec_u (a_{1,n}, \ldots, a_{n-k,n}, a_{k,k}, \ldots, a_{1,1})^T$ for $j \geq k + 1$. Hence we have $w_0a_1 = 0$. Since $T$ is invertible, $a_1 \neq 0$ and this implies $w_0 = 0$. In a similar way we have $w_0a_2 + w_1a_1 = 0$ which implies $w_1 = 0$ and continuing this process we have $w_{i-1} = 0$ for $1 \leq i \leq n - j$. Now we have $w_0a_{j,n} + w_1a_{j+1,n} + \cdots + w_{n-j-1}a_2 + w_{n-j}a_1 = a_1$ hence $w_{n-j} = 1$. Also $w_0a_{j-1,n} + w_1a_{j,n} + \cdots + w_{n-j}a_2 + w_{n-j+1}a_1 = a_2$ which implies $w_{n-j+1} = 0$. Again continuing this process we have $w_{n-j} = \cdots = w_{n-j+k-1} = 0$. Hence $W = [0 \ \cdots \ 0 \ \cdots \ 0 \ \cdots \ 0 \ \cdots \ w_{n-j+k} \ w_{n-j+1} \ w_{n-1}]$, where $1$ is in $(n-j)$th position. Now we have $w_0a_{j-k,n} + \cdots + w_{n-j}a_{n-k,n} + w_{n-j+1}a_k + \cdots + w_{n-j+k}a_1 = a_{j-k,j}$. Hence $a_{n-k,n} + w_{n-j+k}a_1 = a_{j-k,j}$, which implies
\[
a_{n-k,n} \leq a_{j-k,j}
\]

Since $T$ is onto, there is $y \in \mathbb{R}^n$ such that $Ty = U_t T e_n$, where $1 \leq t \leq n - k$. Since $T$ strongly preserves $\prec_u T$ and $Ty \prec_u T Te_n$, we have $y \prec_u e_n$. Hence there is an upper triangular substochastic Toeplitz matrix $W = [w_0 \ \cdots \ w_{n-1}]$ such that $y = We_n$. Consequently $U_t T e_n = Ty = TW e_n$. We have:

\[
TW e_n = \begin{pmatrix} \sum_{j=1}^{k} a_j w_{n-j+1} + \sum_{j=k+1}^{n} a_{1,j} w_{n-j+1} \\ \sum_{j=2}^{k+1} a_{j-1} w_{n-j+1} + \sum_{j=k+2}^{n} a_{2,j} w_{n-j+1} \\ \vdots \\ a_1 w_{k+1} + a_2 w_k + \cdots + a_k w_2 + a_{n-k,n} w_1 \\ a_1 w_k + a_2 w_{k-1} + \cdots + a_k w_1 \\ \vdots \\ a_1 w_2 + a_2 w_1 \\ a_1 w_1 \end{pmatrix} = \begin{pmatrix} a_{1,n} \\ \vdots \\ a_{n-k,n} \\ a_k \\ \vdots \\ a_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = U_t T e_n
\]

Since $a_1 w_1 = 0$ implies $w_1 = 0$ and $a_1 w_2 + a_2 w_1 = 0$ implies $w_2 = 0$, continuing this process, we have $w_1 = \cdots = w_{n-1} = 0$. Now $a_1 w_t + a_2 w_{t-1} + \cdots + a_k w_{t-k+1} + a_{n-t+1,n-t+k+1} w_{t-k} + \cdots + a_{n-t+1,n} w_1 = a_1$. Hence $w_t = 1$ and like the above argument we conclude $w_{t+1} = \cdots = w_{1+k-1} = 0$. We have $a_1 w_{t+k} + a_2 w_{t+k-1} + \cdots + a_k w_{t+1} + a_{n-t-k+1,n-t-k+1} w_1 + \cdots + a_{n-t-k+1,n} w_1 = a_{n-k,n}$. Hence $a_{n-t-k+1,n-t+1} \leq a_{n-k,n}$. If we consider $j = n - t + 1$, then
\[
a_{j-k,j} \leq a_{n-k,n}.
\]

By inequalities (2) and (3) we have $a_{j-k,j} = a_{n-k,n}$, $\forall k + 1 \leq j \leq n$, and the proof is completed.

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