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QUANTITATIVE ESTIMATES FOR MODIFIED BETA OPERATORS

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Abstract. In this paper, we introduce a sequence of positive linear Beta type operators based on a function τ having certain properties. We firstly give some approximation properties of these operators. Next, we establish Voronovskaja type and Grüss-Voronovskaja type theorems in quantitative form with the help of the first order Ditzian-Totik modulus of smoothness.

1. Introduction

It is well-known that the classical Bernstein polynomials for $f \in C[0,1]$,

$$B_n f(x) := B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right); \quad x \in [0,1], \quad n \in \mathbb{N},$$

reproduce the functions $e_i(x) = x^i$ $(i = 0, 1)$, namely, $B_n(e_i;x) = e_i(x)$. For these polynomials $B_n(e_2;x) \neq e_2(x) = x^2$.

In [15], as a generalization of the classical Bernstein polynomials, King introduced a sequence of positive linear operators given by

$$V_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} (r_n(x))^k (1 - r_n(x))^{n-k} f\left(\frac{k}{n}\right); \quad x \in [0,1], \quad n \in \mathbb{N},$$

where $0 \leq r_n(x) \leq 1$ are continuous functions and $f \in C[0,1]$. For the function $r_n^*(x)$ which is a special case of $r_n(x)$, the author proved that $V_n(e_0;x) = e_0(x)$, $V_n(e_2;x) = e_2(x)$ and $\lim_{n\to\infty} V_n(f;x) = f(x)$ for each $f \in C[0,1]$, $x \in [0,1]$. Moreover, King obtained some estimates in terms of the usual modulus of continuity and also showed that the order of approximation of $V_n(f;x)$ to $f(x)$ is at least as good as for the classical Bernstein polynomials whenever $x \in [0, \frac{1}{3})$. Thereafter, Gonska and Piţul [11] discussed different properties and iterates of $V_n(f;x)$ with $r_n^*(x)$. Later,

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in [12] Gonska et al., via a function τ such that $\tau \in C[0,1]$ is strictly increasing and $\tau(0)=0$, $\tau(1)=1$, constructed a sequence of King-type operators as follows:

$$V_n^\tau f := (B_n f)\circ \tau_n = (B_n f)\circ (B_n \tau)^{-1}\circ \tau, \quad f \in C[0,1].$$

The authors proved convergence and Voronovskaja type theorems and studied global smoothness and shape preservation properties of such operators. Recently, Cárdenas-Morales et al. [6] introduced the following sequence of Bernstein type operators for $f \in C[0,1]$ and any functions τ being ∞-times continuously differentiable on $[0,1]$ such that $\tau(0)=0$, $\tau(1)=1$ and $\tau'(x)>0$ on $[0,1]$,

$$B_n^\tau (f;x) = \sum_{k=0}^{n} \binom{n}{k} (\tau(x))^k (1-\tau(x))^{n-k} \left(f\circ \tau^{-1}\right)\left(\frac{k}{n}\right).$$

They investigated their shape preserving and convergence properties and also their asymptotic behavior and saturation. Furthermore, for a particular case of τ by comparing B_n, V_n^τ and B_n^τ with each other, it was shown that in certain classes of functions B_n^τ has a better estimate than B_n. We remark that Cárdenas-Morales et al. [7] established a Voronovskaja type asymptotic formula for the operators B_n^τ. Motivated by the above ideas several authors studied modified approximation operators of these types, which can lead to a better error estimate than the original ones depending on the choice of τ (for instance see [2,4]).

In 1932, Voronovskaja [19] obtained the following asymptotic result on the difference $f(x) - B_n(f;x)$.

Theorem 1.1 ([8, p. 307]). *If f is bounded on $[0,1]$, differentiable in some neighborhood of x, and has second derivative $f''(x)$ for some $x \in [0,1]$, then*

$$\lim_{n\to\infty} n\left[B_n(f;x) - f(x)\right] = \frac{x(1-x)}{2} f''(x).$$

If $f \in C^2[0,1]$, the convergence is uniform.

In [14], Grüss proved the following.

Theorem 1.2 ([17]). *Suppose $f, g : [a,b] \to \mathbb{R}$ are integrable, $m_f \le f(x) \le M_f$ and $m_g \le g(x) \le M_g$ for all $x \in [a,b]$. Then*

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \frac{1}{b-a}\int_a^b g(x)\,dx \right| \le \frac{1}{4}(M_f - m_f)(M_g - m_g).$$

This theorem gives an estimate for the difference between the integral of the product of two functions and the product of integrals of these functions.

Applications of Grüss inequality in approximation theory were initially introduced by Acu et al. in [3]. In this paper, the authors gave Grüss-type inequalities for positive linear operators with the help of the least concave majorant of modulus of continuity. Later, in [13], Gonska and Tachev, using the second order moduli of smoothness, presented Grüss-type inequalities for positive linear operators and obtained some impressive results for Bernstein polynomials. In 2015, Gal and Gonska [10] introduced Voronovskaja type estimates via Grüss-type inequality for Bernstein operators and called them Grüss-Voronovskaja type estimates. Inspired by this paper, several authors studied Grüss-Voronovskaja type theorems for some sequences of positive linear operators.

Let, as in the definition of B_n^τ, τ be any ∞-times continuously differentiable function such that $\tau(0)=0$, $\tau(1)=1$ and $\tau'(x)>0$ on $[0,1]$. In this paper, we consider the sequence of positive linear operators β_n^τ defined by

$$\beta_n^\tau(f;x)=\begin{cases} f(0), & x=0 \\ P_n^\tau(x)\displaystyle\int_0^1 t^{n\tau(x)-1}(1-t)^{n(1-\tau(x))-1}\left(f\circ\tau^{-1}\right)(t)\,dt, & x\in(0,1) \\ f(1), & x=1 \end{cases} \tag{1}$$

where $P_n^\tau(x):=\dfrac{1}{B\big(n\tau(x),n\big(1-\tau(x)\big)\big)}$ with the well-known beta function $B(.,.)$, $x\in[0,1]$, $n\in\mathbb{N}$ and $f\in C[0,1]$. We remark that if we set $\tau(x)=x$ the operators given by (1) reduce to the Beta operators (see, e.g. [5]),

$$\beta_n(f;x)=\begin{cases} f(0), & x=0 \\ \dfrac{1}{B(nx,n(1-x))}\displaystyle\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1}f(t)\,dt, & x\in(0,1) \\ f(1), & x=1 \end{cases}$$

which are a slight modification of the Beta operators presented by Lupaş [18]. It is obvious that $\beta_n^\tau f=\left(\beta_n\left(f\circ\tau^{-1}\right)\right)\circ\tau$.

In the present article, we firstly introduce some approximation properties of β_n^τ. Further, we prove Voronovskaja type and Grüss Voronovskaja type theorems in quantitative form with the help of the first order Ditzian-Totik modulus of smoothness.

2. Auxiliary results

In order to derive our main results we first give some necessary lemmas. The first one can be easily proved by using the beta function $B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\,dt$, $u,v>0$ and the definition of the operators β_n^τ.

Lemma 2.1. *For the operators β_n^τ we have*

$$\beta_n^\tau(1;x)=1,\qquad \beta_n^\tau(\tau;x)=\tau(x),$$

$$\beta_n^\tau\left(\tau^2;x\right)=\frac{n\tau^2(x)+\tau(x)}{n+1},$$

$$\beta_n^\tau\left(\tau^3;x\right)=\frac{n^2\tau^3(x)+3n\tau^2(x)+2\tau(x)}{(n+1)(n+2)}$$

and

$$\beta_n^\tau\left(\tau^4;x\right)=\frac{n^3\tau^4(x)+6n^2\tau^3(x)+11n\tau^2(x)+6\tau(x)}{(n+1)(n+2)(n+3)}.$$

LEMMA 2.2. *For the operators* β_n^τ *we have*

$$\mu_{n,0}^\tau(x) = 1, \qquad \mu_{n,1}^\tau(x) = 0,$$

$$\mu_{n,2}^\tau(x) = \frac{\varphi_\tau^2(x)}{n+1}$$

and

$$\mu_{n,4}^\tau(x) = \frac{\left[3(n-6)\varphi_\tau^2(x)+6\right]\varphi_\tau^2(x)}{(n+1)(n+2)(n+3)},$$

where $\varphi_\tau^2(x) = \tau(x)\left(1-\tau(x)\right)$ *and* $\mu_{n,m}^\tau(x) = \beta_n^\tau\big(\left(\tau(t)-\tau(x)\right)^m; x\big)$, $m = 0,1,2,4$.

Proof. In terms of Lemma 2.1 and the equality

$$\mu_{n,m}^\tau(x) = \frac{1}{B\big(n\tau(x), n\left(1-\tau(x)\right)\big)} \int_0^1 t^{n\tau(x)-1}\left(1-\tau(x)\right)^{n(1-\tau(x))-1}\left(t-\tau(x)\right)^m dt$$

we can prove easily Lemma 2.2. $\square$

THEOREM 2.3. *Let* $f \in C[0,1]$. *Then* $\beta_n^\tau f$ *converges uniformly to* f *on* $[0,1]$.

Proof. When we use the definition of β_n^τ, Lemma 2.1 and the well-known Korovkin theorem (see [16]), by taking into account the fact that $\left\{e_0, \tau, \tau^2\right\}$ is an extended complete Tchebychev system on $[0,1]$ (see [6]), we can say that $\beta_n^\tau f$ converges uniformly to f on $[0,1]$. $\square$

Further on, throughout the paper, we suppose that $\inf_{x\in[0,1]} \tau'(x) \geq a$, $a \in \mathbb{R}^+$.

THEOREM 2.4. *For any* $f \in C^2[0,1]$ *and* $x \in [0,1]$ *one has*

$$\left|\beta_n^\tau(f;x) - f(x)\right| \leq \frac{\varphi_\tau^2(x)}{2(n+1)}\left(\frac{\|f''\|}{a^2} + \frac{\|f'\|\|\tau''\|}{a^3}\right),$$

where $\|\cdot\|$ *denotes the usual* sup *norm on* $C[0,1]$.

Proof. Applying β_n^τ to the both sides of the Taylor expansion of $f\circ\tau^{-1}$,

$$\begin{aligned} f(t) = \left(f\circ\tau^{-1}\right)(\tau(t)) &= \left(f\circ\tau^{-1}\right)(\tau(x)) + \left(f\circ\tau^{-1}\right)'(\tau(x))\left(\tau(t)-\tau(x)\right) \\ &\quad + \int_{\tau(x)}^{\tau(t)} \left(\tau(t)-u\right)\left(f\circ\tau^{-1}\right)''(u)\,du \end{aligned} \tag{2}$$

and using Lemma 2.2, we can write

$$\left|\beta_n^\tau(f;x) - f(x)\right| \leq \beta_n^\tau\left(\left|\int_{\tau(x)}^{\tau(t)} \left(\tau(t)-u\right)\left(f\circ\tau^{-1}\right)''(u)\,du\right|; x\right). \tag{3}$$

If we take into consideration the following inequality given in [2],

$$\int_{\tau(x)}^{\tau(t)} \left(\tau(t)-u\right)\left(f\circ\tau^{-1}\right)''(u)\,du =$$

$$\int_{\tau(x)}^{\tau(t)} \left(\tau(t)-u\right)\frac{f''(\tau^{-1}(u))}{\left[\tau'(\tau^{-1}(u))\right]^2}\,du - \int_{\tau(x)}^{\tau(t)} \left(\tau(t)-u\right)\frac{f'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{\left[\tau'(\tau^{-1}(u))\right]^3}\,du,$$

then from (3) we reach the desired result. Thus the proof is completed. $\square$

REMARK 2.5. If we take $\tau(x) = x$ in Theorem 2.4, then we have

$$\left|\beta_n(f;x) - f(x)\right| \le \frac{x(1-x)}{2(n+1)}\|f''\|.$$

3. Voronovskaja and Grüss-Voronovskaja type theorems

In this part, we present Voronovskaja and Grüss-Voronovskaja type quantitative estimates in terms of the first order Ditzian-Totik modulus of smoothness. For this aim, we introduce some needful definitions and notations given in [4] and [9]. The first order Ditzian-Totik modulus of smoothness $\omega_{\varphi_\tau}(f;t)$ and the corresponding K-functional $K_{\varphi_\tau}(f;t)$ for $f \in C[0,1]$ are defined by

$$\omega_{\varphi_\tau}(f;t) = \sup_{0<h\le t}\left\{\left|f\left(x+\frac{h\varphi_\tau(x)}{2}\right) - f\left(x-\frac{h\varphi_\tau(x)}{2}\right)\right|, x \mp \frac{h\varphi_\tau(x)}{2} \in [0,1]\right\}$$

and $\quad K_{\varphi_\tau}(f;t) = \inf\limits_{g\in W_{\varphi_\tau}[0,1]}\left\{\|f-g\| + t\|\varphi_\tau g'\|\right\}(t>0),$

where $\varphi_\tau(x) := \sqrt{\tau(x)(1-\tau(x))}$ and $W_{\varphi_\tau}[0,1] = \left\{g : g \in AC_{loc}[0,1], \|\varphi_\tau g'\| < \infty\right\}$. $g \in AC_{loc}[0,1]$ means that g is absolutely continuous on every closed finite interval $[c,d] \subset (0,1)$. It is well-known that for a constant $M > 0$ the following inequality holds

$$K_{\varphi_\tau}(f;t) \le M\omega_{\varphi_\tau}(f;t). \tag{4}$$

We now give the following quantitative Voronovskaja type result.

THEOREM 3.1. *For any $f \in C^2[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$, one has*

$$n\left|\beta_n^\tau(f;x) - f(x) - \frac{\varphi_\tau^2(x)}{2(n+1)}\left(f\circ\tau^{-1}\right)''(\tau(x))\right| \le M\varphi_\tau(x)\omega_{\varphi_\tau}\left(\left(f\circ\tau^{-1}\right)'';t_n^\tau(x)\right),$$

where $M > 0$ is a constant and $t_n^\tau(x) := \frac{2}{a}\sqrt{\frac{3(n-6)\varphi_\tau^2(x)+6}{(n+2)(n+3)}}$.

Proof. Since

$$\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(f\circ\tau^{-1}\right)''(\tau(x))\,du = \frac{1}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))(\tau(t)-\tau(x))^2$$

by the Taylor's expansion (2), we can write

$$f(t) - f(x) - \left(f\circ\tau^{-1}\right)'(\tau(x))(\tau(t)-\tau(x)) - \frac{1}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))(\tau(t)-\tau(x))^2$$

$$= \int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(f\circ\tau^{-1}\right)''(u)\,du - \int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(f\circ\tau^{-1}\right)''(\tau(x))\,du$$

$$= \int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left[\left(f\circ\tau^{-1}\right)''(u) - \left(f\circ\tau^{-1}\right)''(\tau(x))\right]du$$

and so

$$\begin{aligned}&\left|\beta_n^\tau(f;x)-f(x)-\frac{\mu_{n,2}^\tau(x)}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))\right|\\ \le&\beta_n^\tau\left(\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right|du\right|;x\right).\end{aligned}$$

Using the following inequality computed in [4] for $g\in W_{\varphi_\tau}[0,1]$,

$$\begin{aligned}&\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right|du\right|\\ \le&\left\|\left(f\circ\tau^{-1}\right)''-g\right\|(\tau(t)-\tau(x))^2+\frac{2\|\varphi_\tau g'\|}{a\varphi_\tau(x)}|\tau(t)-\tau(x)|^3\end{aligned}$$

we can write

$$\begin{aligned}&\left|\beta_n^\tau(f;x)-f(x)-\frac{\mu_{n,2}^\tau(x)}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))\right|\\ \le&\left\|\left(f\circ\tau^{-1}\right)''-g\right\|\mu_{n,2}^\tau(x)+\frac{2\|\varphi_\tau g'\|}{a\varphi_\tau(x)}\beta_n^\tau\left(|\tau(t)-\tau(x)|^3;x\right).\end{aligned}$$

If we apply the Cauchy-Schwarz inequality and use Lemma 2.2, then we get

$$\begin{aligned}&\left|\beta_n^\tau(f;x)-f(x)-\frac{\mu_{n,2}^\tau(x)}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))\right|\\ \le&\left\|\left(f\circ\tau^{-1}\right)''-g\right\|\mu_{n,2}^\tau(x)+\frac{2\|\varphi_\tau g'\|}{a\varphi_\tau(x)}\sqrt{\mu_{n,2}^\tau(x)}\sqrt{\mu_{n,4}^\tau(x)}\\ =&\frac{1}{n+1}\left\{\left\|\left(f\circ\tau^{-1}\right)''-g\right\|\varphi_\tau^2(x)+\frac{2\|\varphi_\tau g'\|}{a}\varphi_\tau(x)\sqrt{\frac{3(n-6)\varphi_\tau^2(x)+6}{(n+2)(n+3)}}\right\}.\end{aligned}$$

Since $\varphi_\tau^2(x)\le\varphi_\tau(x)$ for $x\in[0,1]$ it follows that

$$\begin{aligned}&\left|\beta_n^\tau(f;x)-f(x)-\frac{\varphi_\tau^2(x)}{2(n+1)}\left(f\circ\tau^{-1}\right)''(\tau(x))\right|\\ \le&\frac{\varphi_\tau(x)}{n+1}\left\{\left\|\left(f\circ\tau^{-1}\right)''-g\right\|+\frac{2\|\varphi_\tau g'\|}{a}\sqrt{\frac{3(n-6)\varphi_\tau^2(x)+6}{(n+2)(n+3)}}\right\}.\end{aligned}$$

and thus

$$\begin{aligned}&n\left|\beta_n^\tau(f;x)-f(x)-\frac{\varphi_\tau^2(x)}{2(n+1)}\left(f\circ\tau^{-1}\right)''(\tau(x))\right|\\ \le&\frac{n\varphi_\tau(x)}{n+1}\left\{\left\|\left(f\circ\tau^{-1}\right)''-g\right\|+\frac{2\|\varphi_\tau g'\|}{a}\sqrt{\frac{3(n-6)\varphi_\tau^2(x)+6}{(n+2)(n+3)}}\right\}.\end{aligned}$$

Taking the infimum on the right-hand side of the last inequality over all $g\in W_{\varphi_\tau}[0,1]$, we obtain

$$n\left|\beta_n^\tau(f;x)-f(x)-\frac{\varphi_\tau^2(x)}{2(n+1)}\left(f\circ\tau^{-1}\right)''(\tau(x))\right|\le\varphi_\tau(x)K_{\varphi_\tau}\left(\left(f\circ\tau^{-1}\right)'';t_n^\tau(x)\right)$$

Hence, using the relation (4) the proof is completed. □

Remark 3.2. From Theorem 3.1, it follows that

$$\lim_{n\to\infty} n\big[\beta_n^\tau(f;x)-f(x)\big]=\frac{\varphi_\tau^2(x)}{2}\left(f\circ\tau^{-1}\right)''(\tau(x)),$$

and if we take $\tau(x)=x$, then we have

$$\lim_{n\to\infty} n\big[\beta_n(f;x)-f(x)\big]=\frac{x(1-x)}{2}f''(x)$$

which was obtained in [1].

Finally, we introduce the following quantitative Grüss-Voronovskaja type theorem.

Theorem 3.3. *For any $f,g\in C^2[0,1]$, $x\in[0,1]$ and $n\in\mathbb{N}$, one has*

$$\begin{aligned}
&n\left|\beta_n^\tau(fg;x)-\beta_n^\tau(f;x)\,\beta_n^\tau(g;x)-\frac{\varphi_\tau^2(x)}{n+1}\frac{f'(x)g'(x)}{[\tau'(x)]^2}\right|\\
\le{}& C\varphi_\tau(x)\Bigg\{\omega_{\varphi_\tau}\left(\left(fg\circ\tau^{-1}\right)'';t_n^\tau(x)\right)+\|g\|\,\omega_{\varphi_\tau}\left(\left(f\circ\tau^{-1}\right)'';t_n^\tau(x)\right)\\
&+\|f\|\,\omega_{\varphi_\tau}\left(\left(g\circ\tau^{-1}\right)'';t_n^\tau(x)\right)\\
&+\frac{n}{(n+1)^2}\left(\frac{\|f''\|}{a^2}+\frac{\|f'\|\|\tau''\|}{a^3}\right)\left(\frac{\|g''\|}{a^2}+\frac{\|g'\|\|\tau''\|}{a^3}\right)\Bigg\},
\end{aligned}$$

where $C>0$ is a constant.

Proof. As an analogue of the decomposition formula given in [10], we can write

$$\begin{aligned}
&\beta_n^\tau(fg;x)-\beta_n^\tau(f;x)\,\beta_n^\tau(g;x)-\mu_{n,2}^\tau(x)\frac{f'(x)g'(x)}{[\tau'(x)]^2}\\
={}&\left[\beta_n^\tau(fg;x)-(fg)(x)-\frac{\mu_{n,2}^\tau(x)}{2}\left(fg\circ\tau^{-1}\right)''(\tau(x))\right]\\
&-g(x)\left[\beta_n^\tau(f;x)-f(x)-\frac{\mu_{n,2}^\tau(x)}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))\right]\\
&-f(x)\left[\beta_n^\tau(g;x)-g(x)-\frac{\mu_{n,2}^\tau(x)}{2}\left(g\circ\tau^{-1}\right)''(\tau(x))\right]\\
&+\left[g(x)-\beta_n^\tau(g;x)\right]\left[\beta_n^\tau(f;x)-f(x)\right].
\end{aligned}$$

Using Lemma 2.2, this equality leads to

$$\begin{aligned}
&\left|\beta_n^\tau(fg;x)-\beta_n^\tau(f;x)\,\beta_n^\tau(g;x)-\frac{\varphi_\tau^2(x)}{n+1}\frac{f'(x)g'(x)}{[\tau'(x)]^2}\right|\\
\le{}&\left|\beta_n^\tau(fg;x)-(fg)(x)-\frac{\varphi_\tau^2(x)}{2(n+1)}\left(fg\circ\tau^{-1}\right)''(\tau(x))\right|
\end{aligned}$$

$$+\left|g(x)\right|\left|\beta_n^\tau\left(f;x\right)-f(x)-\frac{\varphi_\tau^2(x)}{2(n+1)}\left(f\circ\tau^{-1}\right)''\left(\tau(x)\right)\right|$$
$$+\left|f(x)\right|\left[\beta_n^\tau\left(g;x\right)-g(x)-\frac{\varphi_\tau^2(x)}{2(n+1)}\left(g\circ\tau^{-1}\right)''\left(\tau(x)\right)\right]$$
$$+\left|g(x)-\beta_n^\tau\left(g;x\right)\right|\left|\beta_n^\tau\left(f;x\right)-f(x)\right|.$$

Hence, from Theorems 2.4 and 3.1 it follows that

$$n\left|\beta_n^\tau\left(fg;x\right)-\beta_n^\tau\left(f;x\right)\beta_n^\tau\left(g;x\right)-\frac{\varphi_\tau^2(x)}{n+1}\frac{f'(x)g'(x)}{\left[\tau'(x)\right]^2}\right|$$
$$\le M\varphi_\tau(x)\Bigg\{\omega_{\varphi_\tau}\left(\left(fg\circ\tau^{-1}\right)'';t_n^\tau(x)\right)+\|g\|\,\omega_{\varphi_\tau}\left(\left(f\circ\tau^{-1}\right)'';t_n^\tau(x)\right)$$
$$+\|f\|\,\omega_{\varphi_\tau}\left(\left(g\circ\tau^{-1}\right)'';t_n^\tau(x)\right)\Bigg\}$$
$$+\frac{n\varphi_\tau^4(x)}{4(n+1)^2}\left(\frac{\|f''\|}{a^2}+\frac{\|f'\|\|\tau''\|}{a^3}\right)\left(\frac{\|g''\|}{a^2}+\frac{\|g'\|\|\tau''\|}{a^3}\right),$$

If we use the inequality $\varphi_\tau^4(x)\le\varphi_\tau(x)$ for $x\in[0,1]$ and take $C:=\max\{M,\frac{1}{4}\}$, then we obtain

$$n\left|\beta_n^\tau\left(fg;x\right)-\beta_n^\tau\left(f;x\right)\beta_n^\tau\left(g;x\right)-\frac{\varphi_\tau^2(x)}{n+1}\frac{f'(x)g'(x)}{\left[\tau'(x)\right]^2}\right|$$
$$\le C\varphi_\tau(x)\Bigg\{\omega_{\varphi_\tau}\left(\left(fg\circ\tau^{-1}\right)'';t_n^\tau(x)\right)+\|g\|\,\omega_{\varphi_\tau}\left(\left(f\circ\tau^{-1}\right)'';t_n^\tau(x)\right)$$
$$+\|f\|\,\omega_{\varphi_\tau}\left(\left(g\circ\tau^{-1}\right)'';t_n^\tau(x)\right)$$
$$+\frac{n}{(n+1)^2}\left(\frac{\|f''\|}{a^2}+\frac{\|f'\|\|\tau''\|}{a^3}\right)\left(\frac{\|g''\|}{a^2}+\frac{\|g'\|\|\tau''\|}{a^3}\right)\Bigg\},$$

which is the desired result. □

Remark 3.4. From Theorem 3.3, it follows that

$$\lim_{n\to\infty}n\left[\beta_n^\tau\left(fg;x\right)-\beta_n^\tau\left(f;x\right)\beta_n^\tau\left(g;x\right)\right]=\varphi_\tau^2(x)\frac{f'(x)g'(x)}{\left[\tau'(x)\right]^2},$$

and if we take $\tau(x)=x$, then we have

$$\lim_{n\to\infty}n\left[\beta_n\left(fg;x\right)-\beta_n^\tau\left(f;x\right)\beta_n^\tau\left(g;x\right)\right]=x(1-x)f'(x)g'(x).$$

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