EXACT FORMULAE OF GENERAL SUM-CONNECTIVITY INDEX FOR SOME GRAPH OPERATIONS

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Abstract. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $a \in V(G)$ is denoted by $d_G(a)$. The general sum-connectivity index of $G$ is defined as $\chi_\alpha(G) = \sum_{ab \in E(G)} (d_G(a) + d_G(b))^{\alpha}$, where $\alpha$ is a real number. In this paper, we compute exact formulae for general sum-connectivity index of several graph operations. These operations include tensor product, union of graphs, splices and links of graphs and Hajós construction of graphs. Moreover, we also compute exact formulae for general sum-connectivity index of some graph operations for positive integral values of $\alpha$. These operations include cartesian product, strong product, composition, join, disjunction and symmetric difference of graphs.

1. Introduction and preliminary results

Let $G = (V(G), E(G))$ be a simple and connected graph. An edge with end vertices $a$ and $b$ is denoted by $ab$ (or $ba$). The order and size of graph $G$ are denoted by $n_G$ and $m_G$, respectively. The degree of a vertex $a \in V(G)$, denoted by $d_G(a)$, is the number of vertices incident with $a$. A path $P_n$ of length $n - 1$ is a graph with vertex set $\{a_i \mid i = 1, \ldots, n\}$ and edge set $\{a_ia_{i+1} \mid i = 1, \ldots, n - 1\}$. A cycle $C_n$ of length $n$ is a graph with vertex set $\{a_i \mid i = 1, \ldots, n\}$ and edge set $\{a_ia_{i+1} \mid i = 1, \ldots, n - 1\} \cup \{a_na_1\}$. A complete graph of order $n$ is denoted by $K_n$.

A topological index is a mathematical measure which correlate to the chemical structures of any simple finite graph. They play an important role in the study of QSAR/QSPR. There are numerous topological descriptors that have some applications in theoretical chemistry. Among these topological descriptors, the degree based topological indices are of great importance.

The first degree based topological indices that are defined by Gutman and Trinajstić [8] in 1972, are the first and second Zagreb indices. These indices are defined as follows: $M_1(G) = \sum_{a \in V(G)} (d_G(a))^2$, $M_2(G) = \sum_{ab \in E(G)} d_G(a)d_G(b)$. Here $M_1(G)$ and $M_2(G)$

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denote the first and second Zagreb index, respectively. Li and Zhao [13] introduced the first general Zagreb index:

\[ M_\alpha(G) = \sum_{a \in V(G)} (d_G(a))^\alpha, \quad \alpha \in \mathbb{R}. \]

The general Randić index (product-connectivity index), introduced by Li and Gutman [12], is defined in the following way:

\[ R_\alpha(G) = \sum_{ab \in E(G)} (d_G(a)d_G(b))^\alpha, \quad \alpha \in \mathbb{R}. \]

Then \( R_{-1/2}(G) \) is called the Randić index which was defined by Randić [15] in 1975. The general sum-connectivity index is introduced by Zhou and Trinajstić [18] and is defined as:

\[ \chi_\alpha(G) = \sum_{ab \in E(G)} (d_G(a) + d_G(b))^\alpha, \quad \alpha \in \mathbb{R}. \]

Then \( \chi_{-1/2}(G) \) is the classical sum-connectivity index which was defined by Zhou and Trinajstić [17] in 2009. Another variant of the Randić index of \( G \) is the harmonic index, denoted by \( H(G) \) and is defined as follows:

\[ H(G) = \sum_{ab \in E(G)} \frac{2}{d_G(a) + d_G(b)} = 2\chi_{-1}(G). \]

Recently Ashrafi et al. [1] computed the exact formulae for Zagreb coindices of some graph operations. Ashrafi et al. [2] calculated some topological indices of splices and links of graphs. In [16], Yarahmadi computed some topological indices of tensor product of graphs. For a detailed study on topological indices of graph operations, we refer the reader to [3, 4, 6, 7, 9–11, 14].

This paper is organized as follows. In Section 2, we present some known graph operations. In Section 3, we give exact formulae of the general sum-connectivity index for several graph operations. These graph operations include tensor product, union of graphs, splices and links of graphs and Hajós construction of graphs for real values of \( \alpha \). In Section 4, we compute exact formulae of the general sum-connectivity index of some graph operations including cartesian product, strong product, composition, join, disjunction and symmetric difference of graphs for positive integral values of \( \alpha \).

2. Graph operations

Let \( G \) and \( H \) be two simple connected graphs whose vertex sets are disjoint. The set of real numbers and the set of positive integers are denoted by \( \mathbb{R} \) and \( \mathbb{Z}^+ \), respectively.

The tensor product of \( G \) and \( H \), denoted by \( G \otimes H \), is the graph with vertex set \( V(G \otimes H) = V(G) \times V(H) \) and \((a, b)(c, d) \in E(G \otimes H)\) whenever \( ac \in E(G) \) and \( bd \in E(H) \). The order and the size of \( G \otimes H \) are \( n_Gn_H \) and \( 2m_Gm_H \), respectively. The degree of a vertex \((a, b) \in V(G \otimes H)\) is given by

\[ d_{G\otimes H}(a, b) = d_G(a)d_H(b). \]
The union of $G$ and $H$, denoted by $G \cup H$, is the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. The order and size of $G \cup H$ are $n_G + n_H$ and $m_G + m_H$, respectively. For $a \in V(G \cup H)$, either $a \in V(G)$ or $a \in V(H)$ but not both. The degree of $a$ in $G \cup H$ is given by

$$d_{G \cup H}(a) = \begin{cases} d_G(a) & \text{if } a \in V(G), \\ d_H(a) & \text{if } a \in V(H). \end{cases}$$

Splices of graphs (also known as coalescences of graphs) were introduced by Đosić [5] in 2005. A splice of $G$ and $H$ for given vertices $a \in V(G)$ and $b \in V(H)$, denoted by $(G \bullet H)(a, b)$, is defined by identifying the vertices $a$ and $b$ in the union of $G$ and $H$. The order and the size of $(G \bullet H)(a, b)$ are $n_G + n_H - 1$ and $m_G + m_H$, respectively. Let $x$ be the vertex obtained by identifying $a \in V(G)$ and $b \in V(H)$. Then $V((G \bullet H)(a, b)) = \{x\} \cup (V(G) \cup V(H))$ and $E((G \bullet H)(a, b)) = E(G) \cup E(H)$, respectively. The degree of a vertex $c \in V((G \bullet H)(a, b))$ is given by

$$d_{(G \bullet H)(a, b)}(c) = \begin{cases} d_G(c) & \text{if } c \in V(G), c \neq a, \\ d_H(c) & \text{if } c \in V(H), c \neq b, \\ d_G(c) + d_H(b) & \text{if } c = x. \end{cases}$$

The motivation for considering these graphs comes from chemistry, where splices of cycles serve as models of spirane molecules and models of complex molecules are built from simpler building blocks by iterating and/or combining the splice and link operations.

Links of graphs were introduced by Đosić [5] in 2005. The link of $G$ and $H$ for given vertices $a \in V(G)$ and $b \in V(H)$, denoted by $(G \sim H)(a, b)$, is obtained by joining the vertices $a$ and $b$ by an edge in the union of $G$ and $H$. The order and the size of $(G \sim H)(a, b)$ are $n_G + n_H$ and $m_G + m_H + 1$, respectively. The set of vertices and edges are $V((G \sim H)(a, b)) = V(G) \cup V(H)$ and $E((G \sim H)(a, b)) = (E(G) \cup V(H)) \cup \{ab\}$, respectively. The degree of a vertex $c \in V((G \sim H)(a, b))$ is given by

$$d_{(G \sim H)(a, b)}(c) = \begin{cases} d_G(c) & \text{if } c \in V(G), c \neq a, \\ d_H(c) & \text{if } c \in V(H), c \neq b, \\ d_G(c) + 1 & \text{if } c = a, \\ d_H(c) + 1 & \text{if } c = b. \end{cases}$$

Let $a\hat{a}$ be an edge in $G$ and $b\hat{b}$ be an edge in $H$. Then the Hajós construction of graphs, denoted by $(G \triangle H)(a\hat{a}, b\hat{b})$, is obtained by identifying $a$ and $b$, deleting the edges $a\hat{a}$ and $b\hat{b}$, and adding an edge $\hat{a}\hat{b}$. The order and size of $(G \triangle H)(a\hat{a}, b\hat{b})$ are $n_G + n_H - 1$ and $m_G + m_H - 1$, respectively. Let $x$ be the vertex obtained by identifying $a \in V(G)$ and $b \in V(H)$. Then $V((G \triangle H)(a\hat{a}, b\hat{b})) = \{x\} \cup (V(G) \setminus \{a\}) \cup (V(H) \setminus \{b\})$ and $E((G \triangle H)(a\hat{a}, b\hat{b})) = (E(G) \setminus \{a\hat{a}\}) \cup (V(H) \setminus \{b\hat{b}\}) \cup \{\hat{a}\hat{b}\}$, respectively. The degree of a vertex $c \in V((G \triangle H)(a\hat{a}, b\hat{b}))$ is given by

$$d_{(G \triangle H)(a\hat{a}, b\hat{b})}(c) = \begin{cases} d_G(c) & \text{if } c \in V(G), c \neq a, \\ d_H(c) & \text{if } c \in V(H), c \neq b, \\ d_G(a) + d_H(b) - 2 & \text{if } c = x. \end{cases}$$

The cartesian product of $G$ and $H$, denoted by $G \square H$, is the graph with vertex set
Exact formulae of general sum-connectivity index

V(G□H) = V(G)×V(H) and (a,b)(c,d) ∈ E(G□H) whenever ∣a = c and bd ∈ E(H)∣ or ∣ac ∈ E(G) and b = d∣. The order and size of G□H are n_Gn_H and m_Gn_H + m_Hn_G, respectively. The degree of a vertex (a,b) ∈ V(G□H) is given by

\[ d_{G□H}(a,b) = d_G(a) + d_H(b). \]

The strong product of G and H, denoted by G ⊠ H, is the graph with vertex set \(V(G ⊠ H) = V(G) \times V(H)\) and (a,b)(c,d) ∈ E(G ⊠ H) whenever [a = c and bd ∈ E(H)] or [ac ∈ E(G) and b = d] or [ac ∈ E(G) and bd ∈ E(H)]. The order and size of G ⊠ H are n_Gn_H and n_Gm_H + n_Hm_G + 2m_Gm_H, respectively. The degree of a vertex (a,b) ∈ V(G ⊠ H) is given by

\[ d_{G ⊠ H}(a,b) = d_G(a) + d_H(b) + d_G(a)d_H(b). \]

The composition (lexicographic product) of G and H, denoted by G[H], is the graph with vertex set \(V(G[H]) = V(G) \times V(H)\) and (a,b)(c,d) ∈ E(G[H]) whenever [a is adjacent to c in G] or [a = c and b is adjacent to d in H]. The order and size of G[H] are n_Gn_H and m_Gn_H^2 + n_Gm_H, respectively. The degree of a vertex (a,b) ∈ V(G[H]) is given by

\[ d_{G[H]}(a,b) = n_Hd_G(a) + d_H(b). \]

The join of G and H, denoted by G + H, is the graph union G ∪ H together with all the edges joining V(G) and V(H). The order and size of G + H are n_G + n_H and m_G + m_H + n_Gn_H, respectively. The degree of a vertex a in G + H is given by

\[ d_{G + H}(a) = \begin{cases} d_G(a) + n_H & \text{if } a \in V(G), \\ d_H(a) + n_G & \text{if } a \in V(H). \end{cases} \]

The disjunction of G and H, denoted by G ∨ H, is the graph with vertex set \(V(G ∨ H) = V(G) \times V(H)\) and (a,b)(c,d) ∈ E(G ∨ H) whenever ac ∈ E(G) or bd ∈ E(H). The order and size of G ∨ H are n_Gn_H and m_Gn_H^2 + m_Hn_G^2 + 2m_Gm_H, respectively. The degree of a vertex (a,b) in G ∨ H is given by

\[ d_{G ∨ H}(a,b) = n_Hd_G(a) + n_Gd_H(b) - d_G(a)d_H(b). \]

The symmetric difference of G and H, denoted by G ⊕ H, is the graph with vertex set \(V(G ⊕ H) = V(G) \times V(H)\) and (a,b)(c,d) ∈ E(G ⊕ H) whenever ac ∈ E(G) or bd ∈ E(H) but not both. The order and size of G ⊕ H are n_Gn_H and m_Gn_H^2 + m_Hn_G^2 + 4m_Gm_H, respectively. The degree of a vertex (a,b) in G ⊕ H is given by

\[ d_{G ⊕ H}(a,b) = n_Hd_G(a) + n_Gd_H(b) - 2d_G(a)d_H(b). \]

3. Formulae of general sum-connectivity index when α ∈ \(\mathbb{R}\)

In this section, we derive exact formulae of general sum-connectivity index for some graph operations defined in Section 2. In the following theorem, we compute the general sum-connectivity index of \(G ⊗ H\).

**Theorem 3.1.** Let G and H be two graphs such that either G or H is regular. Then
the general sum-connectivity index of $G \otimes H$ is given by the formula:

$$\chi_\alpha(G \otimes H) = \frac{1}{2^{\alpha-1}} \chi_\alpha(G) \chi_\alpha(H).$$

**Proof.** By equation (1) and the definition of general sum-connectivity index, we have

$$\chi_\alpha(G \otimes H) = 2 \sum_{a \in E(G)} \sum_{b \in E(H)} (d_G(a)d_H(b) + d_G(c)d_H(d))^\alpha.$$  \hspace{1cm} (10)

Without loss of generality, assume that $G$ is a regular graph.

For each $(a,b)(c,d) \in E(G \otimes H)$, we have $d_G(a) = d_G(b)$. Thus

$$d_G(a)d_H(b) + d_G(c)d_H(d) = \frac{1}{2}(d_G(a)d_H(b) + d_G(a)d_H(b) + d_G(c)d_H(d) + d_G(c)d_H(d))$$
$$= \frac{1}{2}(d_G(a)d_H(b) + d_G(c)d_H(b) + d_G(a)d_H(d) + d_G(c)d_H(d))$$ \hspace{1cm} (11)
$$= \frac{1}{2}(d_G(a) + d_G(c))(d_H(b) + d_H(d)).$$

Then (10) together with (11) gives

$$\chi_\alpha(G \otimes H) = 2 \sum_{a \in E(G)} \sum_{b \in E(H)} (d_G(a)d_H(b) + d_G(c)d_H(d))^\alpha$$
$$= 2 \sum_{a \in E(G)} \sum_{b \in E(H)} \left(\frac{1}{2}(d_G(a) + d_G(c))(d_H(b) + d_H(d))\right)^\alpha$$
$$= \left(\frac{2}{2^\alpha}\right) \sum_{a \in E(G)} (d_G(a) + d_G(c))^\alpha \sum_{b \in E(H)} (d_H(b) + d_H(d))^\alpha = \frac{1}{2^{\alpha-1}} \chi_\alpha(G) \chi_\alpha(H).$$ \hspace{1cm} \square

**Example 3.2.** Applying Theorem 3.1, the general sum-connectivity index for tensor product of some graphs is given below:

1) $\chi_\alpha(C_n \otimes C_m) = 2^{\alpha+1}mn$, \hspace{1cm} 2) $\chi_\alpha(K_n \otimes K_m) = \frac{1}{2^{\alpha+1}}mn[(n+1)(m+1)]^\alpha$.

In the following theorem, we give the general sum-connectivity index of union of a finite number of graphs. The proof is omitted since it can be easily derived.

**Theorem 3.3.** Let $G_1, G_2, \ldots, G_n$ be vertex-disjoint graphs. Then the general sum-connectivity index of $\bigcup_{i=1}^k G_i$ is given by the following formula:

$$\chi_\alpha\left(\bigcup_{i=1}^k G_i\right) = \chi_\alpha(G_1) + \chi_\alpha(G_2) + \ldots + \chi_\alpha(G_n).$$

**Example 3.4.** Using Theorem 3.3, the general sum-connectivity index for union of some graphs is given below:

1) $\chi_\alpha\left(\bigcup_{i=1}^k C_{n_i}\right) = 4^\alpha \sum_{i=1}^k n_i$, \hspace{1cm} 2) $\chi_\alpha\left(\bigcup_{i=1}^k P_{n_i}\right) = 2k(3^\alpha + 4^\alpha) + 4^\alpha \sum_{i=1}^k n_i$.

In the following theorem, we compute the general sum-connectivity index of splices of two graphs $G$ and $H$ for given vertices $a$ and $b$. 

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THEOREM 3.5. Let \( G \) and \( H \) be two graphs. Then the general sum-connectivity index of \((G \bullet H)(a, b)\) is given by the formula:

\[
\chi_\alpha ((G \bullet H)(a, b)) = \chi_\alpha (G) \sum_{d \in N_G(a)} [(d_G(a) + d_H(b) + d_G(d))^\alpha - (d_G(a) + d_G(d))^\alpha] \\
+ \chi_\alpha (H) \sum_{d \in N_H(b)} [(d_H(b) + d_G(a) + d_H(d))^\alpha - (d_H(b) + d_H(d))^\alpha].
\]

Proof. By equation (2) and the definition of general sum-connectivity index, we have

\[
\chi_\alpha ((G \bullet H)(a, b)) = \sum_{c,d \in E(G)} (d_G(c) + d_G(d))^\alpha + \sum_{c,d \in E(H)} (d_H(c) + d_H(d))^\alpha \\
+ \sum_{c,d \in E(G), c \neq a, d \in V(G)} (d_G(c) + d_G(d) + d_H(b))^\alpha + \sum_{c,d \in E(H), c \neq b, d \in E(G)} (d_H(c) + d_H(d) + d_G(a))^\alpha \\
= \chi_\alpha (G) \sum_{d \in N_G(a)} [(d_G(a) + d_G(d))^\alpha - (d_G(a) + d_G(d))^\alpha] \\
+ \chi_\alpha (H) \sum_{d \in N_H(b)} [(d_H(b) + d_G(a) + d_H(d))^\alpha - (d_H(b) + d_H(d))^\alpha].
\]

EXAMPLE 3.6. Applying Theorem 3.5, the general sum-connectivity index for splices of \( K_n \) and \( C_m \) is given below:

\[
\chi_\alpha ((K_n \bullet C_m)(a, b)) = 2^{n-1}n(n-1)^\alpha + 4^\alpha (m-2) + 2^\alpha (n-1)(n^\alpha + (n-1)^\alpha) + 2(n+3)^\alpha.
\]

In the following theorem, we compute the general sum-connectivity index of links of two graphs \( G \) and \( H \) for given vertices \( a \) and \( b \).

THEOREM 3.7. Let \( G \) and \( H \) be two graphs. Then the general sum-connectivity index of \((G \sim H)(a, b)\) is given by the formula:

\[
\chi_\alpha ((G \sim H)(a, b)) = \chi_\alpha (G) + \chi_\alpha (H) \sum_{d \in N_G(a)} [(d_G(a) + 1 + d_G(d))^\alpha - (d_G(a) + d_G(d))^\alpha] \\
+ (d_G(a) + d_H(b) + 2)^\alpha + \sum_{d \in N_H(b)} [(d_H(b) + 1 + d_H(d))^\alpha - (d_H(b) + d_H(d))^\alpha].
\]

Proof. Equation (3) and the definition of general sum-connectivity index give

\[
\chi_\alpha ((G \sim H)(a, b)) = \sum_{c,d \in E(G)} (d_G(c) + d_G(d))^\alpha + \sum_{c,d \in E(H)} (d_H(c) + d_H(d))^\alpha
\]
Theorem

By equation (4) and the definition of general sum-connectivity index, we get

\[ \chi_\alpha(G) = \sum_{cd \in \text{E}(G), e = a} \frac{(d_G(c) + d_G(d))^\alpha}{d \in V(G)} + \sum_{cd \in \text{E}(H), e = b} \frac{(d_H(c) + d_H(d))^\alpha}{d \in V(H)} \]

Proof.

\[ \chi_\alpha(G) - \sum_{d \in N_G(a)} (d_G(c) + d_G(d))^\alpha + \chi_\alpha(H) - \sum_{d \in N_H(b)} (d_H(b) + d_H(d))^\alpha \]

\[ + (d_G(a) + d_H(b) + 2)^\alpha + \sum_{d \in N_G(a)} (d_G(a) + 1 + d_G(d))^\alpha + \sum_{d \in N_H(b)} (d_H(b) + 1 + d_H(d))^\alpha \]

\[ = \chi_\alpha(G) + \chi_\alpha(H) + \sum_{d \in N_G(a)} [(d_G(a) + 1 + d_G(d))^\alpha - (d_G(a) + d_G(d))^\alpha] \]

\[ + (d_G(a) + d_H(b) + 2)^\alpha + \sum_{d \in N_H(b)} (d_H(b) + 1 + d_H(d))^\alpha - (d_H(b) + d_H(d))^\alpha] \]

\[ = \chi_\alpha(G) + \chi_\alpha(H) + \sum_{d \in N_G(a)} [(d_G(a) + 1 + d_G(d))^\alpha - (d_G(a) + d_G(d))^\alpha] \]

\[ + (d_G(a) + d_H(b) + 2)^\alpha + \sum_{d \in N_H(b)} (d_H(b) + 1 + d_H(d))^\alpha - (d_H(b) + d_H(d))^\alpha] \]

\[ \square \]

Example 3.8. Applying Theorem 3.7, the general sum-connectivity index of \((P_n \sim C_m)(a, b)\) is given by \(\chi_\alpha((P_n \sim C_m)(a, b)) = 2^{n+1} (2 \times 3^m + m(n - 3)4^m)\).

Theorem 3.9. Let \(G\) and \(H\) be two graphs. Then the general sum-connectivity index of \((G \triangle H)(a\hat{a}, b\hat{b})\) is given by the formula:

\[ \chi_\alpha((G \triangle H)(a\hat{a}, b\hat{b})) = \chi_\alpha(G) + \chi_\alpha(H) + (d_G(a) + d_H(b))^\alpha \]

\[ + \sum_{\hat{a} \neq d \in N(a)} (d_G(a) + d_H(b) - 2 + d_G(d))^\alpha - \sum_{d \in N_G(a)} (d_G(a) + d_G(d))^\alpha \]

\[ + \sum_{\hat{b} \neq d \in N(b)} (d_H(b) + d_G(a) - 2 + d_H(d))^\alpha - \sum_{d \in N_H(b)} (d_H(b) + d_H(d))^\alpha \]

Proof. By equation (4) and the definition of general sum-connectivity index, we get

\[ \chi_\alpha((G \triangle H)(a\hat{a}, b\hat{b})) = \sum_{cd \in \text{E}(G)} \frac{(d_G(c) + d_G(d))^\alpha}{c \neq d \in V(G)} + \sum_{cd \in \text{E}(H)} \frac{(d_H(c) + d_H(d))^\alpha}{c \neq d \in V(H)} + (d_G(a) + d_H(b))^\alpha \]

\[ + \sum_{\hat{a} \neq d \in N(a)} (d_G(a) + d_H(b) - 2 + d_G(d))^\alpha - \sum_{d \in N_G(a)} (d_G(a) + d_G(d))^\alpha \]

\[ = \chi_\alpha(G) - (d_G(a) + d_G(d))^\alpha - \sum_{\hat{b} \neq d \in N_H(b)} (d_H(b) + d_G(a) - 2 + d_H(d))^\alpha \]

\[ + \sum_{\hat{b} \neq d \in N_H(b)} (d_H(b) + d_G(a) - 2 + d_H(d))^\alpha + (d_G(a) + d_H(b))^\alpha \]

\[ = \chi_\alpha(G) + \chi_\alpha(H) + (d_G(a) + d_H(b))^\alpha + \sum_{\hat{a} \neq d \in N_G(a)} (d_G(a) + d_H(b) - 2 + d_G(d))^\alpha \]

\[ + \sum_{\hat{b} \neq d \in N_H(b)} (d_G(a) + d_H(b) - 2 + d_H(d))^\alpha - \sum_{d \in N_G(a)} (d_G(a) + d_G(d))^\alpha \]

\[ - \sum_{\hat{a} \neq d \in N_G(a)} (d_G(a) + d_G(d))^\alpha - \sum_{\hat{b} \neq d \in N_H(b)} (d_G(a) + d_H(b) - 2 + d_G(d))^\alpha \]

\[ \square \]
Example 3.10. Applying Theorem 3.9, the general sum-connectivity index of 
\((C_n \sim K_m)(aa', bb')\) is given by:
\[
\chi_\alpha((C_n \sim K_m)(aa', bb')) = (n - 2)4^\alpha + 2(m + 1)\alpha + 2^{\alpha-1}(m - 1)\alpha(m^2 - m - 2).
\]

4. Formulae of general sum-connectivity index when \(\alpha \in \mathbb{Z}\)

In this section, we derive exact formulae of general sum-connectivity index for some graph operations defined in Section 2 when \(\alpha \in \mathbb{Z}\).

Theorem 4.1. Let \(G\) and \(H\) be two graphs and \(\alpha \in \mathbb{Z}\). Then the general sum-connectivity index of \(G\square H\) is given by the formula:
\[
\chi_\alpha(G\square H) = \sum_{u \in V(G)} \sum_{v \in V(H)} (2d_G(u) + d_H(b) + d_H(d))^\alpha.
\]

Proof. By equation (5) and the definition of general sum-connectivity index, we have
\[
\chi_\alpha(G\square H) = \sum_{u \in V(G)} \sum_{bd \in E(H)} (2d_G(u) + d_H(b) + d_H(d))^\alpha.
\]

Now, for \(u, a, c \in V(G)\) and \(v, b, d \in V(H)\), using binomial expansion, we obtain
\[
(2d_G(u) + (d_H(b) + d_H(d)))^\alpha = \sum_{n=0}^\alpha 2^n \alpha \alpha - n \alpha \alpha - n (d_G(u)^n (d_H(b) + d_H(d))^\alpha - n), \tag{13}
\]
\[
(2d_H(v) + (d_G(a) + d_G(c)))^\alpha = \sum_{n=0}^\alpha 2^n \alpha \alpha - n \alpha \alpha - n (d_H(v)^n (d_G(a) + d_G(c))^\alpha - n). \tag{14}
\]

Using equations (13) and (14) in equation (12), we get
\[
\chi_\alpha(G\square H) = \sum_{u \in V(G)} \sum_{bd \in E(H)} \alpha \alpha - n \alpha \alpha - n d_G(u)^n (d_H(b) + d_H(d))^\alpha - n
\]
\[
+ \sum_{ac \in E(G)} \sum_{v \in V(H)} \alpha \alpha - n \alpha \alpha - n d_H(v)^n (d_G(a) + d_G(c))^\alpha - n
\]
\[
= \sum_{n=0}^\alpha 2^n \alpha \alpha - n \alpha \alpha - n \sum_{u \in V(G)} d_G(u)^n \sum_{bd \in E(H)} (d_H(b) + d_H(d))^\alpha - n
\]
\[
+ \sum_{n=0}^\alpha 2^n \alpha \alpha - n \alpha \alpha - n \sum_{v \in V(H)} d_H(v)^n \sum_{ac \in E(G)} (d_G(a) + d_G(c))^\alpha - n
\]
\[
= \sum_{n=0}^\alpha 2^n \alpha \alpha - n \alpha \alpha - n M_n(G)\chi_{\alpha-n}(H) + \sum_{n=0}^\alpha 2^n \alpha \alpha - n \alpha \alpha - n M_n(H)\chi_{\alpha-n}(G).
\]

□
Similarly, for \(v\), the general sum-connectivity index of cartesian product of \(C_l\) and \(C_m\) is \(\chi_\alpha(C_l \square C_m) = 2^{2\alpha+1}lm\).

In the following theorem, we compute the general sum-connectivity index of \(G \boxtimes H\).

**Theorem 4.3.** Let \(G\) and \(H\) be two graphs such that either \(G\) or \(H\) is regular and \(\alpha \in \mathbb{Z}\). Then the general sum-connectivity index of \(G \boxtimes H\) is given by the formula:

\[
\chi_\alpha(G \boxtimes H) = \frac{1}{2^{n-1}} \sum_{\beta=0}^{\alpha} \binom{\alpha}{\alpha - \beta} \chi_\alpha(G) \sum_{\beta=0}^{n-\beta} \binom{n}{n-\beta} \chi_{\alpha-n+\beta}(H)
\]

\[+\sum_{n=0}^{\alpha} \left(\frac{\alpha}{\alpha - n}\right) (M_a(G) \sum_{\beta=0}^{n-\beta} \binom{n}{n-\beta} \chi_{\alpha+\beta-n}(H) + M_b(H) \sum_{\beta=0}^{n-\beta} \binom{n}{n-\beta} \chi_{\alpha+\beta-n}(G))\]

\[= 2 \sum_{\alpha \in E(G)} \sum_{\beta \in E(H)} (2d_G(u) + (d_H(b) + d_H(d) + d_G(u)(d_H(b) + d_H(d)))^\alpha
\]

**Proof.** By equation (6) and the definition of general sum-connectivity index, we get

\[
\chi_\alpha(G \boxtimes H) = \sum_{u \in V(G)} \sum_{b \in E(H)} (2d_G(u) + (d_H(b) + d_H(d) + d_G(u)(d_H(b) + d_H(d)))^\alpha
\]

\[+ \sum_{a \in E(G)} \sum_{v \in V(H)} (2d_H(v) + (d_G(a) + d_G(c)) + d_H(v)(d_G(a) + d_G(c)))^\alpha
\]

\[+ 2 \sum_{a \in E(G)} \sum_{b \in E(H)} ((d_G(a) + d_G(c)) + (d_H(b) + d_H(d)) + (d_G(a)d_H(b) + d_G(c)d_H(d)))^\alpha
\]

Now, for \(u \in V(G)\) and \(b, d \in V(H)\), using binomial expansion, we obtain

\[
(2d_G(u) + (d_H(b) + d_H(d) + d_G(u)(d_H(b) + d_H(d)))^\alpha
\]

\[= \sum_{n=0}^{\alpha} \left(\frac{\alpha}{\alpha - n}\right) d_G(u)^n \sum_{\beta=0}^{n-\beta} \binom{n}{n-\beta} (d_H(b) + d_H(d))^\alpha-n+\beta.
\]

Similarly, for \(v \in V(H)\) and \(a, c \in V(G)\), we obtain

\[
(2d_H(v) + (d_G(a) + d_G(c)) + d_H(v)(d_G(a) + d_G(c)))^\alpha
\]

\[= \sum_{n=0}^{\alpha} \left(\frac{\alpha}{\alpha - n}\right) d_H(v)^n \sum_{\beta=0}^{n-\beta} \binom{n}{n-\beta} (d_G(a) + d_G(c))^\alpha-n+\beta.
\]

Without loss of generality, assume that \(G\) is a regular graph. Then for \((a,b)(c,d) \in E(G \boxtimes H)\), we obtain

\[
d_G(a)d_H(b) + d_G(c)d_H(d) = \frac{1}{2} (d_G(a)d_H(b) + d_G(a)d_H(d) + d_G(c)d_H(b) + d_G(c)d_H(d))
\]

\[= \frac{1}{2} (d_G(a)d_H(b) + d_G(c)d_H(b) + d_G(a)d_H(d) + d_G(c)d_H(d))
\]

\[= \frac{1}{2} (d_G(a) + d_G(c))(d_H(b) + d_H(d)).
\]

Using binomial expansion, we get

\[
(d_G(a) + d_G(c)) + (d_H(b) + d_H(d)) + (d_G(a)d_H(b) + d_G(c)d_H(d))
\]
Exact formulae of general sum-connectivity index

\[ \alpha \sum_{n=0}^{\alpha} \frac{1}{2^{n-1}} \left( \frac{\alpha}{\alpha - n} \right)^n d_G(a) d_G(c)^n \sum_{\beta=0}^{n} 2^{n-\beta} \left( \frac{n}{n - \beta} \right)^n (d_H(b) + d_H(d))^{\alpha - n + \beta}. \] (18)

Using equations (16)–(18) in equation (3), we get

\[ \chi_\alpha(G \boxtimes H) = \sum_{w \in V(G)} \sum_{v \in V(H)} \left( \frac{\alpha}{\alpha - n} \right) d_G(u)^n \sum_{\beta=0}^{n} 2^{n-\beta} \left( \frac{n}{n - \beta} \right) (d_H(b) + d_H(d))^{\alpha - n + \beta} \]

\[ + \sum_{a \in E(G)} \sum_{c \in E(H)} \frac{1}{2n-1} \left( \frac{\alpha}{\alpha - n} \right) d_G(a) d_G(c)^n \sum_{\beta=0}^{n} 2^{n-\beta} \left( \frac{n}{n - \beta} \right) (d_H(b) + d_H(d))^{\alpha - n + \beta} \]

\[ = \sum_{n=0}^{\alpha} \left( \frac{\alpha}{\alpha - n} \right) \sum_{w \in V(G)} d_G(u)^n \sum_{\beta=0}^{n} 2^{n-\beta} \left( \frac{n}{n - \beta} \right) \sum_{b \in E(H)} (d_H(b) + d_H(d))^{\alpha - n + \beta} \]

\[ + \sum_{n=0}^{\alpha} \left( \frac{\alpha}{\alpha - n} \right) \sum_{v \in V(H)} d_H(v)^n \sum_{\beta=0}^{n} 2^{n-\beta} \left( \frac{n}{n - \beta} \right) \sum_{a \in E(G)} (d_G(a) + d_G(c))^{\alpha - n + \beta} \]

Thus, (15) holds.

\[ \square \]

**Example 4.4.** Using Theorem 4.3, the general sum-connectivity index of strong product of \( C_7 \) and \( P_m \) is given by:

\[ \chi_\alpha(C_7 \boxtimes P_m) = \sum_{n=0}^{\alpha} \left( \frac{\alpha}{\alpha - n} \right) \sum_{\beta=0}^{n} 2^{n-\beta} \left( \frac{n}{n - \beta} \right) \left[ 2 \times 3^{\alpha + \beta} \left( \frac{1}{3^n} + \frac{2^{n+2}}{3^n} \right) \right. \]

\[ + \left. 4^{\alpha + \beta} (m - 2) \left( \frac{1}{4^m} + \frac{1}{2^{n+1}} \right) + 4^\beta \left( \frac{1}{2^{2n-1}} + m - 2 \right) \right]. \]

In the following theorem, we give the general sum-connectivity index of composition of two graphs.

**Theorem 4.5.** Let \( G \) and \( H \) be two graphs such that either \( G \) or \( H \) is regular and \( \alpha \in \mathbb{Z} \). Then the general sum-connectivity index of \( G[H] \) is given by the formula:

\[ \chi_\alpha(G[H]) = \sum_{k=0}^{\alpha} \left( \frac{\alpha}{\alpha - k} \right) 2^k n_H^k M_k(G) \chi_{\alpha-k}(H) \]

\[ + \sum_{k=0}^{\alpha} \left( \frac{\alpha}{\alpha - k} \right) n_H^k \chi_{\alpha-k}(G) \sum_{\beta=0}^{k} \left( \frac{k}{k - \alpha \beta} \right) M_\beta(H) M_{k-\beta}(H). \] (19)

**Proof.** Equation (7) and the definition of general sum-connectivity index give

\[ \chi_\alpha(G[H]) = \sum_{w \in V(G)} \sum_{u \in V(H)} (2n_H d_G(w) + (d_H(u) + d_H(v)))^\alpha \]
\[ + \sum_{a \in E(G)} \sum_{u \in V(H)} \sum_{v \in V(H)} (n_H(d_G(a) + d_G(c)) + (d_H(u) + d_H(v)))^\alpha. \quad (20) \]

Now, for \( w \in V(G) \) and \( u, v \in V(H) \), by using binomial expansion, we get
\[ (2n_Hd_G(w) + (d_H(u) + d_H(v)))^\alpha = \sum_{k=0}^{\alpha} \binom{\alpha}{\alpha - k} 2^k n_H^k d_G(w)^k (d_H(u) + d_H(v))^{\alpha-k}. \quad (21) \]

Similarly, for \( a, c \in V(G) \) and \( u, v \in V(H) \), we obtain
\[ (n_H(d_G(a) + d_G(c)) + (d_H(u) + d_H(v)))^\alpha = \sum_{k=0}^{\alpha} \binom{\alpha}{\alpha - k} n_H^{\alpha-k} (d_G(a) + d_G(c))^{\alpha-k} \sum_{\beta=0}^{k} \binom{k}{k - \beta} d_H(u)^\beta d_H(v)^{k-\beta}. \quad (22) \]

Equation (20) along with equations (21) and (22) gives
\[
\chi_\alpha(G[H]) = \sum_{w \in V(G)} \sum_{u \in V(H)} \sum_{v \in V(H)} \binom{\alpha}{\alpha - k} 2^k n_H^k d_G(w)^k (d_H(u) + d_H(v))^{\alpha-k} + \sum_{a \in E(G)} \sum_{u \in V(H)} \sum_{v \in V(H)} \binom{\alpha}{\alpha - k} n_H^{\alpha-k} (d_G(a) + d_G(c))^{\alpha-k} \sum_{\beta=0}^{k} \binom{k}{k - \beta} d_H(u)^\beta d_H(v)^{k-\beta} + \sum_{a \in E(G)} \sum_{u \in V(H)} \sum_{v \in V(H)} (d_H(u) + d_H(v))^{\alpha-k} + \sum_{u \in V(H)} \sum_{v \in V(H)} (d_H(u) + d_H(v))^{\alpha-k} \sum_{\beta=0}^{k} \binom{k}{k - \beta} d_H(u)^\beta d_H(v)^{k-\beta}.
\]

Thus the required result is given by (19).

**Example 4.6.** Using Theorem 4.5, we obtain the general sum-connectivity index of the fence graph \( P_n[P_2] \) below:
\[
\chi_\alpha(P_n[P_2]) = \sum_{k=0}^{\alpha} \binom{\alpha}{\alpha - k} 2^{\alpha-k} (2 \times 3^{\alpha-k} + (n-2)4^{\alpha-k}) \sum_{\beta=0}^{k} \binom{k}{k - \alpha \beta} \left[ 1 + 2(4^{k-\beta} + 4^\beta) + 4^{k+1} \right].
\]

In the following theorem, we compute the general sum-connectivity index of join of finite number of graphs for \( \alpha \in \mathbb{Z} \).

**Theorem 4.7.** Let \( \alpha \in \mathbb{Z} \) and \( G_1, G_2, \ldots, G_n \) be vertex-disjoint graphs with \( V_i = V(G_i) \) and \( E_i = E(G_i) \), \( 1 \leq i \leq n \), \( G = G_1 + G_2 + \ldots + G_n \) and \( V = V(G) \). Then the general sum-connectivity index of join of graphs is given by formula:
\[
\chi_\alpha(G) = \sum_{i=1}^{n} \sum_{k=0}^{\alpha} 2^k \binom{\alpha}{\alpha - k} (|V| - |V_i|)^k \chi_{\alpha-k}(G_i)
\]

(23)
+ \frac{1}{2} \sum_{i \neq j, i, j=1}^{n} \sum_{k=0}^{\alpha} \left( \frac{\alpha}{\alpha - k} \right) (2|V| - |V_i| - |V_j|)^{\alpha - k} \sum_{\beta=0}^{k} \left( \frac{k}{k - \beta} \right) M_{\beta}(G_i)M_{k-\beta}(G_j).

Proof. By equation (8) and the definition of general sum-connectivity index, we obtain

$$
\chi_\alpha(G) = \sum_{i=1}^{n} \sum_{u \in E_i} ((d_{G_i}(u) + d_{G_i}(v)) + (2|V| - |V_i|)^\alpha
+ \frac{1}{2} \sum_{i \neq j, i, j=1}^{n} \sum_{u \in V_i} \sum_{v \in V_j} ((d_{G_i}(u) + d_{G_j}(v)) + (2|V| - |V_i| - |V_j|)^\alpha (24)
$$

Now, for \( u, v \in V_i, 1 \leq i \leq n \), we use binomial expansion to obtain

$$
((d_{G_i}(u)+d_{G_i}(v)) + (2|V| - |V_i|)^\alpha = \sum_{k=0}^{\alpha} 2^k \left( \frac{\alpha}{\alpha - k} \right) (|V| - |V_i|)^k (d_{G_i}(u) + d_{G_i}(v))^{\alpha - k}. (25)
$$

Similarly, for \( u \in V(G_i) \) and \( v \in V(G_j) \), \( 1 \leq i, j \leq n \), we obtain

$$
((d_{G_i}(u)+d_{G_j}(v)) + (2|V| - |V_i| - |V_j|)^\alpha = \sum_{k=0}^{\alpha} \left( \frac{\alpha}{\alpha - k} \right) (2|V| - |V_i| - |V_j|)^{\alpha - k} \sum_{\beta=0}^{k} \left( \frac{k}{k - \beta} \right) d_{G_i}(u)^\beta d_{G_j}(v)^{k-\beta}. (26)
$$

Using equations (25) and (26) in equation (24), we get

$$
\chi_\alpha(G) = \sum_{i=1}^{n} \sum_{u \in E_i} \sum_{k=0}^{\alpha} 2^k \left( \frac{\alpha}{\alpha - k} \right) (|V| - |V_i|)^k (d_{G_i}(u) + d_{G_i}(v))^{\alpha - k}
+ \frac{1}{2} \sum_{i \neq j, i, j=1}^{n} \sum_{u \in V_i} \sum_{v \in V_j} \sum_{k=0}^{\alpha} 2^k \left( \frac{\alpha}{\alpha - k} \right) (|V| - |V_i| - |V_j|)^{\alpha - k} \sum_{\beta=0}^{k} \left( \frac{k}{k - \beta} \right) d_{G_i}(u)^\beta d_{G_j}(v)^{k-\beta}
$$

$$
= \sum_{i=1}^{n} \sum_{u \in E_i} \sum_{k=0}^{\alpha} 2^k \left( \frac{\alpha}{\alpha - k} \right) (|V| - |V_i|)^k \sum_{u \in V_j} (d_{G_i}(u) + d_{G_j}(v))^{\alpha - k}
+ \frac{1}{2} \sum_{i \neq j, i, j=1}^{n} \sum_{k=0}^{\alpha} \left( \frac{\alpha}{\alpha - k} \right) (2|V| - |V_i| - |V_j|)^{\alpha - k} \sum_{\beta=0}^{k} \left( \frac{k}{k - \beta} \right) \sum_{u \in V_i} d_{G_i}(u)^\beta \sum_{v \in V_j} d_{G_j}(v)^{k-\beta}.
$$

After simplification, we get exactly (23). \( \square \)

Example 4.8. Using Theorem 4.7, the general sum-connectivity index of join of \( P_n \) and \( P_m \) is given below:

$$
\chi_\alpha(P_n + P_m) = \sum_{k=0}^{\alpha} \left( \frac{\alpha}{\alpha - n} \right) \left[ 2 \times 3^\alpha (m^k + n^k) + 4^\alpha (m^k n^{k-1} + m^{k-1} n^k) \right]
+ \frac{1}{2} (n + m)^{\alpha - k} \sum_{\beta=0}^{k} \left( \frac{k}{k - \beta} \right) (4 + (m - 2)2^{2k-2\beta+1} + 2(n - 2)4^\beta + (n - 2)(m - 2)4^k).$$
Theorem 4.9. Let $G$ and $H$ be two graphs such that either $G$ or $H$ is regular and $\alpha \in \mathbb{Z}$. Then the general sum-connectivity index of $G \lor H$ is given by the formula:

$$\chi_\alpha(G \lor H) =$$

$$ \sum_{k=0}^{\alpha} \sum_{a \in V(G)} \sum_{c \in V(G)} \sum_{b \in E(H)} \sum_{d \in E(H)} (n_H(d_G(a) + d_G(c)) + n_G(d_H(b) + d_H(d)) - (d_G(a) + d_G(c) + d_H(b) + d_H(d)))^\alpha$$

$$+ \sum_{a \in V(G)} \sum_{c \in V(G)} \sum_{b \in E(H)} \sum_{d \in E(H)} (n_H(d_G(a) + d_G(c)) + n_G(d_H(b) + d_H(d)) - (d_G(a) + d_G(c) + d_H(b) + d_H(d)))^\alpha$$

$$- 4 \sum_{a \in V(G)} \sum_{c \in V(G)} \sum_{b \in E(H)} \sum_{d \in E(H)} (n_H(d_G(a) + d_G(c)) + n_G(d_H(b) + d_H(d)) - (d_G(a) + d_G(c) + d_H(b) + d_H(d)))^\alpha.$$

Proof. Equation (9) and the definition of general sum-connectivity index imply

$$\chi_\alpha(G \lor H) =$$

$$\sum_{k=0}^{\alpha} \frac{1}{2\alpha} \left( \begin{array}{c} \alpha \\ \alpha - k \end{array} \right) n_{\alpha-k}^G \sum_{n=0}^{k} \left( \begin{array}{c} k \\ k - n \end{array} \right) M_n(G) M_{k-n}(G) \sum_{n=0}^{k} n_{n_H}^{k-n} (-1)^n 2^{k-n} \left( \begin{array}{c} k \\ k - n \end{array} \right) \chi_{\alpha+n-k}(H)$$

$$+ \sum_{k=0}^{\alpha} \frac{1}{2\alpha} \left( \begin{array}{c} \alpha \\ \alpha - k \end{array} \right) n_{\alpha-k}^H \sum_{n=0}^{k} \left( \begin{array}{c} k \\ k - n \end{array} \right) M_n(H) M_{k-n}(H) \sum_{n=0}^{k} n_{n_H}^{k-n} (-1)^n 2^{k-n} \left( \begin{array}{c} k \\ k - n \end{array} \right) \chi_{\alpha+n-k}(G)$$

$$- 4 \sum_{k=0}^{\alpha} \frac{1}{2\alpha} \left( \begin{array}{c} \alpha \\ \alpha - k \end{array} \right) n_{\alpha-k}^G \chi_k(G) \sum_{n=0}^{k} (-1)^{n} 2^{k-n} n_{H}^{k-n} \left( \begin{array}{c} k \\ k - n \end{array} \right) \chi_{\alpha-n+k}(H).$$

Now, for $a, c \in V(G)$ and $b, d \in V(H)$, we obtain the following by using binomial expansion:

$$(n_H(d_G(a) + d_G(c)) + n_G(d_H(b) + d_H(d)) - (d_G(a) + d_G(c) + d_H(b) + d_H(d)))^\alpha$$

$$= \sum_{k=0}^{\alpha} \frac{1}{2\alpha} \left( \begin{array}{c} \alpha \\ \alpha - k \end{array} \right) n_{\alpha-k}^G \sum_{n=0}^{k} \left( \begin{array}{c} k \\ k - n \end{array} \right) d_G(a)^n d_G(c)^{k-n}$$

$$\sum_{n=0}^{k} n_{n_H}^{k-n} (-1)^n 2^{k-n} \left( \begin{array}{c} k \\ k - n \end{array} \right) (d_H(b) + d_H(d))^{\alpha+n-k}. \quad (28)$$

Similarly, for $a, c \in V(G)$ and $b, d \in V(H)$, using binomial expansion, we obtain

$$(n_G(d_H(b) + d_H(d)) + n_H(d_G(a) + d_G(c)) - (d_G(a) + d_G(c) + d_H(b) + d_H(d)))^\alpha$$

$$= \sum_{k=0}^{\alpha} \frac{1}{2\alpha} \left( \begin{array}{c} \alpha \\ \alpha - k \end{array} \right) n_{\alpha-k}^G \sum_{n=0}^{k} \left( \begin{array}{c} k \\ k - n \end{array} \right) d_H(b)^n d_H(d)^{k-n}$$

$$\sum_{n=0}^{k} n_{n_H}^{k-n} (-1)^n 2^{k-n} \left( \begin{array}{c} k \\ k - n \end{array} \right) (d_G(a) + d_G(c))^{\alpha+n-k}. \quad (29)$$
Without loss of generality, assume that $G$ is a regular graph. Then $(a, b)(c, d) \in E(G \vee H)$, we obtain

$$d_G(a)d_H(b) + d_G(c)d_H(d) = \frac{1}{2}(d_G(a)d_H(b) + d_G(a)d_H(b) + d_G(c)d_H(d) + d_G(c)d_H(d))$$

$$= \frac{1}{2}(d_G(a)d_H(b) + d_G(c)d_H(b) + d_G(a)d_H(d) + d_G(c)d_H(d)) = \frac{1}{2}(d_G(a) + d_G(c))(d_H(b) + d_H(d)).$$

Using binomial expansion, we obtain

$$n_H(d_G(a) + d_G(c)) + n_G(d_H(b) + d_H(d)) - (d_G(a)d_H(b) + d_G(c)d_H(d))^\alpha$$

$$= \sum_{k=0}^{\alpha} \left( \alpha \right)_{\alpha-k} n_G^{\alpha-k}(d_G(a) + d_G(c))^k \sum_{n=0}^{k} (-1)^n 2^{k-n} n_H^{k-n} \binom{k}{k-n} (d_H(b) + d_H(d))^\alpha-(k+n).$$

Using equations (28)–(30) in equation (27), we get

$$\chi_\alpha(G \vee H) = \sum_{a \in V(G)} \sum_{c \in V(G)} \sum_{b \in E(H)} \sum_{d \in E(H)} \frac{1}{2^k} \left( \alpha \right)_{\alpha-k} n_G^{\alpha-k}$$

$$\sum_{n=0}^{k} \binom{k}{k-n} n_H^{n-1} \sum_{n=0}^{k} \binom{k}{k-n} (d_H(b) + d_H(d))^\alpha-n-k$$

$$+ \sum_{a \in E(G)} \sum_{b \in V(H)} \sum_{d \in V(H)} \frac{1}{2^k} \left( \alpha \right)_{\alpha-k} n_G^{\alpha-k}$$

$$\sum_{n=0}^{k} \binom{k}{k-n} d_H(b)^n d_H(d)^{k-n} \sum_{n=0}^{k} \binom{k}{k-n} (d_G(a) + d_G(c))^\alpha-n-k$$

$$- 4 \sum_{a \in E(G)} \sum_{b \in E(H)} \frac{1}{2^k} \left( \alpha \right)_{\alpha-k} n_G^{\alpha-k} \chi_\alpha(G) \sum_{n=0}^{k} \binom{k}{k-n} n_H^{k-n} \sum_{a \in V(G)} \sum_{c \in V(G)} \sum_{b \in E(H)} \sum_{d \in E(H)} (d_H(b) + d_H(d))^\alpha-n-k$$

$$\sum_{a \in V(G)} d_G(a)^n \sum_{c \in V(G)} d_G(c)^{k-n}$$

$$\sum_{n=0}^{k} \binom{k}{k-n} \sum_{b \in E(H)} d_H(b)^n \sum_{d \in E(H)} d_H(d)^{k-n}$$

$$\sum_{n=0}^{k} \binom{k}{k-n} (d_G(a) + d_G(c))^\alpha-n-k$$

$$- 4 \sum_{a \in E(G)} \frac{1}{2^k} \left( \alpha \right)_{\alpha-k} n_G^{\alpha-k}$$

$$\sum_{a \in E(G)} \sum_{b \in V(H)} \sum_{d \in V(H)} (d_G(a) + d_G(c))^\alpha-n-k$$

$$\sum_{b \in E(H)} (d_H(b) + d_H(d))^\alpha-n-k.$$

Thus the statement of the theorem is true. \qed
Example 4.10. Using Theorem 4.9, the general sum-connectivity index of $G_{1} \lor G_{m}$ is given below:

$$
\chi_{\alpha}(G_{1} \lor G_{m}) = \sum_{k=0}^{\alpha} \binom{\alpha}{\alpha-k} \left[ \sum_{n=0}^{k} \binom{k}{k-n} \sum_{n=0}^{k} \binom{k}{k-n} \right] m^{n+1}(-1)^{n}2^{n+2\alpha} + \sum_{k=0}^{k} \binom{k}{k-n} m^{\alpha-k+2} \sum_{n=0}^{k} \binom{k}{k-n} (-1)^{n}2^{n+2\alpha} - 4m^{\alpha-k+1} \sum_{n=0}^{k} \binom{k}{k-n} (-1)^{n}2^{n+2\alpha}.
$$

In the following theorem, we give the general sum-connectivity index of symmetric difference of two graphs for $\alpha \in \mathbb{Z}^{+}$. The proof is similar to the proof of Theorem 4.9, hence omitted.

Theorem 4.11. Let $G$ and $H$ be two graphs such that either $G$ or $H$ is regular and $\alpha \in \mathbb{Z}$. Then the general sum-connectivity index of $G \oplus H$ is given by the formula:

$$
\chi_{\alpha}(G \oplus H) = \sum_{k=0}^{\alpha} \binom{\alpha}{\alpha-k} n_{G}^{\alpha-k} \sum_{n=0}^{k} \binom{k}{k-n} M_{n}(G)M_{k-n}(G) \sum_{n=0}^{k} \binom{k}{k-n} (-1)^{n} \binom{k}{k-n} \chi_{\alpha+n-k}(H)
$$

$$
+ \sum_{k=0}^{\alpha} \binom{\alpha}{\alpha-k} n_{H}^{\alpha-k} \sum_{n=0}^{k} \binom{k}{k-n} M_{n}(H)M_{k-n}(H) \sum_{n=0}^{k} \binom{k}{k-n} (-1)^{n} \binom{k}{k-n} \chi_{\alpha+n-k}(G)
$$

$$
- 2 \sum_{k=0}^{\alpha} \binom{\alpha}{\alpha-k} n_{G}^{\alpha-k} \chi_{\alpha}(G) \sum_{n=0}^{k} \binom{k}{k-n} \sum_{n=0}^{k} \binom{k}{k-n} (-1)^{n} n_{H}^{\alpha-k}(n) \binom{k}{k-n} \chi_{\alpha+n-k}(H).
$$

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