MATEMATIČKI VESNIK
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HYPERBALL PACKINGS IN HYPERBOLIC 3-SPACE

Jenő Szirmai

Abstract. In earlier works we have investigated the densest packings and the thinnest coverings by congruent hyperballs based on the regular prism tilings in n-dimensional hyperbolic space $\mathbb{H}^n$ ($3 \le n \in \mathbb{N}$).

In this paper we study a large class of hyperball packings in $\mathbb{H}^3$ that can be derived from truncated tetrahedron tilings. In order to get an upper bound for the density of the above hyperball packings, it is sufficient to determine this density upper bound locally, e.g. in truncated tetrahedra.

Thus we prove that if the truncated tetrahedron is regular, then the density of the densest packing is ≈ 0.86338. This is larger than the Böröczky-Florian density upper bound for balls and horoballs. Our locally optimal hyperball packing configuration cannot be extended to the entire hyperbolic space $\mathbb{H}^3$, but we describe a hyperball packing construction, by the regular truncated tetrahedron tiling under the extended Coxeter group $[3, 3, 7]$ with maximal density ≈ 0.82251.

Moreover, we show that the densest known hyperball packing, related to the regular p-gonal prism tilings, can be realized by a regular truncated tetrahedron tiling as well.

1. Introduction

Let X^n denote any of n-dimensional spaces of constant curvature: sphere $\mathbb{S}^n$, Euclidean space $\mathbb{E}^n$, or hyperbolic space $\mathbb{H}^n$ ($2 \le n \in \mathbb{N}$).

In the space X^n let $d_n(r)$ be the density of $n+1$ mutually touching spheres of radius r with respect to the simplex spanned by their centres. L. Fejes Tóth and H. S. M. Coxeter conjectured that the packing density of balls of radius r in X^n cannot exceed $d_n(r)$. This conjecture has been proved by C. A. Rogers for Euclidean space $\mathbb{E}^n$. The 2-dimensional spherical case was settled by L. Fejes Tóth in [8]. In $\mathbb{H}^n$ there are 3 kinds of "generalized balls (spheres)": the *usual balls* (spheres), *horoballs* (horospheres) and *hyperballs* (hyperspheres). A hypersphere is the set of all points in

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$\mathbb{H}^n$, lying at a certain distance, called its *height*, from a hyperplane, on both sides of the hyperplane (cf. [27] for the planar case).

K. Böröczky proved the following generalization for *ball and horoball* packings for $n = 3$, and claimed the analogous statement for any n.

THEOREM 1.1. *([5]) In an n-dimensional space of constant curvature consider a packing of spheres of radius r. In spherical space suppose that $r < \frac{\pi}{4}$. Then the density of each sphere in its Dirichlet-Voronoi cell cannot exceed the density of $n+1$ spheres of radius r mutually touching one another with respect to the simplex spanned by their centers.*

The above greatest density in $\mathbb{H}^3$ is ≈ 0.85328 which is not realized by packing with equal balls. However, it is attained by the horoball packing (in this case $r = \infty$) of $\overline{\mathbb{H}}^3$ where the ideal centers of horoballs lie on the absolute figure of $\overline{\mathbb{H}}^3$. This ideal regular tetrahedron tiling is given with Coxeter-Schläfli symbol $[3, 3, 6]$. Ball packings of hyperbolic n-space and of other Thurston geometries are extensively discussed in the literature, see e.g. [1,5,7,15], where the reader can find further references as well.

In a previous paper [11] we proved that the above known optimal horoball packing arrangement in $\mathbb{H}^3$ is not unique. We gave several new examples of horoball packing arrangements based on totally asymptotic Coxeter tilings that yield the above Böröczky–Florian packing density upper bound [6]. Two horoballs in a horoball packing are of the "same type" iff the local densities of the horoballs to the corresponding cell (e.g. D-V cell or ideal simplex) are equal (see [20]).

We have also found that the Böröczky-Florian type density upper bound for horoball packings of different types is no longer valid for fully asymptotic simplices in higher dimensions $n > 3$ (see [19]). For example in $\mathbb{H}^4$, the density of such optimal, locally densest horoball packing is ≈ 0.77038, larger than the analogous Böröczky-Florian type density upper bound of ≈ 0.73046. However, these horoball packing configurations are only locally optimal and cannot be extended to the whole hyperbolic space $\mathbb{H}^4$.

In the paper [12] we have continued our previous investigation in $\mathbb{H}^4$ allowing horoballs of different types. In that paper we considered horoball packings in 4-dimensional hyperbolic space, and showed that it was possible to exceed the conjectured 4-dimensional realizable packing density upper bound due to L. Fejes-Tóth [8]. We gave seven examples of horoball packing configurations that yield higher densities (≈ 0.71645) where horoballs are centered at ideal vertices of certain Coxeter simplices, and are invariant under the actions of their respective Coxeter groups.

In [21–23] we have studied the regular prism tilings and the corresponding optimal hyperball packings. Their metric data and their densities have also been determined.

In hyperbolic plane $\mathbb{H}^2$ the universal upper bound of the hypercycle packing density is $\frac{3}{\pi}$, and the universal lower bound of hypercycle covering density is $\frac{\sqrt{12}}{\pi}$, proved by I. Vermes in [26–28]. Recently, to the best of author's knowledge, candidates for the densest hyperball packings in the 3, 4 and 5-dimensional hyperbolic spaces are derived by the regular prism tilings [21–23].

We observe that some extremal properties of hyperball packings naturally belong to the regular truncated tetrahedron (or simplex, in general, see Lemma 3.2 and Lemma 3.3). Therefore, in this paper we study hyperball packings in truncated tetrahedra, and prove that if the truncated tetrahedron is regular, then the density of the densest packing is ≈ 0.86338 (see Theorem 5.1). However, these hyperball packing configurations are only locally optimal, and cannot be extended to the whole space $\mathbb{H}^3$. Moreover, we show that the densest known hyperball packing, dually related to the regular prism tilings, introduced in [21], can be realized by a regular truncated tetrahedron tiling.

We have an extensive program of finding globally and locally optimal ball packings in the eight Thurston geometries, arising from Thurston's geometrization conjecture [14,18–20,25]. The packing density δ is defined (see [8,21,23,27,28]) as the reciprocal of the ratio of the volume of a fundamental domain for the symmetry group of a tiling to the volume of the ball pieces contained in the fundamental domain ($\delta < 1$). Similarly the covering density $\Delta > 1$ is defined. A large class of truncated tetrahedron (or simplex) tilings is studied, e.g. in [17], on the base of [16].

2. The projective model and saturated hyperball packings of $\mathbb{H}^3$

We use for $\mathbb{H}^3$ (and analogously for $\mathbb{H}^n$, $n \geq 3$) the projective model in the Lorentz space $\mathbb{E}^{1,3}$ that denotes the real vector space $\mathbf{V}^4$ equipped with the bilinear form of signature $(1,3)$, $\langle \mathbf{x},\ \mathbf{y} \rangle = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3$, where the non-zero vectors $\mathbf{x} = (x^0, x^1, x^2, x^3) \in \mathbf{V}^4$ and $\mathbf{y} = (y^0, y^1, y^2, y^3) \in \mathbf{V}^4$, are determined up to real factors, for representing points of $\mathcal{P}^3(\mathbb{R})$. Then $\mathbb{H}^3$ can be interpreted as the interior of the conical quadric $Q = \{(\mathbf{x}) \in \mathcal{P}^3 | \langle \mathbf{x},\ \mathbf{x} \rangle = 0\} =: \partial\mathbb{H}^3$ in the real projective space $\mathcal{P}^3(\mathbf{V}^4, \boldsymbol{V}_4)$ (here $\boldsymbol{V}_4$ is the dual space of $\mathbf{V}^4$). Namely, for an interior point $\mathbf{y}$ there holds $\langle \mathbf{y},\ \mathbf{y} \rangle < 0$. (Restricting this model to the hyperplane $x^0 = 1$ we obtain the usual collinear, i.e., Cayley-Klein model.)

Points of the boundary $\partial\mathbb{H}^3$ in $\mathcal{P}^3$ are called points at infinity, or at the absolute of $\mathbb{H}^3$. Points lying outside $\partial\mathbb{H}^3$ are said to be outer points of $\mathbb{H}^3$ relative to Q. Let $(\mathbf{x}) \in \mathcal{P}^3$, a point $(\mathbf{y}) \in \mathcal{P}^3$ is said to be conjugate to $(\mathbf{x})$ relative to Q if $\langle \mathbf{x},\ \mathbf{y} \rangle = 0$ holds. The set of all points which are conjugate to $(\mathbf{x})$ form a projective (polar) hyperplane $pol(\mathbf{x}) := \{(\mathbf{y}) \in \mathcal{P}^3 | \langle \mathbf{x},\ \mathbf{y} \rangle = 0\}$. Thus the quadric Q induces a bijection (linear polarity $\mathbf{V}^4 \to \boldsymbol{V}_4$) from the points of $\mathcal{P}^3$ onto their polar hyperplanes.

Point $X(\mathbf{x})$ and hyperplane $\alpha(\boldsymbol{a}) = \{(x^0, x^1, x^2, x^3) | \sum_{i=0}^{3} x^i a_i = 0\}$ are incident if $\mathbf{x}\boldsymbol{a} = 0$ ($\mathbf{x} \in \mathbf{V}^4 \setminus \{\mathbf{0}\},\ \boldsymbol{a} \in \boldsymbol{V}_4 \setminus \{\mathbf{0}\}$).

The hypersphere (or equidistant surface) is a quadratic surface at a constant distance from a plane (base plane) in both halfspaces. The infinite body bounded by the hypersphere, containing the base plane, is called *hyperball*.

The *half hyperball* (i.e., the part of the hyperball lying on one side of its base plane) with distance h to a base plane β is denoted by $\mathcal{H}^h_+$. The volume of the intersection of $\mathcal{H}^h_+(\mathcal{A})$ and the right prism with base a 2-polygon $\mathcal{A} \subset \beta$ can be determined by the classical formula (1) of J. Bolyai [2, 3].

$$\mathrm{Vol}(\mathcal{H}^h_+(\mathcal{A})) = \frac{1}{4}\mathrm{Area}(\mathcal{A})\left[k\sinh\frac{2h}{k} + 2h\right], \tag{1}$$

The constant $k = \sqrt{\frac{-1}{K}}$ is the natural length unit in $\mathbb{H}^3$, where K denotes the constant negative sectional curvature. In the following we may assume that $k = 1$.

Let $\{\mathcal{H}^h_i\}$ be a hyperball packing in $\mathbb{H}^3$ with congruent hyperballs of height h. The density of packing can be heuristically improved by adding hyperballs as long as there is sufficient room to do so. *The hypersphere packing is saturated if no new congruent hypersphere can be added to it, retaining the packing property.* We always assume that our packings are saturated. For a packing of hyperballs $\{\mathcal{H}^h_i\}$ their base planes are denoted by β_i. Thus in a hyperball packing the distance between two ultraparallel base planes $d(\beta_i, \beta_j)$ is at least $2h$ (where for the natural indices there holds $i < j$ and d is the hyperbolic distance function). Furthermore, if the packing is saturated, then there does not exist a hyperplane at distance at least $2h$ from all base planes.

3. On hyperball packings in a truncated tetrahedron

We consider a saturated hyperball packing $\{\mathcal{H}^h_i\}$ of hyperballs in $\mathbb{H}^3$ which can be derived from a truncated tetrahedron tiling $\mathcal{T}$ (see [21–24,27]). One truncated tetrahedron of $\mathcal{T}$ is $\mathcal{S}$=$C^1_1C^1_2C^1_3\ C^2_1C^2_2C^2_3\ C^3_1C^3_2C^3_3\ C^4_1C^4_2C^4_3$ illustrated in Figure 1a. The vertices B_i $(i = 1, 2, 3, 4)$ lie outside of the model, and the truncating facets $C^i_1C^i_2C^i_3 \subset \beta_i$ are orthogonal to the edges of the tetrahedron, "joining" the vertices B^i to the other vertices B^j of the tetrahedron.

The ultraparallel base planes of $\mathcal{H}^h_i$ $(i = 1, 2, 3, 4)$ are denoted by β_i. The distance between two base planes $d(\beta_i, \beta_j) =: e_{ij} \geq 2h$, $i < j \in \{1, 2, 3, 4\}$. Moreover, let the volume of the truncated tetrahedron $\mathcal{S}$ be $\mathrm{Vol}(\mathcal{S})$. We introduce the local density function $\delta(\mathcal{S}(h))$ related to $\mathcal{S}$:

Definition 3.1. $\delta(\mathcal{S}(h)) := \frac{\sum_{i=1}^4 \mathrm{Vol}(\mathcal{H}^h_i \cap \mathcal{S})}{\mathrm{Vol}(\mathcal{S})}$.

It is clear that $\sup_{\mathcal{S}\in\mathcal{T}} \delta(\mathcal{S}(h))$ provides a universal upper bound to the density, associated to this cell decomposition and to this density, of the considered hyperball packing $\{\mathcal{H}^h_i\}$ in space $\mathbb{H}^3$. The problem of determining $\sup_{\mathcal{S}} \delta(\mathcal{S})$ seems to be complicated in general, but we can formulate some important assertions.

1. The area of each rectangular hexagon face, e.g. $\mathrm{Area}(C^1_1C^1_2C^3_1C^3_2C^2_2C^2_1)$ is π.

2. If we restrict ourselves to the above rectangular hexagon $\mathcal{F} = C^1_1C^1_2\ C^3_1C^3_2C^2_2C^2_1$ then the intersections of $\mathcal{H}^h_i$ $(i = 1, 2, 3)$ with $\mathcal{F}$ form in $\mathcal{F}$ a partial hypercycle packing (see Figure 1b).

   It is clear that the density $\delta(\mathcal{F}(h))$ of the hypercycle packing in $\mathcal{F}$ is maximal if the area $\sum_{i=1}^3 \mathrm{Area}(\mathcal{H}^h_i \cap \mathcal{F})$ is maximal, because $\mathrm{Area}(\mathcal{F}) = \pi$ is fixed.

I. Vermes in [27] noticed that the density $\delta(\mathcal{F}(h))$ is maximal if the lengths of the common perpendiculars are equal $e_{12} = e_{23} = e_{13} = 2h$. We note here, that in this "regular" case $\sum_{i=1}^{3} b_i$ is maximal as well, where b_i are the "base segments" of the hypercycle domains $\mathcal{H}_i^h \cap \mathcal{F}$ (Figure 1b). I. Vermes proved in [27] that

$$\delta(\mathcal{F}(h)) = \frac{6\sinh(h)\mathrm{arcsinh}\frac{1}{2\sinh(h)}}{\pi}, \quad \lim_{h\to\infty}(\delta(\mathcal{F}(h))) = \frac{3}{\pi}, \text{ increasingly.}$$

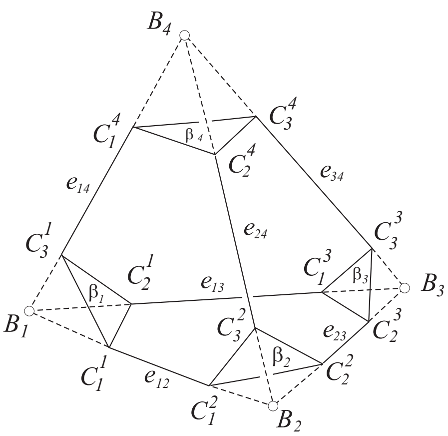
The above statement holds of course for the other regular hexagonal facets of $\mathcal{S}$, as well. From the above considerations there follows

LEMMA 3.2. *The "regular" truncated tetrahedron provides the densest hypercycle packing in the rectangular hexagons of $\mathcal{S}$. The density of the hypercycle packing in the regular hexagonal facets of $\mathcal{S}$ is at most that of the above hypercycle packings, i.e. $\delta(\mathcal{F}(h))$, which is an increasing function of $h \in (0,\infty)$ where the distance between two base planes is $e_{ij} = 2h$ (for each $i < j \in \{1,2,3,4\}$).*

The dihedral angles of the truncated tetrahedron $\mathcal{S}$ at the edges B_iB_j, ($i, j \in \{1,2,3,4\}$ where $i < j$) are denoted by ω_{ij}. If we assume that the sum of the dihedral angles ω_{ij} is constant: Ω, then the surface area of $\mathcal{S}$ is $8\pi - 2\Omega$ constant as well. (At the truncations the other dihedral angles of $\mathcal{S}$ are $\frac{\pi}{2}$). We obtain the following lemma as a consequence of the above assertions and formula (1).

LEMMA 3.3. $\sum_{k=1}^{4} \mathrm{Vol}(\mathcal{H}_k^h \cap \mathcal{S})$ *is maximal if* $e_{ij} = 2h$ ($i < j \in \{1,2,3,4\}$).

Although this lemma does not provide an explicit estimate yet (h depends on the ω_{ij}'s), it motivates the following additional assumption: let the truncated tetrahedron be regular. Then h can also be calculated as will follow in Section 5.



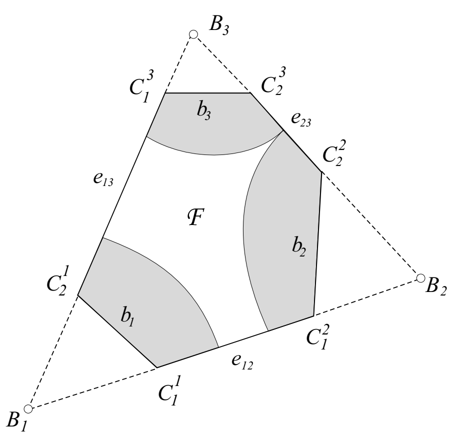


Figure 1: Truncated tetrahedron and one of its rectangular hexagonal faces

4. Characteristic orthoschemes for the volume of a truncated regular tetrahedron

An orthoscheme $\mathcal{O}$ *in* $\mathbb{H}^n$ $n \geq 2$ *in classical sense* is a simplex bounded by $n+1$ hyperplanes $H_0, \dots, H_n$ such that [4] $H_i \perp H_j$, for $j \neq i-1, i, i+1$.

Remark 4.1. This definition is equivalent to the following (see [10]): A simplex $\mathcal{O}$ in $\mathbb{H}^n$ is an orthoscheme iff the $n+1$ vertices of $\mathcal{O}$ can be labelled by $R_0, R_1, \dots, R_n$ in such a way that $\text{span}(R_0, \dots, R_i) \perp \text{span}(R_i, \dots, R_n)$ for $0 < i < n-1$.

Geometrically, complete orthoschemes of degree $m = 0, 1, 2$ can be described as follows:

1. For $m = 0$, they coincide with the class of classical orthoschemes introduced by L. Schläfli. The initial and final vertices, R_0 and R_n of the orthogonal edge-path $R_i R_{i+1}$, $i = 0, \dots, n-1$, are called principal vertices of the orthoscheme (see Remark 4.1).
2. A complete orthoscheme of degree $m = 1$ can be constructed from an orthoscheme with one outer principal vertex, say R_n, which is truncated by its polar plane $pol(R_n)$ (see Figure 2b). In this case the orthoscheme is called simply truncated with outer vertex R_n.
3. A complete orthoscheme of degree $m = 2$ can be constructed from an orthoscheme with two outer principal vertices R_0, R_n truncated by its polar hyperplanes $pol(R_0)$ and $pol(R_n)$. In this case the orthoscheme is called doubly truncated (see [10]).

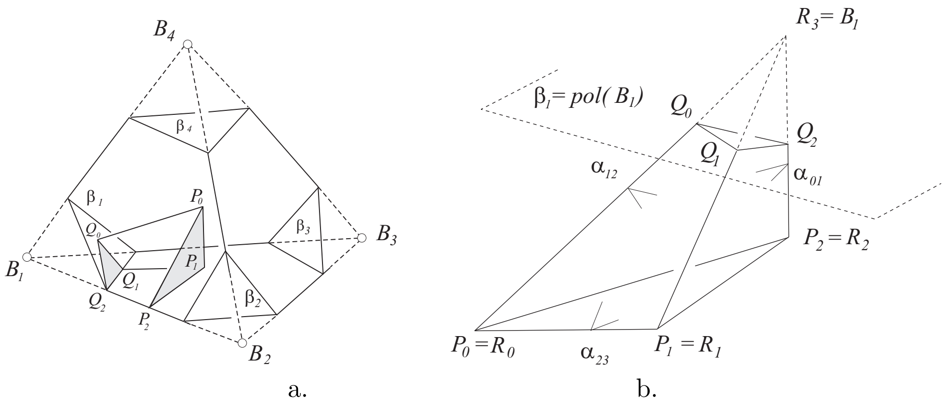


Figure 2: Truncated tetrahedron with a complete orthoscheme of degree $m = 1$

In the following we use the "3-dimensional simply truncated orthoschemes" whose volume formula is derived by the next Theorem of R. Kellerhals (extending the formula of N.I. Lobachevsky [13] for classical orthoschemes).

Theorem 4.2. *([10, Theorem II]) The volume of a three-dimensional hyperbolic complete orthoscheme (except Lambert cube cases, i.e. complete orthoschemes of degree $m = 2$ with outer edge) $\mathcal{O} \subset \mathbb{H}^3$ is expressed by the essential angles $\alpha_{01}, \alpha_{12}, \alpha_{23}$, $(0 \le \alpha_{ij} \le \frac{\pi}{2})$ (Figure 2b) in the following form:*

$$\mathrm{Vol}(\mathcal{O}) = \frac{1}{4}\{\mathcal{L}(\alpha_{01}+\theta) - \mathcal{L}(\alpha_{01}-\theta) + \mathcal{L}(\frac{\pi}{2}+\alpha_{12}-\theta)+ \\ + \mathcal{L}(\frac{\pi}{2}-\alpha_{12}-\theta) + \mathcal{L}(\alpha_{23}+\theta) - \mathcal{L}(\alpha_{23}-\theta) + 2\mathcal{L}(\frac{\pi}{2}-\theta)\},$$

where $\theta \in [0, \frac{\pi}{2})$ is defined by:

$$\tan(\theta) = \frac{\sqrt{\cos^2\alpha_{12} - \sin^2\alpha_{01}\sin^2\alpha_{23}}}{\cos\alpha_{01}\cos\alpha_{23}},$$

and where $\mathcal{L}(x) := -\int\limits_0^x \log|2\sin t|dt$ denotes the Lobachevsky function.

In the following we assume that the ultraparallel base planes β_i of $\mathcal{H}_i^{h(p)}$ $(i = 1,2,3,4$, and $6 < p \in \mathbb{R})$ generate a “regular truncated tetrahedron” $\mathcal{S}^r$ with outer vertices B_i (see Figure 2a) whose non-orthogonal dihedral angles are equal to $\frac{2\pi}{p}$, and the distances between two base planes $d(\beta_i, \beta_j) =: e_{ij}$ $(i < j \in \{1,2,3,4\})$ are equal to $2h(p)$ depending on the angle $\frac{\pi}{p}$.

The truncated regular tetrahedron $\mathcal{S}^r$ can be decomposed into 24 congruent simply truncated orthoschemes; one of them $\mathcal{O} = Q_0Q_1Q_2P_0P_1P_2$ is illustrated in Figure 2a where P_0 is the center of the “regular tetrahedron” $\mathcal{S}^r$, P_1 is the center of a hexagonal face of $\mathcal{S}^r$, P_2 is the midpoint of a “common perpendicular” edge of this face, Q_0 is the center of an adjacent regular triangle face of $\mathcal{S}^r$, Q_1 is the midpoint of an appropriate edge of this face and one of its endpoints is Q_2.

In our case the essential dihedral angles of orthoschemes $\mathcal{O}$ are the following: $\alpha_{01} = \frac{\pi}{p}$, $\alpha_{12} = \frac{\pi}{3}$, $\alpha_{23} = \frac{\pi}{3}$ (see Figure 2b). Therefore, the volume $\mathrm{Vol}(\mathcal{O})$ of the orthoscheme $\mathcal{O}$ and the volume $\mathrm{Vol}(\mathcal{S}^r) = 24 \cdot \mathrm{Vol}(\mathcal{O})$ can be computed for any given parameter p $(6 < p \in \mathbb{R})$ by Theorem 4.2.

5. Packing with congruent hyperballs in a regular truncated tetrahedron

In this case for a given parameter p the length of the common perpendiculars $h(p) = \frac{1}{2}e_{ij}$ $(i < j,\ i,j \in \{1,2,3,4\})$ can be determined by the machinery of projective metric geometry. (In the following $\mathbf{x} \sim c\cdot\mathbf{x}$ with $c \in \mathbb{R}\setminus\{\mathbf{0}\}$ represent the same point $X = (\mathbf{x} \sim c\cdot\mathbf{x})$ of $\mathcal{P}^3$.)

The points $P_2(\mathbf{p}_2)$ and $Q_2(\mathbf{q}_2)$ are proper points of hyperbolic 3-space and Q_2 lies on the polar hyperplane $pol(B_1)(\boldsymbol{b}^1)$ of the outer point B_1 thus

$$\mathbf{q}_2 \sim c\cdot\mathbf{b}_1 + \mathbf{p}_2 \in \boldsymbol{b}^1 \Leftrightarrow c\cdot\mathbf{b}_1\boldsymbol{b}^1 + \mathbf{p}_2\boldsymbol{b}^1 = 0 \Leftrightarrow c = -\frac{\mathbf{p}_2\boldsymbol{b}^1}{\mathbf{b}_1\boldsymbol{b}^1} \Leftrightarrow$$

$$\mathbf{q}_2 \sim -\frac{\mathbf{p}_2\boldsymbol{b}^1}{\mathbf{b}_1\boldsymbol{b}^1}\mathbf{b}_1 + \mathbf{p}_2 \sim \mathbf{p}_2(\mathbf{b}_1\boldsymbol{b}^1) - \mathbf{b}_1(\mathbf{p}_2\boldsymbol{b}^1) = \mathbf{p}_2h_{33} - \mathbf{b}_1h_{23},$$

where h_{ij} is the inverse of the Coxeter-Schläfli matrix

$$(c^{ij}) := \begin{pmatrix} 1 & -\cos\frac{\pi}{p} & 0 & 0 \\ -\cos\frac{\pi}{p} & 1 & -\cos\frac{\pi}{3} & 0 \\ 0 & -\cos\frac{\pi}{3} & 1 & -\cos\frac{\pi}{3} \\ 0 & 0 & -\cos\frac{\pi}{3} & 1 \end{pmatrix} \tag{2}$$

of the orthoscheme $\mathcal{O}$. The hyperbolic distance $h(p)$ can be calculated by the following formula:

$$\cosh h(p) = \cosh P_2Q_2 = \frac{-\langle \mathbf{q}_2, \mathbf{p}_2\rangle}{\sqrt{\langle \mathbf{q}_2, \mathbf{q}_2\rangle\langle \mathbf{p}_2, \mathbf{p}_2\rangle}} = \frac{h_{23}^2 - h_{22}h_{33}}{\sqrt{h_{22}\langle \mathbf{q}_2, \mathbf{q}_2\rangle}} = \sqrt{\frac{h_{22}\ h_{33} - h_{23}^2}{h_{22}\ h_{33}}}.$$

We get that the volume $\mathrm{Vol}(\mathcal{S}^r)$, the maximal height $h(p)$ of the congruent hyperballs lying in $\mathcal{S}^r$ and $\sum_{i=1}^4 \mathrm{Vol}(\mathcal{H}_i^h \cap \mathcal{S}^r))$ all depend only on the parameter p of the truncated regular tetrahedron $\mathcal{S}^r$.

Therefore, the density $\delta(\mathcal{S}^r(h(p)))$ depends only on p ($6 < p \in \mathbb{R}$). Moreover, the total volume of the parts of the four hyperballs lying in $\mathcal{S}^r$ can be computed by formula (1), and the volume of $\mathcal{S}^r$ can be determined by Theorem 4.2.

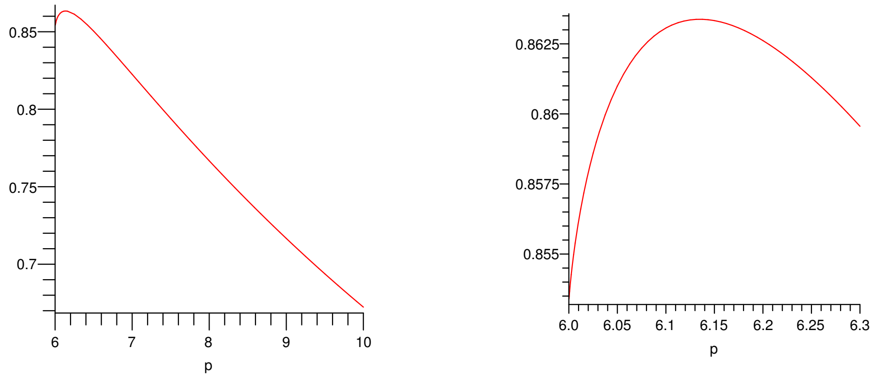


Figure 3: The density function $\delta(\mathcal{S}^r(h(p)))$, $p \in (6, \infty)$

Finally, we obtain the plot after careful analysis of the smooth density function (cf. Figure 3) and we obtain the following.

THEOREM 5.1. *The density function* $\delta(\mathcal{S}^r(h(p)))$, $p \in (6, \infty)$ *attains its maximum at* $p^{opt} \approx 6.13499$, *and* $\delta(\mathcal{S}^r(h(p)))$ *is strictly increasing in the interval* $(6, p^{opt})$, *and strictly decreasing in* (p^{opt}, ∞). *Moreover, the optimal density* $\delta^{opt}(\mathcal{S}^r(h(p^{opt}))) \approx 0.86338$ *(see Figure 3).*

REMARK 5.2. 1. In our case $\lim_{p\to 6}(\delta(\mathcal{S}^r(h(p))))$ is equal to the Böröczky-Florian upper bound of the ball and horoball packings in $\mathbb{H}^3$ [6] (observe that the dihedral angles of $\mathcal{S}^r$ for the case of the horoball equal $2\pi/6$).

2. $\delta^{opt}(\mathcal{S}^r(h(p^{opt}))) \approx 0.86338$ is larger than the Böröczky-Florian upper bound $\delta_{BF} \approx 0.85328$; but these hyperball packing configurations are only locally optimal and cannot be extended to the entire hyperbolic space $\mathbb{H}^3$.

5.1 Tilings with regular truncated tetrahedra in hyperbolic 3-space

In the papers [21–24] we have studied the hyperball packings and coverings associated to regular prism tilings in n-dimensional ($n = 3, 4, 5$) hyperbolic space and determined the corresponding densest hyperball packings and thinnest hyperball coverings. From the definitions of the prism tilings and the complete orthoschemes of degree $m = 1$ it follows that a regular prism tiling exists in space $\mathbb{H}^n$ if and only if there exists a complete Coxeter orthoscheme of degree $m = 1$ with two ultraparallel faces (in Figure 2a these are $P_0P_1P_2$ and $Q_0Q_1Q_2$). The complete Coxeter orthoschemes were classified by Im Hof in [9] by generalizing the methods of Coxeter and Böhm appropriately. The truncated tetrahedron tilings are studied in [17] on the base of [16].

The hyperball packings in the regular truncated tetrahedra under the extended reflection groups with Coxeter-Schläfli symbol $[3, 3, p]$, investigated in this paper, can be extended to the entire hyperbolic space if p is an integer parameter bigger than 6. They coincide with the hyperball packings given by the regular p-gonal prism tilings in $\mathbb{H}^3$ with extended Coxeter-Schläfli symbols $[p, 3, 3]$, see in [21]. As we know, $[3, 3, p]$ and $[p, 3, 3]$ are dually isomorphic extended reflection groups, just with the above frustum of orthoscheme as fundamental domain (Figure 2b, matrix (c^{ij}) in formula (2)).

In the following table we summarize the data of the hyperball packings for some parameters p ($6 < p \in \mathbb{N}$), where $\mathcal{A}$ is a trigonal face of the truncated tetrahedron (Figure 1).

Table 1: Hyperball packings for various parameters p

p	h	Vol$(\mathcal{O})$	Vol$(\mathcal{H}_+^h(\mathcal{A}))$	δ
7	0.78871	0.08856	0.07284	0.82251
8	0.56419	0.10721	0.08220	0.76673
9	0.45320	0.11825	0.08474	0.71663
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	0.16397	0.14636	0.06064	0.41431
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	0.06325	0.15167	0.02918	0.19240
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	0.03147	0.15241	0.01549	0.10165
$p \to \infty$	0	0.15266	0	0

The problems of the densest horoball and hyperball packings in hyperbolic n-space $n \geq 3$ with horoballs of different types and hyperballs have not been settled yet, in general (see e.g. [11, 12, 19, 20]).

Optimal ball (sphere) packings in other homogeneous Thurston geometries represent another huge class of open problems. For these non-Euclidean geometries only very few results are known (e.g. [25] and the references given there). Detailed studies are the objective of ongoing research. Applications of the above projective method seem to be interesting in (non-Euclidean) crystallography as well, a topic of much current interest.

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Budapest University of Technology and Economics, Institute of Mathematics, Department of Geometry, Budapest, Hungary

E-mail: szirmai@math.bme.hu