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HEMI-SLANT $\xi^{\perp}$-LORENTZIAN SUBMERSIONS FROM $(LCS)_n$-MANIFOLDS

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Abstract. The present paper introduce a study of hemi-slant $\xi^{\perp}$-Lorentzian submersion from $(LCS)_n$-manifolds with an example. We obtain some results and investigate the geometry of foliations. Necessary and sufficient conditions for such submersion to be totally geodesic have been obtained. Finally, we study such submersions with totally umbilical fibers.

1. Introduction

Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) were introduced in [23]. They are a generalization of LP-Sasakian manifold [20]. This manifold has many applications, see [7, 25]. In [19], it has shown that Lorentzian concircular spacetime coincide with generalized Robertson-Walker space-time. So, these manifolds are interesting for geometry as well as for physics. For detailed study of $(LCS)_n$-manifolds we refer to [24] and for study of submanifolds of these manifolds we refer to [5, 8, 14–16].

O'Neill [21, 22] and Gray [11] introduced the study of semi-Riemannain submersions between semi-Riemannain manifolds and the study of Lorentzian submersion was introduced by Majid [18] and Falcitelli et al. [10], respectively. Recently, Gündüzalp et al. [13] studied para contact semi-Riemannain submersions, Faghfouri et al. [9] studied anti-invariant semi-Riemannian submersions, Akyol et al. [2] studied semi-invariant semi-Riemannian submersions and Gündüzalp et al. [12] studied slant submersions from Lorentzian almost paracontact manifolds. On the other hand, Akyol et al. [3] studied semi-slant $\xi^{\perp}$-Riemannian submersions as a generalization of anti-invariant $\xi^{\perp}$-Riemannian submersions [17] and semi-invariant $\xi^{\perp}$-Riemannian submersions [4]. Also, Tastan et al. [26] studied hemi-slant submersions from Kählerian manifolds.

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Here, we have studied hemi-slant $\xi^{\perp}$-Lorentzian submersions from $(LCS)_n$-manifolds and the structure of the paper is as follows. Section 2 studies $(LCS)_n$-manifolds and semi-Riemannian submersions. In Section 3, we define hemi-slant $\xi^{\perp}$-Lorentzian submersions, present an example, find the integrability conditions for distributions and investigate the geometry of leaves of different distributions including horizontal and vertical distribution. In Section 4, we find a necessary and sufficient condition for a hemi-slant $\xi^{\perp}$-Lorentzian submersion to be totally geodesic. In this section we also study hemi-slant $\xi^{\perp}$-Lorentzian submersions with totally umbilical fibers.

2. Preliminaries

$(LCS)_n$-manifold is a Lorentzian manifold $\overline{M}$ of dimension n endowed with the unit timelike concircular vector field ξ, its associated 1-form η and a $(1,1)$ tensor field ϕ such that $\nabla_X \xi = \alpha\phi X$, α being a non-zero scalar function satisfying $\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X)$, where $\rho = -(\xi\alpha)$ is another scalar, and ∇ is the Levi-Civita connection of the Lorentzian metric g. If $\alpha = 1$, then this manifold reduces to the LP-Sasakian manifold [20].

In a $(LCS)_n$-manifold $(n > 2)$ M, the following relations hold [23, 24]:

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2}$$

$$(\nabla_X \phi)Y = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{3}$$

A differential map $\pi : M \to N$ between a Lorentzian manifold (M, g_M) and a semi-Riemannian manifold (N, g_N) is called a Lorentzian submersion if π_* is onto and it satisfies

(i) The fibers $\pi^{-1}(q)$, $q \in N$, are semi-Riemannian submanifolds of M.

(ii) π_* preserves scalar product of vectors normal to fibers.

For each $q \in N$, $\pi^{-1}(q)$ is a submanifold of M of dimension $k(= \dim M - \dim N)$. The submanifolds $\pi^{-1}(q)$ are called fibers, and a vector field X on M is called *horizontal* (resp. *vertical*) if it is always *orthogonal* (resp. *tangent*) to fibers. If X is horizontal and π-related to a vector field X_* on N then X is called *basic*. The projection morphisms on the vertical distribution $\ker \pi_*$ and the horizontal distribution $(\ker \pi_*)^{\perp}$ are denoted by $\mathcal{V}$ and $\mathcal{H}$, respectively [10]. The O'Neill's tensors [21] on M are

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \quad \mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \tag{4}$$

for $E, F \in \chi(M)$, where ∇ is the Levi-Civita connection of (M, g_M). For $U,\ V \in \ker \pi_*$ and $X,\ Y \in (\ker \pi_*)^{\perp}$ on M, we have $\mathcal{T}_U V = \mathcal{T}_V U$, $\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X,Y]$. Also from (4), we have

$$\nabla_U V = \mathcal{T}_U V + \hat{\nabla}_U V, \qquad \nabla_U X = \mathcal{H}\nabla_U X + \mathcal{T}_U X, \tag{5}$$

$$\nabla_X U = \mathcal{A}_X U + \mathcal{V}\nabla_X U, \qquad \nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{6}$$

for $X, Y \in (\ker \pi_*)^{\perp}$ and $U, V \in \ker \pi_*$, where $\hat{\nabla}_U V = \mathcal{V}\nabla_U V$ and $\mathcal{H}\nabla_U X = \mathcal{A}_X U$, if

X is basic. Clearly $\mathcal{T}$ acts on the fiber as the second fundamental form and $\mathcal{A}$ acts on the horizontal distribution. If $\mathcal{T} \equiv 0$, then π is said to be a submersion with totally geodesic fibers and it is said to be a submersion with totally umbilical fibers if

$$\mathcal{T}_E F = g_M(E, F)H, \tag{7}$$

for any $E, F \in \ker \pi_*$. If $H \equiv 0$, then π is said to be minimal [10]. Now, we recall that if (M, g_M) and (N, g_N) bare semi-Riemannian manifolds and $\pi : M \to N$ is a smooth map, then the second fundamental form of π is given by

$$(\nabla\pi_*)(E, F) = \nabla^{\pi}_{E}\pi_* F - \pi_*(\nabla_E F), \tag{8}$$

for $E,\ F \in \Gamma(TM)$, where ∇^{π} is the pull back connection and for convenience we denote by ∇ the Levi-Civita connection of the metrics g_M and g_N. π is said to be harmonic if $trace(\nabla\pi_*) = 0$ and it is called a totally geodesic map if $(\nabla\pi_*)(E, F) = 0$, for $E,\ F \in \Gamma(TM)$ [6]. Throughout the paper we consider (M, g_M) to be an $(LCS)_n$-manifold and (N, g_N) a semi-Riemannian manifold.

A Lorentzian submersion $\pi : M \to N$ is said to be anti-invariant [9] if $\phi(\ker \pi_*) \subseteq (\ker \pi_*)^{\perp}$ and is said to be slant (or θ-slant) [13] if the angle $\theta(X)$ between ϕX and $(\ker \pi_* - \{\xi_p\})$ is constant, i.e., it is independent of the choice of the non-zero vector $X \in \ker \pi_* - \{\xi_p\}$ and $p \in M$. θ is known as the slant angle of the slant submersion. Also, π is said to be hemi-slant [26] if $\ker \pi_*$ admits two complementary orthogonal distributions $\mathcal{D}^{\theta}$ and $\mathcal{D}^{\perp}$ such that $\mathcal{D}^{\theta}$ is slant and $\mathcal{D}^{\perp}$ is anti-invariant, i.e.,

$$\ker \pi_* = \mathcal{D}^{\theta} \oplus \mathcal{D}^{\perp}. \tag{9}$$

Hemi-slant submersion is natural generalization of anti-invariant, semi-invariant and slant submersion. If the dimensions of $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ are n_1 and n_2, then π is:

(i) an anti-invariant submersion, if $n_2 = 0$,

(ii) an invariant submersion, if $n_1 = 0,\ \theta = 0$,

(iii) a proper slant submersion with slant angle θ, if $n_1 = 0$ and $\theta \neq 0, \frac{\pi}{2}$,

(iv) a semi-invariant submersion, if $\theta = 0, n_1 \neq 0$.

A hemi-slant submersion is proper if $n_1 \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$.

3. Hemi-slant $\xi^{\perp}$-Lorentzian submersion

A hemi-slant Lorentzian submersion $\pi : M \to N$ is said to be a hemi-slant $\xi^{\perp}$-Lorentzian submersion if ξ is orthogonal to $\ker \pi_*$. Now we will construct an example of a hemi-slant $\xi^{\perp}$-Lorentzian submersion from an $(LCS)_n$-manifold onto a semi-Riemannian manifold.

Example 3.1. Let $(\mathbb{R}^9, \phi, \xi, \eta, g)$ denote the manifold $\mathbb{R}^9$ with the (LCS)-structure given by

$$\eta = \frac{1}{3}(-dz + \sum_{i=1}^{n} y^i dx^i),\ \xi = 3\frac{\partial}{\partial z},\ g = -\eta \otimes \eta + \frac{1}{9}\sum_{i=1}^{n} dx^i \otimes dx^i \oplus dy^i \otimes dy^i,$$

$$\phi(\frac{\partial}{\partial x^1}) = \frac{\partial}{\partial y^1},\ \phi(\frac{\partial}{\partial x^2}) = \frac{\partial}{\partial y^2},\ \phi(\frac{\partial}{\partial x^3}) = \frac{\partial}{\partial x^3},\ \phi(\frac{\partial}{\partial x^4}) = \frac{\partial}{\partial x^4},$$
$$\phi(\frac{\partial}{\partial y^1}) = \frac{\partial}{\partial x^1},\ \phi(\frac{\partial}{\partial y^2}) = \frac{\partial}{\partial x^2},\ \phi(\frac{\partial}{\partial y^3}) = -\frac{\partial}{\partial y^3},\ \phi(\frac{\partial}{\partial y^4}) = -\frac{\partial}{\partial y^4},\ \phi(\frac{\partial}{\partial z}) = 0,$$

where $(x^1, \dots, x^4, y^1, \dots, y^4, z)$ are Cartesian coordinates. For $\alpha, \beta \in \mathbb{R}$, let $\pi : \mathbb{R}^9 \to \mathbb{R}^5$ be a submersion defined by

$$(x^1, x^2, x^3, x^4, y^1, y^2, y^3, y^4, z) \mapsto (\cos\alpha x^1 + \sin\alpha x^2, \cos\beta y^1 + \sin\beta y^2, \frac{x^3 - y^3}{\sqrt{3}}, \frac{x^4 - y^4}{\sqrt{3}}, 3z).$$

Then it follows that $\ker\pi_* = \mathrm{span}\{J_1, J_2, J_3, J_4\}$, where $J_1 = \sin\alpha\frac{\partial}{\partial x^1} - \cos\alpha\frac{\partial}{\partial x^2}$, $J_2 = \sin\beta\frac{\partial}{\partial y^1} - \cos\beta\frac{\partial}{\partial y^2}$, $J_3 = \frac{\partial}{\partial x^3} + \frac{\partial}{\partial y^3}$, $J_4 = \frac{\partial}{\partial x^4} + \frac{\partial}{\partial y^4}$ and $(\ker\pi_*)^\perp = \mathrm{span}\{L_1, L_2, L_3, L_4, \xi\}$, where $L_1 = \cos\alpha\frac{\partial}{\partial x^1} + \sin\alpha\frac{\partial}{\partial x^2}$, $L_2 = \cos\beta\frac{\partial}{\partial y^1} + \sin\beta\frac{\partial}{\partial y^2}$, $L_3 = \frac{\partial}{\partial x^3} - \frac{\partial}{\partial y^3}$, $L_4 = \frac{\partial}{\partial x^4} - \frac{\partial}{\partial y^4}$. Then, $g(\phi J_1, J_2) = \frac{1}{9}\cos(\alpha - \beta)$,$\phi J_3 = L_3$ and $\phi J_4 = L_4$. Thus $\mathrm{span}\{J_1, J_2\}$ is a slant distribution with slant angle $|\alpha - \beta|$ and $\mathrm{span}\{J_3, J_4\}$ is an anti-invariant distribution.

Also, by direct decomposition, we find that $g_N(\pi_* L_1, \pi_* L_1) = g_M(L_1, L_1)$, $g_N(\pi_* L_2, \pi_* L_2) = g_M(L_2, L_2)$, $g_N(\pi_* L_3, \pi_* L_3) = g_M(L_3, L_3)$, $g_N(\pi_* L_4, \pi_* L_4) = g_M(L_4, L_4)$, $g_N(\xi, \xi) = g_M(\xi, \xi)$, where g_M and g_N are the metrics of $\mathbb{R}^9$ and $\mathbb{R}^5$. Thus π is a hemi-slant $\xi^\perp$-Lorentzian submersion.

For any $E \in \ker\pi_*$, let $E = \mathcal{P}E + \mathcal{Q}E$, where $\mathcal{P}E \in \mathcal{D}^\theta$ and $\mathcal{Q}E \in \mathcal{D}^\perp$ and take

$$\phi E = tE + \omega E, \tag{10}$$

where $tE \in \ker\pi_*$ and $\omega E \in (\ker\pi_*)^\perp$. Also for any $X \in (\ker\pi_*)^\perp$, we have

$$\phi X = bX + cX, \tag{11}$$

where $bX \in \ker\pi_*$ and $cX \in (\ker\pi_*)^\perp$ and hence $(\ker\pi_*)^\perp = \omega\mathcal{D}^\theta \oplus \phi\mathcal{D}^\perp \oplus \mu$, where μ is a ϕ-invariant distribution of $(\ker\pi_*)^\perp$.

The proof of the following theorem is similar to [5, Theorem 3.1].

Theorem 3.2. *Let π be a $\xi^\perp$-Lorentzian submersion from (M, g_M) onto (N, g_N). Then π is a hemi-slant Lorentzian submersion if and only if there exist a constant $\lambda \in [0, 1]$ and a distribution $\mathcal{D}$ on $\ker\pi_*$ such that*
(i) $\mathcal{D} = \{V \in \ker\pi_* | t^2 V = \lambda V\}$,

(ii) $\phi V = \omega V$, for any $\in \ker\pi_*$ and orthogonal to $\mathcal{D}$.
Furthermore, if θ is the slant angle of π, then $\lambda = \cos^2\theta$.

For any $U \in \mathcal{D}^\theta$, we get

$$t^2 U = \cos^2\theta U. \tag{12}$$

Consequently, we obtain $g(tU, tV) = \cos^2\theta g(U, V)$ and $g(\omega U, \omega V) = \sin^2\theta g(U, V)$ for every $U, V \in \mathcal{D}^\theta$.

Lemma 3.3. *Let π be a hemi-slant $\xi^\perp$-Lorentzian submersion from (M, g_M) onto (N, g_N). Then we have $t^2 + b\omega = I$, $\omega t + c\omega = 0$, $c^2 + \omega b = I + \eta\otimes\xi$, $tb + bc = 0$.*

Proof. Proof of this lemma follows from (10), (11) and (2). □

LEMMA 3.4. *Let π be a hemi-slant $\xi^\perp$-Lorentzian submersion from (M, g_M) onto (N, g_N). Then we have*
(i) $t\mathcal{D}^\theta = \mathcal{D}^\theta$, *(ii)* $t\mathcal{D}^\perp = \{0\}$, *(iii)* $b\omega\mathcal{D}^\theta = \mathcal{D}^\theta$, *(iv)* $b\phi\mathcal{D}^\perp = \mathcal{D}^\perp$.

By using (3), (5), (6), (10) and (11), we can easily obtain the following assertions.

LEMMA 3.5. *Let π be a hemi-slant $\xi^\perp$-Lorentzian submersion from (M, g_M) onto (N, g_N). Then*

$$\hat{\nabla}_E tF + \mathcal{T}_E\omega F = b\mathcal{T}_E F + t\hat{\nabla}_E F, \tag{13}$$

$$\mathcal{T}_E tF + \mathcal{H}\nabla_E\omega F = c\mathcal{T}_E F + \omega\hat{\nabla}_E F + \alpha g(E,F)\xi, \tag{14}$$

$$\mathcal{T}_E bX + \mathcal{H}\nabla_E cX = c\mathcal{H}\nabla_E X + \omega\mathcal{T}_E X,$$

$$\hat{\nabla}_E bX + \mathcal{T}_E cX = b\mathcal{H}\nabla_E X + t\mathcal{T}_E X + \alpha\eta(X)E,$$

$$\mathcal{A}_X bY + \mathcal{H}\nabla_X cY = c\mathcal{H}\nabla_X Y + \omega\mathcal{A}_X Y + \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X]$$

$$\omega\nabla_X bY + \mathcal{A}_X cY = b\mathcal{H}\nabla_X Y + t\mathcal{A}_X Y$$

Now, the covariant derivatives of t and ω are defined by $(\nabla_E t)F = \hat{\nabla}_U tF - t\hat{\nabla}_E F$ and $(\nabla_E\omega)F = \mathcal{H}\nabla_E\omega F - \omega\hat{\nabla}_E F$, for $E, F \in \ker\pi_*$. Then from (13) and (14), we get the following.

COROLLARY 3.6. *Let π be a hemi-slant $\xi^\perp$-Lorentzian submersion from (M, g_M) onto (N, g_N). Then t is parallel if and only if $\mathcal{T}_E\omega F = b\mathcal{T}_E F$ and ω is parallel if and only if $\mathcal{T}_E tF = c\mathcal{T}_U F + \alpha g(E,F)\xi$, where $E, F \in \ker\pi_*$.*

THEOREM 3.7. *Let π be a hemi-slant $\xi^\perp$-Lorentzian submersion from (M, g_M) onto (N, g_N). Then $\mathcal{D}^\theta$ is integrable if and only if*

$$g_M(\mathcal{H}\nabla_U\omega V - \mathcal{H}\nabla_V\omega U, \phi Z) = g_M(\mathcal{T}_V\omega tU - \mathcal{T}_U\omega tV, Z),$$

for $U,\ V \in \mathcal{D}^\theta$ and $Z \in \mathcal{D}^\perp$.

Proof. For $U,\ V \in \mathcal{D}^\theta$ and $Z \in \mathcal{D}^\perp$, we have from (1) that

$$\begin{aligned} g_M(\nabla_U V, Z) &= g_M(\nabla_U tV, \phi Z) + g_M(\nabla_U\omega V, \phi Z) = g_M(\nabla_U\phi tV, Z) + g_M(\nabla_U\omega V, \phi Z) \\ &= g_M(\nabla_U t^2 V, Z) + g_M(\nabla_U\omega tV, Z) + g_M(\nabla_U\omega V, \phi Z). \end{aligned}$$

By virtue of (5) and (12), the above equation yields

$$\sin^2\theta g_M(\nabla_U V, Z) = g_M(\mathcal{T}_U\omega tV, Z) + g_M(\mathcal{H}\nabla_U\omega V, \phi Z). \tag{15}$$

Thus we obtain

$$\sin^2\theta g_M([U,V], Z) = g_M(\mathcal{T}_U\omega tV - \mathcal{T}_V\omega tU, Z) + g_M(\mathcal{H}\nabla_U\omega V - \mathcal{H}\nabla_V\omega U, \phi Z). \qquad \square$$

COROLLARY 3.8. *Let π be a hemi-slant $\xi^\perp$-Lorentzian submersion from (M, g_M) onto (N, g_N). If $\mathcal{H}\nabla_U\omega V - \mathcal{H}\nabla_V\omega U \in \omega\mathcal{D}^\theta \oplus \mu$ and $\mathcal{T}_U\omega tV - \mathcal{T}_V\omega tU \in \mathcal{D}^\theta$, for every $U,\ V \in \mathcal{D}^\theta$ and $Z \in \mathcal{D}^\perp$, then $\mathcal{D}^\theta$ is integrable.*

THEOREM 3.9. *Let π be a hemi-slant $\xi^\perp$-Lorentzian submersion from (M, g_M) onto (N, g_N). Then $\mathcal{D}^\perp$ is integrable if and only if*

$$g_M(\mathcal{H}\nabla_Z\phi W - \mathcal{H}\nabla_W\phi Z, \omega U) = g_M(\mathcal{T}_W Z - \mathcal{T}_Z W, \omega tU),$$

for every $Z,\ W\in\mathcal{D}^{\perp}$ *and* $U\in\mathcal{D}^{\theta}$.

Proof. For $Z,\ W\in\mathcal{D}^{\perp}$ and $U\in\mathcal{D}^{\theta}$, we have from (1), (3) and (10) that

$$\begin{aligned}g_M(\nabla_Z W,U)&=g_M(\nabla_Z W,\phi tU)+g_M(\nabla_Z\phi W,\omega U)\\&=g_M(\nabla_Z W,t^2U)+g_M(\nabla_Z W,\omega tU)+g_M(\nabla_Z\phi W,\omega U).\end{aligned}$$

By virtue of (5) and (12), the above equation yields

$$\sin^2\theta g_M(\nabla_Z W,U)=g_M(\mathcal{H}\nabla_Z\phi W,\omega U)+g_M(\mathcal{T}_Z W,\omega tU). \tag{16}$$

Thus we find

$$\sin^2\theta g_M([Z,W],U)=g_M(\mathcal{H}\nabla_Z\phi W-\mathcal{H}\nabla_W\phi Z,\omega U)+g_M(\mathcal{T}_Z W-\mathcal{T}_W Z,\omega tU).\qquad\square$$

Corollary 3.10. *Let* π *be a hemi-slant* $\xi^{\perp}$*-Lorentzian submersion from* (M,g_M) *onto* (N,g_N). *If* $\mathcal{H}\nabla_Z\phi W-\mathcal{H}\nabla_W\phi Z$ *and* $\mathcal{T}_Z W-\mathcal{T}_W Z$ *both belong to* $\phi\mathcal{D}^{\perp}\oplus\mu$, *for every* $Z,\ W\in\mathcal{D}^{\perp}$ *and* $U\in\mathcal{D}^{\theta}$, *then* $\mathcal{D}^{\perp}$ *is integrable.*

Theorem 3.11. *Let* π *be a hemi-slant* $\xi^{\perp}$*-Lorentzian submersion from* (M,g_M) *onto* (N,g_N). *Then* $\mathcal{D}^{\theta}$ *describes a totally geodesic foliation if and only if*

$$g_M(\mathcal{H}\nabla_U\omega V,\phi Z)+g_M(\mathcal{T}_U\omega tV,Z)=0 \tag{17}$$

and

$$g_M(\mathcal{H}\nabla_U\omega tV,X)+g_M(\mathcal{H}\nabla_U\omega V,cX)+g_M(\mathcal{A}_U\omega V,bX)=0, \tag{18}$$

for every $U,\ V\in\mathcal{D}^{\theta}$, $Z\in\mathcal{D}^{\perp}$ *and* $X\in(\ker\pi_*)^{\perp}$.

Proof. Since $\theta\neq 0,\frac{\pi}{2}$, the relation (17) follows from (15). Also, for $U,\ V\in\mathcal{D}^{\theta}$ and $X\in(\ker\pi_*)^{\perp}$, we have from (1), (3), (10) and (11) that

$$\begin{aligned}g_M(\nabla_U V,X)&=g_M(\nabla_U t^2V,X)+g_M(\nabla_U\omega tV,X)-g_M((\nabla_U\phi)tV,X)\\&\quad+g_M(\nabla_U\omega V,bX)+g_M(\nabla_U\omega V,cX)+\alpha\eta(X)g_M(\phi U,V)\end{aligned}$$

By virtue of (3), (5) and (12), the above relation yields

$$\sin^2\theta g_M(\nabla_U V,X)=g_M(\mathcal{H}\nabla_U\omega tV,X)+g_M(\mathcal{A}_U\omega V,bX)+g_M(\mathcal{H}\nabla_U\omega V,cX). \tag{19}$$

which gives (18). The converse part also follows from (15) and (19). $\square$

Theorem 3.12. *Let* π *be a hemi-slant* $\xi^{\perp}$*-Lorentzian submersion from* (M,g_M) *onto* (N,g_N). *Then* $\mathcal{D}^{\perp}$ *describes a totally geodesic foliation if and only if*

$$g_M(\mathcal{H}\nabla_Z\phi W,\omega U)+g_M(\mathcal{T}_Z W,\omega tU)=0 \tag{20}$$

and

$$g_M(\mathcal{H}\nabla_Z\phi W,cX)=g_M(\hat{\nabla}_Z tbX+\mathcal{T}_Z\omega bX,W), \tag{21}$$

for every $U\in\mathcal{D}^{\theta}$, $Z,\ W\in\mathcal{D}^{\perp}$ *and* $X\in(\ker\pi_*)^{\perp}$.

Proof. Since $\theta\neq 0,\frac{\pi}{2}$, (20) follows from (16). Also, for $Z,\ W\in\mathcal{D}^{\theta}$ and $X\in(\ker\pi_*)^{\perp}$, from (3), (10) and (11), we get

$$\begin{aligned}g_M(\nabla_Z W,X)&=g_M(\nabla_Z W,tbX)+g_M(\nabla_Z W,\omega bX)+g_M(\nabla_Z\phi W,cX)\\&=-g_M(\nabla_Z tbX,W)-g_M(\nabla_Z\omega bX,W)+g_M(\nabla_Z\phi W,cX)\end{aligned}$$

which by virtue of (5), yields

$$g_M(\nabla_Z W,X)=-g_M(\hat{\nabla}_Z tbX,W)-g_M(\mathcal{T}_Z\omega bX,W)+g_M(\mathcal{H}\nabla_Z\phi W,cX), \tag{22}$$

from which (21) follows. The converse part follows from (16) and (22). $\square$

Proposition 3.13. *Let π be a hemi-slant $\xi^{\perp}$-Lorentzian submersion from (M, g_M) onto (N, g_N). Then $\ker \pi_*$ becomes a direct product of $\mathcal{D}^{\theta}$ and $\mathcal{D}^{\perp}$ if and only if* (17), (18), (20) *and* (21) *hold simultaneously.*

Theorem 3.14. *Let π be a hemi-slant $\xi^{\perp}$-Lorentzian submersion from (M, g_M) onto (N, g_N). Then the following assertions are equivalent:*
(i) $\ker \pi_*{}^{\perp}$ *is integrable*

(ii) the following relations hold:

$$g_M(\mathcal{H}\nabla_Y \phi Z, cX) - g_M(\mathcal{H}\nabla_X \phi Z, cY) = g_M(\mathcal{A}_Y bX - \mathcal{A}_X bY, \phi Z) - \alpha[\eta(X) g_M(Y, \phi Z) - \eta(Y) g_M(X, \phi Z)] \tag{23}$$

and

$$g_M(\mathcal{H}\nabla_X Y - \mathcal{H}\nabla_Y X, \omega t U) = g_M(\mathcal{A}_Y bX - \mathcal{A}_X bY, \omega U) + g_M(\mathcal{H}\nabla_Y cX - \mathcal{H}\nabla_X cY, \omega U) - \alpha[\eta(X) g_M(Y, \omega U) - \eta(Y) g_M(X, \omega U)], \tag{24}$$

for $X,\ Y \in (\ker \pi_*)^{\perp}$, $Z \in \mathcal{D}^{\perp}$ and $U \in \mathcal{D}^{\theta}$.

(iii) the following relations hold:

$$g_N((\nabla \pi_*)(Y, bX) - (\nabla \pi_*)(X, bY), \pi_* \phi Z) = g_M(\mathcal{H}\nabla_X \phi Z, cY) - g_M(\mathcal{H}\nabla_Y \phi Z, cX) - \alpha[\eta(X) g_M(Y, \phi Z) - \eta(Y) g_M(X, \phi Z)]$$

and

$$g_N((\nabla \pi_*)(Y, bX) - (\nabla \pi_*)(X, bY), \pi_* \omega U) = g_M(\mathcal{H}\nabla_Y X - \mathcal{H}\nabla_X Y, \omega t U) + g_M(\mathcal{H}\nabla_Y cX - \mathcal{H}\nabla_X cY, \omega U) - \alpha[\eta(X) g_M(Y, \omega U) - \eta(Y) g_M(X, \omega U)]$$

for $X, Y \in (\ker \pi_*)^{\perp}$, $Z \in \mathcal{D}^{\perp}$ and $U \in \mathcal{D}^{\theta}$.

Proof. For $X, Y \in (\ker \pi_*)^{\perp}$ and $Z \in \mathcal{D}^{\perp}$, we have from (1), (3) and (11) that

$$g_M(\nabla_X Y, Z) = g_M(\nabla_X bY, \phi Z) - g_M(cY, \nabla_X \phi Z) - \alpha \eta(Y) g_M(X, \phi Z).$$

By virtue of (5), the above equation yields

$$g_M(\nabla_X Y, Z) = g_M(\mathcal{A}_X bY, \phi Z) - g_M(\mathcal{H}\nabla_X \phi Z, cY) - \alpha \eta(Y) g_M(X, \phi Z). \tag{25}$$

Thus we find

$$g_M([X, Y], Z) = g_M(\mathcal{A}_X bY - \mathcal{A}_Y bX, \phi Z) - g_M(\mathcal{H}\nabla_X \phi Z, cY) + g_M(\mathcal{H}\nabla_Y \phi Z, cX) - \alpha[\eta(Y) g_M(X, \phi Z) - \eta(X) g_M(Y, \phi Z)]. \tag{26}$$

Also, for $X, Y \in (\ker \pi_*)^{\perp}$ and $U \in \mathcal{D}^{\theta}$, we have from (1), (3), (10) and (11) that

$$g_M(\nabla_X Y, U) = g_M(\nabla_X Y, t^2 U) + g_M(\nabla_X Y, \omega t U) + g_M(\nabla_X bY, \omega U) + g_M(\nabla_X cY, \omega U) - \alpha \eta(Y) g_M(X, \omega U).$$

Using (5) and (12) in the above equation, we obtain

$$\sin^2 \theta g_M(\nabla_X Y, U) = g_M(\mathcal{H}\nabla_X Y, \omega t U) + g_M(\mathcal{A}_X bY, \omega U) + g_M(\mathcal{H}\nabla_X cY, \omega U) - \alpha \eta(Y) g_M(X, \omega U). \tag{27}$$

Thus we get

$$\sin^2 \theta g_M([X, Y], U) = g_M(\mathcal{H}\nabla_X Y - \mathcal{H}\nabla_Y X, \omega t U) + g_M(\mathcal{A}_X bY - \mathcal{A}_Y bX, \omega U) + g_M(\mathcal{H}\nabla_X cY - \mathcal{H}\nabla_Y cX, \omega U) - \alpha[\eta(Y) g_M(X, \omega U) - \eta(X) g_M(Y, \omega U)]. \tag{28}$$

From (26) and (28), we get (i)⇔(ii).

Now, from (8), we have

$$g_M(\mathcal{A}_X bY, \phi Z) = -g_N((\nabla\pi_*)(X, bY), \pi_*\phi Z) \tag{29}$$

and

$$g_M(\mathcal{A}_Y bX, \phi Z) = -g_N((\nabla\pi_*)(Y, bX), \pi_*\phi Z) \tag{30}$$

Using (29) and (30) in (23) and (24), respectively, we get (ii)⇔(iii). □

Theorem 3.15. *Let π be a hemi-slant $\xi^\perp$-Lorentzian submersion from (M, g_M) onto (N, g_N). Then the following statements are equivalent:*
(i) $(\ker\pi_*)^\perp$ describes a totally geodesic foliation,

(ii) the following relations hold:

$$g_M(\mathcal{A}_X bY, \phi Z) = g_M(\mathcal{H}\nabla_X\phi Z, cY) + \alpha\eta(Y)g_M(X, \phi Z)$$

and $g_M(\mathcal{A}_X bY, \omega U) = -g_M(\mathcal{H}\nabla_X Y, \omega tU) + g_M(\mathcal{H}\nabla_X cY, \omega U) - \alpha\eta(Y)g_M(X, \omega U)$,

for $X,\ Y \in (\ker\pi_*)^\perp$, $Z \in \mathcal{D}^\perp$ and $U \in \mathcal{D}^\theta$.

(iii) the following relations hold:

$$g_N((\nabla\pi_*)(X, bY), \pi_*\phi Z) = -\, g_M(\mathcal{H}\nabla_X\phi Z, cY) - \alpha\eta(Y)g_M(X, \phi Z)$$

and $g_N((\nabla\pi_*)(X, bY), \pi_*\omega U) = g_M(\mathcal{H}\nabla_X Y, \omega tU) + g_M(\nabla_X cY, \omega U) - \alpha\eta(Y)g_M(X, \phi Z)$,

for every $X,\ Y \in (\ker\pi_*)^\perp$, $Z \in \mathcal{D}^\perp$ and $U \in \mathcal{D}^\theta$.

Proof. From (25) and (27), it is clear that (i)⇔(ii). Using (29) in (25) and (30) in (27), we obtain (ii)⇔(iii). □

Theorem 3.16. *Let π be a hemi-slant $\xi^\perp$-Lorentzian submersion from (M, g_M) onto (N, g_N). Then the following assertions are equivalent:*
(i) $\ker\pi_*$ describes a totally geodesic foliation,

(ii) the following relation holds:

$$g_M(\mathcal{T}_E bX, \omega F) - \cos^2\theta g_M(\mathcal{T}_E\mathcal{P}F, X) = g_M(\mathcal{H}\nabla_E\omega t\mathcal{P}F, X) + g_M(\mathcal{H}\nabla_E\omega F, cX).$$

(iii) the following relation holds:

$$\cos^2\theta g_N((\nabla\pi_*)(E, \mathcal{P}F), \pi_*X) - g_N((\nabla\pi_*)(E, bX), \pi_*\omega F) = g_M(\mathcal{H}\nabla_E\omega t\mathcal{P}F, X) + g_M(\mathcal{H}\nabla_E\omega F, cX),$$

for $E, F \in (\ker\pi_*)$, and $X \in (\ker\pi_*)^\perp$.

Proof. For $E, F \in (\ker\pi_*)$, and $X \in (\ker\pi_*)^\perp$, we have from (1), (3), (9)–(11) that

$$g_M(\nabla_E F, X) = g_M(\nabla_E\phi t\mathcal{P}F, X) + g_M(\nabla_E\omega\mathcal{P}F, bX) + g_M(\nabla_E\omega\mathcal{P}F, cX) + g_M(\nabla_E\phi\mathcal{Q}F, cX) + g_M(\nabla_E\phi\mathcal{Q}F, bX).$$

By virtue of (5) and (12), the above relation yields

$$g_M(\nabla_E F, X) = \cos^2\theta g_M(\mathcal{T}_U\mathcal{P}F, X) + g_M(\mathcal{H}\nabla_E\omega t\mathcal{P}F, X) + g_M(\mathcal{H}\nabla_E\omega\mathcal{P}F, cX) + g_M(\mathcal{H}\nabla_E\phi\mathcal{Q}F, cX) - g_M(\mathcal{T}_E bX, \omega\mathcal{P}F) - g_M(\mathcal{T}_E bX, \phi\mathcal{Q}F).$$

Since $\omega F = \omega\mathcal{P}F \oplus \phi\mathcal{Q}F$, we obtain

$$g_M(\nabla_E F, X) = \cos^2\theta g_M(\mathcal{T}_E\mathcal{P}F, X) + g_M(\mathcal{H}\nabla_E\omega t\mathcal{P}F, X)$$

$$+ g_M(\nabla_E \omega F, cX) - g_M(\mathcal{T}_E bX, \omega F). \tag{31}$$

From (31), we obtain (i)⇔(ii) and using (8) in (31), we get (ii)⇔(iii). □

4. Totally geodesicness and totally umbilical fibers

Theorem 4.1. *Let π be a hemi-slant $\xi^{\perp}$-Lorentzian submersion from (M, g_M) onto (N, g_N). Then π is a totally geodesic map if and only if*

$$g_M(\mathcal{A}_X \mathcal{P}E, Y) = -\sec^2\theta\{g_M(\mathcal{H}\nabla_X \omega t\mathcal{P}E, Y) + g_M(\mathcal{H}\nabla_X \omega E, cY) + g_M(\mathcal{A}_X \omega E, bY) + \alpha\eta(Y) g_M(\phi X, E)\} \tag{32}$$

and

$$g_M(\mathcal{T}_E \mathcal{P}F, X) = -\sec^2\theta\{g_M(\mathcal{H}\nabla_E \omega t\mathcal{P}F, X) + g_M(\mathcal{H}\nabla_E \omega F, cX) + g_M(\mathcal{T}_E \omega F, bX)\}, \tag{33}$$

for $E, F \in \ker\pi_*$ and $X, Y \in (\ker\pi_*)^{\perp}$.

Proof. For $E \in \ker\pi_*$ and $X \in (\ker\pi_*)^{\perp}$, from (8) we have

$$g_N((\nabla\pi_*)(X, E), \pi_* Y) = -g_M(\nabla_X E, Y). \tag{34}$$

Using (1), (3), (10) and (11) in (34), we get

$$\begin{aligned} g_N((\nabla\pi_*)(X, E), \pi_* Y) = &-g_M(\nabla_X t^2\mathcal{P}E, Y) - g_M(\nabla_X \omega t\mathcal{P}E, Y) - g_M(\nabla_X \omega\mathcal{P}E, bY) \\ &- g_M(\nabla_X \omega\mathcal{P}E, cY) - g_M(\nabla_X \phi\mathcal{Q}E, bY) \\ &- g_M(\nabla_X \phi\mathcal{Q}E, cY) - \alpha\eta(Y) g_M(\phi X, Y). \end{aligned}$$

Using (5), (12) and the fact that $\omega E = \omega\mathcal{P}E \oplus \phi\mathcal{Q}E$, we find

$$\begin{aligned} g_N((\nabla\pi_*)(X, E), \pi_* Y) = &-\cos^2\theta g_M(\mathcal{A}_X \mathcal{P}E, Y) - g_M(\mathcal{H}\nabla_X \omega t\mathcal{P}E, Y) \\ &- g_M(\mathcal{A}_X \omega E, bY) - g_M(\mathcal{H}\nabla_X \omega E, cY) - \alpha\eta(Y) g_M(\phi X, Y). \end{aligned} \tag{35}$$

Thus (32) follows from (35), and (33) can be obtained in a similar way. □

Theorem 4.2. *Let π be a hemi-slant $\xi^{\perp}$-Lorentzian submersion from (M, g_M) onto (N, g_N). If ω is parallel with respect to ∇ on $\ker\pi_*$, then*
(i) $c\mathcal{T}_Z W = -\alpha g_M(Z, W)\xi \in \mu$, (ii) $c\mathcal{T}_U Z = 0$, i.e., $\phi\mathcal{T}_U Z \in \ker\pi_*$,
(iii) $\mathcal{T}_Z U = \sec^2\theta c\mathcal{T}_Z tU$, (iv) $\mathcal{T}_V U = \sec^2\theta\,[c\mathcal{T}_V tU + \alpha g_M(tU, V)\xi]$,
for $U, V \in \mathcal{D}^{\theta}$ and $Z, W \in \mathcal{D}^{\perp}$.

Proof. If ω is parallel, then for $E,\ F \in \ker\pi_*$, we have from Corollary 3.6 that

$$\mathcal{T}_E tF - c\mathcal{T}_E F = \alpha g_M(E, F)\xi. \tag{36}$$

Now, for $Z, W \in \mathcal{D}^{\perp}$, we have $tZ = tW = 0$. Thus for $U \in \mathcal{D}^{\theta}$, we get (i) and (ii). Also, from (36), we find $\mathcal{T}_Z tU = c\mathcal{T}_Z U$ and $\mathcal{T}_V tU = c\mathcal{T}_V U + \alpha g_M(U, V)\xi$. Replacing U by tU, we get (iii) and (iv), respectively. □

Corollary 4.3. *Let π be a hemi-slant $\xi^{\perp}$-Lorentzian submersion from (M, g_M) onto (N, g_N). If ω is parallel with respect to ∇ on $\ker\pi_*$, then*
(i) the fibers of π are not geodesic in $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$,

(ii) the fibers of π are mixed geodesic if and only if $c \equiv 0$.

THEOREM 4.4. *Let π be a hemi-slant $\xi^{\perp}$-Lorentzian submersion with totally umbilical fibers from (M, g_M) onto (N, g_N). Then one of the following holds:*
(i) Fibers of π are minimal. *(ii)* $\dim \mathcal{D}^{\perp} = 1$. *(iii)* $H \in \Gamma(\omega\mathcal{D}^{\theta} \oplus \mu)$.

Proof. For $W, Z \in \mathcal{D}^{\perp}$, we have from (3) that

$$\nabla_W \phi Z - \phi(\nabla_W Z) = \alpha g_M(W, Z)\xi. \tag{37}$$

Using (5) in (37), then taking inner product with W, we obtain

$$g_M(\phi Z, \mathcal{T}_W W) = g_M(\mathcal{T}_W Z, \phi W). \tag{38}$$

Using (7) in (38), we find

$$g_M(H, \phi Z) = \frac{g_M(W, Z)}{g_M(W, W)} g_M(H, \phi W). \tag{39}$$

Interchanging W and Z in (39), we get

$$g_M(H, \phi W) = \frac{g_M(W, Z)}{g_M(Z, Z)} g_M(H, \phi Z). \tag{40}$$

Substituting (39) in (40), we obtain

$$\left(1 - \frac{g_M(Z, W)^2}{g_M(W, W) g_M(Z, Z)}\right) g_M(H, \phi W) = 0,$$

from which the theorem follows. □

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