AN EXISTENCE RESULT FOR A CLASS OF \( p \)-BIHARMONIC PROBLEM INVOLVING CRITICAL NONLINEARITY

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Abstract. This paper is concerned with the following elliptic equation with Hardy potential and critical Sobolev exponent

\[
\Delta(|\Delta u|^{p-2}\Delta u) - \lambda \frac{|u|^{p-2}u}{|x|^{2p}} = \mu h(x)|u|^{q-2}u + |u|^{p^* - 2}u \quad \text{in} \; \Omega, \\
u \in W_0^{2,p}(\Omega).
\]

By means of the variational approach, we prove that the above problem admits a nontrivial solution.

1. Introduction

In recent years, a large number of papers have dealt with the existence of solutions of nonlinear problems involving Sobolev critical and Hardy exponents. See [4,6,13,16,18] and the references therein.

The importance of \( p \)-biharmonic operator has been recognized by several authors, see, e.g., [7,11]. Furthermore, this type of equation furnishes a model for studying traveling waves in suspension bridges [14]. In [19], the authors considered a \( p \)-biharmonic problem involving the Hardy term, and they proved the existence of infinitely many solutions for their problem. In the same spirit, the authors in [10] were interested in the existence of solutions for this type of singular elliptic problems. When \( p = 2 \), this kind of the problem was studied by several authors, we quote [2,12,17].

We cannot apply the standard variational arguments directly, because of the lack of compactness of the inclusion of \( W^{2,p}(\Omega) \) into \( L^p(\Omega) \), i.e., in general, the Palais-Smale condition is not satisfied.

In this note, we consider the problem

\[
\Delta(|\Delta u|^{p-2}\Delta u) - \lambda \frac{|u|^{p-2}u}{|x|^{2p}} = \mu h(x)|u|^{q-2}u + |u|^{p^* - 2}u \quad \text{in} \; \Omega, \\
u \in W_0^{2,p}(\Omega),
\]  

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p^* = Np/(N - 2p)$ is the critical Sobolev exponent, $1 < p < \frac{N}{2}$, $N \geq 5$ and $h \in L^{p^*/p^*-2}(\Omega)$, $\lambda^* = [N(p - 1)(N - 2p)/p^2]^\frac{1}{p^*} > \lambda \geq 0$, $\mu \geq 0$.

Let $L^s(\Omega)$ be the Lebesque space equipped with the norm $|u|_s = (\int_\Omega |u|^s dx)^{1/s}$, $1 \leq s < \infty$ and let $W^{2,p}_0(\Omega)$ be the usual Sobolev space with respect to the norm $\|u\| = (\int_\Omega |\Delta u|^p dx)^{1/p}$.

Define the constant $S_\lambda = \inf_{u \in W^{2,p}_0(\Omega)} \int_\Omega (|\Delta u|^p - \lambda \frac{|u|^p}{|x|^{2p}}) dx$, with $\lambda \in [0, \lambda^*)$.

By the Hardy-Rellich inequality (see [17]), we denote the norm

$$||u||_1 = \left( \int_\Omega (|\Delta u|^p - \lambda \frac{|u|^p}{|x|^{2p}}) dx \right)^{1/p},$$

which is equivalent to the standard norm $\|u\|_p$, for $0 \leq \lambda < \lambda^*$.

Let $u \in W^{2,p}_0(\Omega)$ be a weak solution of (1) if

$$\int_\Omega |\Delta u|^{p-2} \Delta u \varphi dx - \lambda \int_\Omega \frac{|u|^{p-2} u \varphi}{|x|^{2p}} dx - \mu \int_\Omega h(x)|u|^q \varphi dx - \int_\Omega |u|^{p^*-2}u \varphi dx = 0,$$

for any $\varphi \in W^{2,p}_0(\Omega)$.

Now we state the main results:

**Theorem 1.1.** Assume that $q \in (p, p^*)$ and $h$ is a nonnegative function with $h \in L^{\frac{2p}{p+q}}(\Omega)$ and $\lambda^* > \lambda \geq 0$. Then there exists $\mu^* > 0$ such that the problem (1) has a nontrivial solution when $\mu \geq \mu^*$.

In the sequel, one takes $h \equiv 1$.

**Theorem 1.2.** Assume that $q < p$ and $\lambda^* > \lambda \geq 0$. Then there exists $\mu^* > 0$ such that the problem (1) has a nontrivial solution when $\mu \in (0, \mu^*)$.

**2. Proof of the result**

To show the existence of solution, we shall use the Mountain Pass Theorem [3].

We consider the energy functional associated to the problem (1),

$$\phi(u) = \frac{1}{p} \left( \int_\Omega |\Delta u|^p dx \right) - \frac{\lambda}{p} \int_\Omega \frac{|u|^p}{|x|^{2p}} dx - \frac{\mu}{q} \int_\Omega h(x)|u|^q dx - \frac{1}{p^*} \int_\Omega |u|^{p^*} dx. \quad (2)$$

It is well known that the functional $\phi \in C^1(W^{2,p}_0(\Omega), \mathbb{R})$ and for any $\varphi \in W^{2,p}_0(\Omega)$, there holds

$$\phi'(u) \cdot \varphi = \left( \int_\Omega |\Delta u|^{p-2} \Delta u \varphi dx \right) - \lambda \int_\Omega \frac{|u|^{p-2} u \varphi}{|x|^{2p}} dx - \mu \int_\Omega h(x)|u|^{q-2} u \varphi dx - \int_\Omega |u|^{p^*-2} u \varphi dx. \quad (3)$$
LEMMA 2.1. Under the assumptions of Theorem 1.1 we have the following assertions:

(i) There exist two positive constants \( r \) and \( p \), such that \( \phi(u) \geq r \) for \( \|u\| = p \).

(ii) There is \( c \in W^{2,p}_0(\Omega) \) with \( \phi(c) < 0 \) and \( \|c\| > 0 \).

Proof. (i) From the formula for \( \phi \), there exist positive constants \( C_0, C_1, C_2 \) and \( C_3 \), such that

\[
\phi(u) \geq \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda}{p} \int_{\Omega} \frac{|u|^p}{|x|^{2p}} \, dx - C_0 |h|^{\frac{p^*}{p-1}} \left( \int_{\Omega} |u|^{p^*} \, dx \right) \frac{1}{p^{\frac{1}{p}} - \frac{1}{p^*}} - \frac{1}{p} \int_{\Omega} |u|^{p^*} \, dx
\]

\[
\geq \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda}{p} \int_{\Omega} \frac{|u|^p}{|x|^{2p}} \, dx - C_0 |h|^{\frac{p^*}{p-1}} \left( \int_{\Omega} |u|^{p^*} \, dx \right) \frac{1}{p^{\frac{1}{p}} - \frac{1}{p^*}} - \frac{1}{p} \int_{\Omega} |u|^{p^*} \, dx
\]

\[
\geq C_1 ||u||_1^p - C_2 ||u||_1^p - C_3 ||u||_1^p.
\]

Since \( q \in (p, p^*) \) then for \( \rho > 0 \) sufficiently small, we may find \( r > 0 \) such that

\[
\inf_{||u||=\rho} \phi(u) \geq r > 0.
\]

(ii) Taking \( \omega \in C^{\infty}_0(\Omega) \), then for \( t > 0 \)

\[
\phi(t\omega) \leq \frac{t^p}{p} ||u||^p - \frac{t^{p^*}}{p^*} \int_{\Omega} |\omega|^{p^*} - \mu t^q \int_{\Omega} h(x)|\omega|^q \, dx \to -\infty,
\]

when \( t \to \infty \). \( \square \)

LEMMA 2.2. If \( (u_n)_n \) is a Palais-Smale sequence \( (PS)_c \) of the functional \( \phi \), then \( (u_n)_n \) is bounded and the functional \( \phi \) satisfies \( (PS)_c \) condition provided \( c < \frac{1}{N} S^N \).

Proof. From the hypothesis, \( (u_n)_n \) is bounded in \( W^{2,p}_0(\Omega) \). In fact, from (2) and (3)

\[
\phi(u_n) = c + o(1),
\]

and

\[
\phi'(u_n).u_n = o(1)||u_n||.
\]

Combining (4) with (5) we get

\[
o(1)(1 + ||u_n||) + c \geq \phi(u_n) - \frac{1}{p} \phi'(u_n).u_n \geq (1 - \frac{1}{p})||u_n||^p - C||u||^q.
\]

It shows that \( (u_n)_n \) is bounded in \( W^{2,p}_0(\Omega) \). Therefore, there exists a subsequence, denoted also by \( (u_n)_n \), satisfying

\[
u_n \to u, \text{ in } W^{2,p}_0(\Omega), \quad \frac{|u_n|^{p-2}u_n}{|x|^{2p}} \to \frac{|u|^{p-2}u}{|x|^{2p}}, \text{ in } L^p(\Omega),
\]

\[
|u_n|^{p^* - 2}u_n \to |u|^{p^* - 2}u, \text{ in } L^{p^*}(\Omega), \quad u_n \to u, \text{ a.e. in } \Omega.
\]

Furthermore, \( u_n \to u, \text{ in } L^q(\Omega) \), so by the Lebesgue dominated convergence theorem,

\[
\int_{\Omega} h(x)|u_n|^q \, dx \to \int_{\Omega} h(x)|u|^q \, dx.
\]

A standard argument shows that the weak limit \( u \) of \( (u_n)_n \) is a critical point of \( \phi \) and then \( \phi'(u) = 0 \).

Meanwhile, let \( \omega_n = u_n - u \). Then by Brezis-Lieb lemma in [5] we get

\[
||\omega_n||^p = ||u_n||^p + ||u||^p + o_n(1), \quad \phi'(u).\omega_n = ||u_n||^p - ||u||^p + o_n(1), \quad ||\omega_n||^p \to ||u||^p + o_n(1),
\]

(7)

(8)
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\[ \int_{\Omega} \frac{|u_n|^p}{|x|^{2p}} \, dx = \int_{\Omega} \frac{|u|^p}{|x|^{2p}} \, dx + \int_{\Omega} \frac{|\omega_n|^p}{|x|^{2p}} \, dx + o_n(1). \]

From (7), (8) and (6) we have

\[ \|\omega_n\|^p = |\omega_n|_{W^{2,p}}^p + o_n(1) \quad (9) \]

and

\[ \frac{1}{p} \|\omega_n\|^p - \frac{1}{p^{*}} |\omega_n|_{W^{2,p}}^{p^{*}} = c - \phi(u) + o_n(1). \quad (10) \]

In view of the boundedness of \((u_n)_n\) in \(W^{2,p}_0(\Omega)\) we may assume that there exists \(l \geq 0\) with

\[ \|\omega_n\|^p \to l. \quad (11) \]

It follows from (9) and (11) that

\[ |\omega_n|_{W^{2,p}}^{p^{*}} \to l, \quad (12) \]

and using the definition of \(S_{\lambda}\), we have \(\|\omega_n\|^p \geq S_{\lambda} \left( |\omega_n|_{W^{2,p}}^{p^{*}} \right)^{\frac{1}{p^{*}}};\) so we infer that \(l \geq S_{\lambda} l^{\frac{p}{p^{*}}};\) and thus we claim that \(l = 0\). Indeed, if \(l > 0\) from the previous inequality we have \(l \geq S_{\lambda} \frac{p}{p^{*}}.\) From (10), (11) and (12), we have \(\phi(u) + \frac{c}{l} = c < \frac{1}{l} S_{\lambda} \frac{p}{p^{*}}\), which implies that \(\phi(u) < 0\).

Meanwhile, we know that \(\phi'(u) \cdot \varphi = 0, \ \forall \varphi \in W^{2,p}_0(\Omega),\) hence

\[ \|u\|_{W^{1,p}}^p = \mu \int_{\Omega} h(x)|u|^q \, dx - \int_{\Omega} |u|^p \, dx, \]

so it follows that

\[ \phi(u) = \frac{1}{p} \|u\|_{W^{1,p}}^p - \frac{\mu}{q} \int_{\Omega} h(x)|u|^q \, dx - \int_{\Omega} \frac{1}{p} |u|^p \, dx \geq 0. \]

On the other hand, the assumption \(c < \frac{1}{\mu} S_{\lambda} \frac{p}{p^{*}}\) implies that \(\phi(u) < 0\).

This contradicts \(\phi(u) \geq 0\). Hence \(l = 0\) and it yields \(u_n \to u\) in \(W^{2,p}_0(\Omega).\)

**Proof** (of Theorem 1.1). We will use the Mountain Pass Lemma to prove the existence of a solution for the problem (1). In our case, we have already checked the mountain pass geometry conditions included in Lemma 2.1. It remains to prove that \(c < \frac{p}{p^{*}} S_{\lambda} \frac{p}{p^{*}}\).

We choose \(\omega \in C_0^\infty(\Omega)\) such that \(|\omega|_{p^{*}} = 1, \ \lim_{t \to \infty} \phi(t\omega) = -\infty,\) and thus \(\sup_{t \geq 0} \phi(t\omega) = \phi(t_0\omega),\) for some \(t_0 > 0.\) Further, \(t_0\) satisfies

\[ t_0^{p-1} \int_{\Omega} \left( |\Delta \omega|^p - \lambda \frac{|\omega|^p}{|x|^{2p}} \right) \, dx - t_0^{p-1} \int_{\Omega} |\omega|^{p^{*}} \, dx - \mu t_0^{p-1} \int_{\Omega} h(x)|\omega|^q \, dx = 0, \quad (13) \]

and so

\[ t_0^{p-1} \left( t_0^{p-q} \int_{\Omega} \left( |\Delta \omega|^p - \lambda \frac{|\omega|^p}{|x|^{2p}} \right) \, dx - t_0^{p-q} \int_{\Omega} h(x)|\omega|^q \, dx \right) = 0. \]

Since \(-t_0^{p-q} \mu \int_{\Omega} h(x)|\omega|^q \, dx \to -\infty\) as \(\mu \to \infty,\) we have \(t_0 \to 0\) as \(\mu \to \infty.\) From the continuity of the functional \(\phi\) we entail that \(\sup_{t \geq 0} \phi(t\omega) \to 0\) as \(\mu \to \infty;\) so we may find \(\mu^{*}\) such that for every \(\mu \geq \mu^{*},\) we have \(\sup_{t \geq 0} \phi(t\omega) < \frac{1}{\lambda} S_{\lambda} \frac{p}{p^{*}}.\)

Putting \(v = t\omega,\) we have, for \(t\) large enough, that \(\phi(v) < 0.\) By the definition of the minimax value in the Mountain Pass Lemma, if we take \(\alpha(t) = tv,\) then

\[ c \leq \sup_{t \geq 0} \phi(tv) < \frac{1}{\lambda} S_{\lambda} \frac{p}{p^{*}}. \]
Remark 2.3. (i) In view of the Ekeland variational principle [8], we can prove that there exists a \((P.S)_c\) sequence \((u_n)_n \subset \overline{B}_p(0)\) with \(c = \inf_{\overline{B}_p(0)} \phi < 0\). Hence we obtain a second solution of the problem (1).

(ii) Under the same conditions of Theorem 1.1, it is possible to prove the analogous result for the problem

\[
\Delta(|\Delta u|^{p-2} \Delta u) - \lambda \frac{|u|^{p-2} u}{|x|^{2p}} = \mu h(x)|u|^{q-2} u + |u|^{p^*-2} u + g \quad \text{in } \Omega,
\]

\(u = \nabla u = 0\quad \text{on } \partial \Omega,
\]

\(\lambda, \mu > 0\) and \(g\) is small enough in the norm of \((W_0^{2,p}(\Omega))^*\). With this in mind, the proof is an adaptation of the above argument.

Lemma 2.4. There exist \(\mu^* > 0, \rho > 0\) and \(r > 0\) such that for all \(\mu \in (0, \mu^*)\) we have \(\phi(u) \geq r > 0\) for \(\|u\| = r\).

Proof. From the Hölder’s inequality and the compact embedding theorem, we have

\[
\phi(u) \geq \frac{1}{p} \int_\Omega |\Delta u|^p \, dx - \lambda \int_\Omega \frac{|u|^p}{|x|^{2p}} \, dx - \frac{\mu}{q} \int_\Omega |u|^q \, dx - \frac{1}{p^*} \int_\Omega |u|^{p^*} \, dx
\]

\[
\geq C_1 \|u\|^p - \frac{C_2 \mu}{q} \|u\|^q - \frac{1}{p^* S^*_X} \|u\|^{p^*}
\]

\[
\geq C_3 \|u\|^p - \frac{C_4 \mu}{q} \|u\|^q - \frac{1}{p^* S^*_X} \|u\|^{p^*},
\]

(14)

with \(C_1, C_2, C_3 > 0\). Since \(q < p\) then for \(\|u\| = \rho > 0\), we may find \(r > 0\) where \(\inf_{\|u\| = \rho} \phi(u) \geq r > 0\). \(\square\)

Lemma 2.5. The weak limit \(u_\ast\) of \((u_n)_n\) is a nontrivial solution to (1) for \(\mu \in (0, \mu^*)\).

Proof. It is clear that the functional \(\phi\) is bounded from below in \(\overline{B}_p(0) = \{u \in W_0^{1,p}(\Omega) : \|u\| \leq \rho\}\), with \(\rho > 0\) given by Lemma 2.1. Hence, using the Ekeland’s variational principle [4] with distance \(d(u, v) = \|u - v\|\), a standard argument (see for instance [13]) shows the existence of a \((P.S)_c\) sequence \((u_n)_n \subset \overline{B}_p(0)\) satisfying \(\tilde{c} = \inf_{\overline{B}_p(0)} \phi\). Moreover, \(\tilde{c} = \inf_{\overline{B}_p(0)} \phi < 0\) and

\[
\tilde{c} + o(1) = \phi(u_n) \geq C_1 \|u_n\|^p - C_2 \|u_n\|^p_1 - C_3 \|u_n\|^q.
\]

Therefore, \(C_2 \|u\|^p_1 + C_3 \|u\|^q > -\tilde{c} > 0\) and \(u_\ast \neq 0\).

On the other hand,

\[
\|u_n\|^p \int_\Omega (|\Delta u_n|^{p-2} \Delta u_n - |\Delta u|^{p-2} \Delta u) (\Delta u_n - \Delta u) \, dx =
\]

\[
\phi'(u_n) \cdot (u_n - u) + \mu \int_\Omega |u_n|^{p^*} u_n (u_n - u) \, dx =
\]

\[
\int_\Omega |u_n|^{p^*} u_n (u_n - u) \, dx - \|u_n\|^p \int_\Omega |\Delta u|^{p-2} \Delta u (\Delta u_n - \Delta u) \, dx.
\]
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In view of $u_n \rightharpoonup u$, arguing as in Leray-Lions [15] and in [9], it yields $\Delta u_n(x) \rightarrow \Delta u(x)$ a.e $x \in \Omega$, and $u_n(x) \rightarrow u(x)$ a.e. in $\Omega$. Then

$$
\|u_n\|^p \int_\Omega (|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u) (\Delta u_n - \Delta u) \, dx \rightarrow 0.
$$

Using the following inequalities

$$
|x - y|^\gamma \leq 2^\gamma(|x|^{\gamma - 2}x - |y|^{\gamma - 2}y).(x - y) \quad \text{if } \gamma \geq 2,
$$

$$
|x - y|^2 \leq \frac{1}{\gamma - 1}((x + |y|)^{2-\gamma}(x|^{\gamma - 2}x - |y|^{\gamma - 2}y).(x - y) \quad \text{if } 1 < \gamma < 2,
$$

$\forall x, y \in \mathbb{R}^N$, where $x.y$ is the inner product in $\mathbb{R}^N$, we get

$$
\int_\Omega (|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u) (\Delta u_n - \Delta u) \, dx \rightarrow 0.
$$

Consequently, $\|u_n - u\| \rightarrow 0$, which implies that $u_n \rightarrow u$ in $W_0^{2,p}(\Omega)$. □

**Proof (of Theorem 1.2).** Theorem 1.2 is a direct corollary of Lemma 2.4 and 2.5. □

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**References**


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