MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
72, 2 (2020), 138–145
June 2020

research paper
оригинални научни рад

ON CONFORMAL TRANSFORMATION OF m-th ROOT FINSLER METRIC

Bankteshwar Tiwari and Manoj Kumar

Abstract. The purpose of the present paper is to study the conformal transformation of m-th root Finsler metric. The spray coefficients, Riemann curvature and Ricci curvature of conformally transformed m-th root metrics are shown to be certain rational functions of direction. Further, under certain conditions it is shown that a conformally transformed m-th root metric is locally dually flat if and only if the transformation is a homothety. Moreover the conditions for the transformed metrics to be Einstein and isotropic mean Berwald curvature are also found.

1. Introduction

The m-th root Finsler metric has been developed by Shimada [15], applied to Biology as an ecological metric by Antonelli [3] and studied by several authors [16, 17, 20]. It is regarded as a generalization of Riemannian metric in the sense that the second root metric is a Riemannian metric. For $m = 3$, it is called a cubic Finsler metric [12] and for $m = 4$ quartic metric [10]. In four-dimension, the special fourth root metric in the form $F = \sqrt[4]{y^1y^2y^3y^4}$ is called the Berwald-Moór metric [5, 6] which is considered by physicists as an important subject for a possible model of space time. Recent studies show that m-th root Finsler metrics play a very important role in physics, space-time and general relativity as well as in unified gauge field theory [4, 13]. Z. Shen and B. Li [10] have studied the geometric properties of locally projectively flat fourth root metrics in the form $F = \sqrt[4]{a_{ijkl}(x)y^iy^jy^ky^l}$ and generalized fourth root metrics in the form $F = \sqrt{\sqrt{a_{ijkl}(x)y^iy^jy^ky^l} + b_{ij}(x)y^iy^j}$. Yaoyong Yu and Ying You have shown that an m-th root Einstein Finsler metric is Ricci-flat [20]. In [16], A. Tayebi and B. Najafi have characterized locally dually flat and Antonelli m-th root metrics and in [17] Tayebi, Peyghan and Shahbazi have found a condition under

2010 Mathematics Subject Classification: 53B40, 53C60
Keywords and phrases: Finsler space; m-th root metric; conformal transformation; locally dually flat metric; Einstein metric; Ricci curvature; isotropic mean Berwald curvature.

which a generalized m-th root metric is projectively related to an m-th root metric. In [19], A. Tayebi and M. Shahbazi Nia have studied a new class of projectively flat Finsler metrics with constant flag curvature $K = 1$.

The conformal theory of Finsler metric based on the theory of Finsler spaces by Matsumoto [11] has been developed by M. Hashiguchi [9]. Let F and $\overline{F}$ be two Finsler metrics on a manifold M^n such that $\overline{F} = e^{\alpha(x)}F$, where α is a scalar function on M^n. Then we say that such two metrics F and $\overline{F}$ are conformally related. More precisely, Finsler metric $\overline{F}$ is said to be a conformally transformed Finsler metric [18]. A Finsler metric, which is conformally related to a Minkowski metric, is called conformally flat Finsler metric. The conformal change is said to be a homothety if α is a constant.

2. Preliminaries

Let M^n be an n-dimensional C^∞-manifold, and T_xM denote the tangent space of M^n at x. The tangent bundle TM is the union of tangent spaces $TM := \bigcup_{x\in M} T_xM$. We denote the elements of TM by (x, y), where $x = (x^i)$ is a point of M^n and $y \in T_xM$ is called supporting element.

Definition 2.1. A Finsler metric on M^n is a function $F : TM \to [0, \infty)$ with the following properties:
(i) F is C^∞ on TM_0,

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM,

(iii) the Hessian of F^2 with element $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.
The pair $(M^n, F) := F^n$ is called a Finsler space. F is called the fundamental function and g_{ij} is called the fundamental tensor.

The normalized supporting element l_i and angular metric tensor h_{ij} of F^n are defined respectively as: $l_i = \frac{\partial F}{\partial y^i}$, $h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}$.

Let F be a Finsler metric defined by $F = \sqrt[m]{A}$, where A is given by $A := a_{i_1 i_2 \ldots i_m}(x)y^{i_1}y^{i_2}\ldots y^{i_m}$, with $a_{i_1\ldots i_m}$ symmetric in all its indices [15]. Then F is called an m-th root Finsler metric. Clearly, A is homogeneous of degree m in y.

Let

$$A_i = a_{ii_2\ldots i_m}(x)y^{i_2}\ldots y^{i_m} = \frac{1}{m}\frac{\partial A}{\partial y^i}, \tag{1}$$

$$A_{ij} = a_{iji_3\ldots i_m}(x)y^{i_3}\ldots y^{i_m} = \frac{1}{m(m-1)}\frac{\partial^2 A}{\partial y^i \partial y^j}. \tag{2}$$

The normalized supporting element l_i of F^n is given by

$$l_i := F_{y^i} = \frac{\partial F}{\partial y^i} = \frac{\partial \sqrt[m]{A}}{\partial y^i} = \frac{1}{m}\frac{\frac{\partial A}{\partial y^i}}{A^{\frac{m-1}{m}}} = \frac{A_i}{F^{m-1}}. \tag{3}$$

Consider the conformal transformation $\overline{F}(x, y) = e^{\alpha(x)}F(x, y)$ of m-th root metric $F = \sqrt[m]{A}$. Clearly $\overline{F}$ is also an m-th root Finsler metric on M^n. Throughout the paper we call the Finsler metric $\overline{F}$ as conformally transformed m-th root metric and $(M^n, \overline{F}) = \overline{F}^n$ as conformally transformed Finser space. We restrict ourselves to $m > 2$ and also the quantities corresponding to the transformed Finsler space $\overline{F}^n$ will be denoted by putting overline on the top of that quantity, for instance, $\overline{A} = e^{m\alpha}A$, $\overline{A}_i = e^{m\alpha}A_i$ and $\overline{A}_{ij} = e^{m\alpha}A_{ij}$.

3. Conformal transformation of m-th root metric

The fundamental metric tensor g_{ij} of Finsler space F^n is given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} = FF_{y^i y^j} + F_{y^i}F_{y^j}.$$

In view of (1), (2) and (3), we have $g_{ij} = (m-1)\frac{A_{ij}}{F^{m-2}} - (m-2)\frac{A_iA_j}{F^{2(m-1)}}$. The contravariant metric tensor g^{ij} of Finsler space F^n is given by $g^{ij} = \frac{F^{m-2}}{(m-1)}A^{ij} + \frac{(m-2)}{(m-1)}\frac{y^iy^j}{F^2}$, where matrix (A^{ij}) denotes the inverse of (A_{ij}) [20]. Here we have used $A^{ij}A_j = A^i = y^i$.

Since the covariant and contravariant metric tensor of transformed Finsler space $\overline{F}^n$ are given by $\overline{g}_{ij} = e^{2\alpha}g_{ij}$ and $\overline{g}^{ij} = e^{-2\alpha}g^{ij}$, we have

Theorem 3.1. *The covariant metric tensor $\overline{g}_{ij}$ and contravariant metric tensor $\overline{g}^{ij}$ of transformed m-th root Finsler space $\overline{F}^n$ are given as*

$$\overline{g}_{ij} = e^{2\alpha}\left((m-1)\frac{A_{ij}}{A^{(1-\frac{2}{m})}} - (m-2)\frac{A_iA_j}{A^{2(1-\frac{1}{m})}}\right) \tag{4}$$

and

$$\overline{g}^{ij} = e^{-2\alpha}\left(A^{(1-\frac{2}{m})}(m-1)A^{ij} + \frac{(m-2)}{(m-1)}\frac{y^iy^j}{A^{\frac{2}{m}}}\right). \tag{5}$$

The geodesic curves of $\overline{F}^n$ are characterized by the system of equations $\frac{d^2x^i}{dt^2} + \overline{G}^i\left(x, \frac{dx}{dt}\right) = 0$, where $\overline{G}^i = \frac{1}{4}\overline{g}^{il}\left\{\left[\overline{F}^2\right]_{x^k y^l} y^k - \left[\overline{F}^2\right]_{x^l}\right\}$ are called the spray coefficients of $\overline{F}$.

The spray coefficients $\overline{G}^i$ of $\overline{F}^n$ can be written as

$$\begin{aligned}\overline{G}^i &= \frac{1}{4}e^{-2\alpha}g^{il}\left\{\frac{\partial^2(e^{2\alpha}F^2)}{\partial x^k\partial y^l}y^k - \frac{\partial(e^{2\alpha}F^2)}{\partial x^l}\right\}\\ &= \frac{1}{4}e^{-2\alpha}g^{il}\left\{e^{2\alpha}(F^2{}_{x^ky^l}y^k - F^2_{x^l}) + 2FF_{y^l}e^{2\alpha}2\alpha_{x^k}y^k - F^2e^{2\alpha}2\alpha_{x^l}\right\}\end{aligned}$$

i. e., $$\overline{G}^i = G^i + \frac{1}{2}g^{il}\left\{2FF_{y^l}\alpha_{x^k}y^k - F^2\alpha_{x^l}\right\},$$

where G^i's are given by (see [20]):

$$G^i = \frac{A^{il}}{2(m-1)}\left\{\frac{\partial A_l}{\partial x^k}y^k - \frac{1}{m}\frac{\partial A}{\partial x^l}\right\}. \tag{6}$$

Further, in view of equation (5) we have

$$\overline{G}^i = G^i + \frac{1}{2}\left\{\frac{F^{m-2}}{(m-1)}A^{ij} + \frac{(m-2)}{(m-1)}\frac{y^iy^j}{F^2}\right\}\left\{2Fl_j\alpha_{x^k}y^k - F^2\alpha_{x^j}\right\}$$

$$=G^i + \frac{1}{2}\left\{\frac{2y^i}{(m-1)}\alpha_{x^k}y^k - \frac{F^m}{(m-1)}A^{ij}\alpha_{x^j} + \frac{(m-2)}{(m-1)}\frac{y^iy^j}{F}2l_j\alpha_{x^k}y^k - \frac{(m-2)}{(m-1)}y^iy^j\alpha_{x^j}\right\}$$

$$=G^i + \frac{1}{2}\left\{\frac{2y^i}{(m-1)}\alpha_{x^k}y^k + \frac{(m-2)}{(m-1)}y^i\alpha_{x^j}y^j - \frac{F^m}{(m-1)}A^{ij}\alpha_{x^j}\right\}.$$

Here, we have used $A^{ij}l_j = \frac{y^i}{F^{m-1}}$. Thus

$$\overline{G}^i = G^i + \frac{\alpha_{x^j}}{2(m-1)}\left\{my^iy^j - AA^{ij}\right\}. \tag{7}$$

Proposition 3.2. *The spray coefficients $\overline{G}^i$ of the transformed Finsler space $\overline{F}^n$ are given by (7), where G^i are spray coefficients of Finsler space F^n.*

In view of equation (6), G^i are rational functions of y (see [20]). Hence from equation (7), we have the following corollary.

Corollary 3.3. *The spray coefficients $\overline{G}^i$ of the transformed Finsler space $\overline{F}^n$ are rational functions with respect to y.*

4. Locally dually flat conformally transformed m-th root metric

In [1], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they studied the information geometry on Riemannian manifolds. In Finsler geometry, Shen extended the notion of locally dually flatness for Finsler metrics [14]. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which play a very important role in studying flat Finsler information structure. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and information theory [1, 2].

A transformed Finsler metric $\overline{F} = \overline{F}(x, y)$ on a manifold M^n is said to be locally dually flat if at any point there is a standard coordinate system (x^i, y^i) in TM such that $[\overline{F}^2]_{x^ky^l}y^k = 2[\overline{F}^2]_{x^l}$. In this case, the coordinate (x^i) is called an adapted local coordinate system [14]. Every locally Minkowskian metric is locally dually flat.

Consider the conformal transformation $\overline{F} = e^\alpha F$, where F is an m-th root metric. Since $\overline{F}^2_{x^k} = 2e^{2\alpha}\alpha_k F^2 + e^{2\alpha}F^2_{x^k} = e^{2\alpha}\left[F^2_{x^k} + 2F^2\alpha_k\right]$, where $\alpha_k := \frac{\partial\alpha}{\partial x^k}$, we have $\overline{F}^2_{x^ky^l} = e^{2\alpha}\left[F^2_{x^ky^l} + 2F^2_{y^l}\alpha_k\right]$ and $\overline{F}^2_{x^ky^l}y^k = e^{2\alpha}\left[F^2_{x^ky^l}y^k + 2Fl_l\alpha_ky^k\right]$. Therefore $2\overline{F}^2_{x^l} - \overline{F}^2_{x^ky^l}y^k = e^{2\alpha}\left[2F^2_{x^l} + 4F^2\alpha_l - F^2_{x^ky^l}y^k - 2y_l\alpha_0\right]$, where $\alpha_0 := \alpha_ky^k$.

Thus $\overline{F}$ is locally dually flat metric if and only if

$$A^{\frac{(2-2m)}{m}}\left\{m(\frac{2}{m}-1)A_lA_0+AA_{0l}-2AA_{x^l}\right\}+m\left(\alpha_0y_l-2A^{\frac{2}{m}}\alpha_l\right)=0, \tag{8}$$

where $A_0 := A_{x^k}y^k$ and $A_{0l} := A_{x^ky^l}y^k$.

The equation (8) can be rewritten as

$$A_{x^l}=\frac{1}{2A}\left[\left\{m(\frac{2}{m}-1)A_lA_0+AA_{0l}\right\}+m\left(\alpha_0y_l-2A^{\frac{2}{m}}\alpha_l\right)A^{\frac{(2m-2)}{m}}\right]. \tag{9}$$

THEOREM 4.1. *Let $\overline{F}$ be a conformally transformed m-th root Finsler metric on a manifold M^n. Then, $\overline{F}$ is locally dually flat metric if and only if* (9) *holds.*

COROLLARY 4.2. *If F is locally dually flat metric then the conformally transformed m-th root Finsler metric $\overline{F}$ is also locally dually flat if and only if conformal transformation is homothetic.*

Proof. In view of [20], F is locally dually flat if and only if $A_{x^l} = \frac{1}{2A}\left[(\frac{2}{m}-1)A_lA_0 + AA_{0l}\right].$
Hence $\overline{F}$ is locally dually flat if and only if

$$\alpha_0y_l - 2A^{\frac{2}{m}}\alpha_l = 0. \tag{10}$$

Contracting (10) with y^l, we have $\alpha_0F^2 - 2F^2\alpha_0 = 0$, i.e. $\alpha_0 = 0$.

Hence from the equation (10), $\alpha_l = 0$, i.e. $\frac{\partial\alpha}{\partial x^l} = 0$. So α is constant and the transformation is homothetic. The converse is trivial. □

5. Conformally transformed Einstein m-th root metric

In Finsler geometry, the flag curvature is an analogue of sectional curvature in Riemannian geometry. A natural problem is to study and characterize Finsler metrics of constant flag curvature. There are only three local Riemannian metrics of constant sectional curvature, up to a scaling. However there are lots of non-Riemannian Finsler metrics of constant flag curvature. For example, the Funk metric is positively complete and non-reversible with $K = \frac{-1}{4}$ and the Hilbert-Klein metric is complete and reversible with $K = -1$ [7,8].

For a Finsler metric $\overline{F}$, the Riemann curvature $\overline{R}_y : T_xM \to T_xM$ is defined by $\overline{R}_y(u) = \overline{R}^i_k(x,y)u^k\frac{\partial}{\partial x^i}, \quad u = u^k\frac{\partial}{\partial x^i}$, where

$$\overline{R}^i_k = 2\frac{\partial\overline{G}^i}{\partial x^k} - y^j\frac{\partial^2\overline{G}^i}{\partial x^j\partial y^k} + 2\overline{G}^j\frac{\partial^2\overline{G}^i}{\partial y^j\partial y^k} - \frac{\partial\overline{G}^i}{\partial y^j}\frac{\partial\overline{G}^j}{\partial y^k}. \tag{11}$$

The Finsler metric $\overline{F}$ is said to be of scalar flag curvature if there is a scalar function $\overline{K} = \overline{K}(x,y)$ such that

$$\overline{R}^i_k = \overline{K}(x,y)\overline{F}^2\left\{\delta^i_k - \frac{\overline{F}_{y_k}y^i}{\overline{F}}\right\}. \tag{12}$$

Moreover $\overline{F}$ is said to be of constant flag curvature if $\overline{K}$ in the equation (12) is constant.

The Ricci curvature of a transformed Finsler metric $\overline{F}$ on a manifold is a scalar function $\overline{\mathrm{Ric}} : TM \to R$, defined to be the trace of $\overline{R}_y$, i.e., $\overline{\mathrm{Ric}}(y) := \overline{R}^k_k(x, y)$ satisfying the homogeneity $\overline{\mathrm{Ric}}(\lambda y) = \lambda^2 \overline{\mathrm{Ric}}(y)$, for $\lambda > 0$. A Finsler metric $\overline{F}$ on an n-dimensional manifold M^n is called an Einstein metric if there is a scalar function $\overline{K} = \overline{K}(x)$ on M^n such that $\overline{\mathrm{Ric}} = \overline{K}(n-1)\overline{F}^2$. A Finsler metric is said to be Ricci-flat if $\overline{\mathrm{Ric}} = 0$. By formula (11) and Corollary 3.3, we get the following lemma.

Lemma 5.1. *$\overline{R}^i_k$ and $\overline{\mathrm{Ric}} = R^k_k$ are rational functions in y.*

Proposition 5.2. *Let $\overline{F}$ be a non-Riemannian conformally transformed m-th root Finsler metric with $m > 2$ on a manifold M^n. If $\overline{F}$ is an Einstein metric, then it is Ricci-flat.*

Proof. If $\overline{F}$ is an Einstein metric, $\overline{\mathrm{Ric}} = \overline{K}(n-1)\overline{F}^2$, and $\overline{F}^2$ is an irrational function, as $m > 2$ and $\overline{\mathrm{Ric}}$ are rational function of y. Therefore $\overline{K} = 0$ and $\overline{\mathrm{Ric}} = 0$. □

Corollary 5.3. *Let $\overline{F} = e^{\alpha(x)}F$ be a non-Riemannian transformed m-th root Finsler metric with $m > 2$ on a manifold M^n. If $\overline{F}$ is of constant flag curvature $\overline{K}$, then $\overline{K} = 0$.*

6. Conformally transformed m-th root metric with isotropic E-curvature

If $\overline{G}^i$ are spray coefficients of a Finsler space $\overline{F}^n$ then the Berwald curvature of $\overline{F}^n$ is defined as

$$\overline{B}^i_{jkl} = \frac{\partial^3 \overline{G}^i}{\partial y^j \partial y^k \partial y^l}.$$

A transformed Finsler metric $\overline{F}$ is called a Berwald metric if spray coefficients $\overline{G}^i$ are quadratic in $y \in T_xM$, for any $x \in M^n$ or equivalently, the Berwald curvature vanishes. The E-curvature is defined by the trace of the Berwald curvature, i.e., $\overline{E}_{ij} = \frac{1}{2}\overline{B}^m_{mij}$. A Finsler metric $\overline{F}$ on an n-dimensional manifold M^n is said to be isotropic mean Berwald curvature or of isotropic E-curvature if

$$\overline{E}_{ij} = \frac{c(n+1)}{2\overline{F}}\overline{h}_{ij}, \tag{13}$$

where $\overline{h}_{ij} = \overline{g}_{ij} - \overline{g}_{ip}y^p\overline{g}_{jq}y^q$ is the angular metric and $c = c(x)$ is a scalar function on M^n. If $c = 0$, then $\overline{F}$ is called weakly Berwald metric.
From equation (4), we have $\overline{g}_{ij} = e^{2\alpha}g_{ij} = e^{2\alpha}\left((m-1)\frac{A_{ij}}{F^{m-2}} - (m-2)\frac{A_iA_j}{F^{2(m-1)}}\right)$.
The angular metric is given by

$$\overline{h}_{ij} = \overline{g}_{ij} - \overline{l}_i\overline{l}_j = e^{2\alpha}\left((m-1)\frac{A_{ij}}{F^{m-2}} - (m-1)\frac{A_iA_j}{F^{2(m-1)}}\right). \tag{14}$$

From equation (13) and (14), we have

$$\overline{E}_{ij} = \frac{(n+1)c}{2\overline{F}}e^{2\alpha}(m-1)\left(\frac{A_{ij}}{F^{m-2}} - \frac{A_iA_j}{F^{2(m-1)}}\right)$$

$$\begin{aligned}\overline{E}_{ij} &= \frac{(n+1)c}{2}e^{\alpha}(m-1)\left(\frac{A_{ij}}{F^{m-1}} - \frac{A_iA_j}{F^{(2m-1)}}\right) \quad \left(\overline{F} = e^{\alpha}F\right)\\ &= \frac{(n+1)c}{2}e^{\alpha}(m-1)F\left(\frac{A_{ij}}{A} - \frac{A_iA_j}{A^2}\right) \quad \left(F = A^{\frac{1}{m}}\right)\\ &= \frac{(n+1)}{2A^2}A^{\frac{1}{m}}e^{\alpha}(m-1)c\left\{(A_{ij}A - A_iA_j)\right\}. \end{aligned} \tag{15}$$

In view of equation (7), we see that $\overline{E}_{ij}$ are rational functions with respect to y. Thus from equation (15), we have either $c = 0$ or $(A_{ij}A - A_iA_j) = 0$. Suppose that $c \neq 0$. Contracting the previous equation with A^{jk} yields $A\delta_i^k - A_iy^k = 0$, which implies that $nA = A$. This contradicts our assumption $n > 1$. Therefore $c = 0$ and consequently $\overline{E}_{ij} = 0$. Thus we have

Proposition 6.1. *Let $\overline{F} = e^cF$ be the conformal change of Finsler metric F. Suppose that $\overline{F}$ has isotropic mean Berwald curvature. Then it reduces to a weakly Berwald metric.*

Acknowledgement. The authors would like to thank the referees for their valuable comments and suggestions. The second author is thankful to the DST-CIMS, BHU for financial support as Senior Research Fellow.

References

[1] S.-I. Amari, H. Nagaoka, *Methods of Information Geometry*, AMS Translation of Math. Monographs, Oxford University Press, (2000).

[2] S.-I. Amari, *Differential-geometrical Methods in Statistics*, Springer Lecture Notes in Statistics, Springer-Verlag, (1985).

[3] P. L. Antonelli, R. S. Ingarden, M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Academic Publishers Netherlands, **58**, (1993).

[4] G. S. Asanov, *Finslerian Extension of General Relativity*, Reidel, Dordrecht, (1984).

[5] V. Balan, *Notable submanifolds in Berwald-Moór spaces*, BSG Proc. **17**, Geometry Balkan Press, (2010), 21–30.

[6] V. Balan, *CMC and minimal surfaces in Berwald-Moór spaces*, Hypercomplex Numbers Geom. Phys., **3 (6-2)** (2006), 113–122.

[7] S. S. Chern, Z. Shen, *Riemann-Finsler Geometry*, Nankai Tracts in Mathematics, World Scientific, **6** (2004).

[8] P. Funk, *Über Geometrien, bei denen die Geraden die Kürzesten sind*, Math. Ann., **101** (1929), 226–237.

[9] M. Hashiguchi, *On conformal transformation of Finsler metrics*, J. Math. Kyoto University, **16** (1976), 25–50.

[10] B. Li, Z. Shen, *Projectively flat fourth root Finsler metrics*, Can. Math. Bull., **55** (2012), 138–145.

[11] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaiseisha Press, Saikawa, Otsu, Japan, (1986).

[12] M. Matsumoto, S. Numata, *On Finsler spaces with a cubic metric*, Tensor, N. S., **38** (1982), 153–162.

[13] D. G. Pavlov, *Space-Time Structure, Algebra and Geometry*, Collected papers, TETRU, (2006).

[14] Z. Shen, *Riemann–Finsler geometry with applications to information geometry*, Chin. Ann. Math., **27** (2006), 73–94.

[15] H. Shimada, *On Finsler spaces with the metric* $L = \sqrt[m]{a_{i_1\dots i_m}y^{i_1}\dots y^{i_m}}$, Tensor N. S., **33** (1979), 365–372.

[16] A. Tayebi, B. Najafi, *On m-th root Finsler metrics*, J. Geom. Phys., **61**, (2011), 1479–1484.

[17] A. Tayebi, E. Peyghan, M. Shahbazi Nia, *On generalized m-th root Finsler metrics*, Linear Algebra Appl., **437**, (2012), 675–683.

[18] A. Tayebi, M. Razgordani, *On conformally flat fourth root* (α, β)*-metrics*, Differ. Geom. Appl., **62**, (2019), 253–266.

[19] A. Tayebi, M. Shahbazi Nia, *A new class of projectively flat Finsler metrics with constant flag curvature* $K = 1$, Differ. Geom. Appl., **41**, (2015), 123–133.

[20] Y. Yu, Y. You, *On Einstein m-th root metrics*, Differ. Geom. Appl., **28**, (2010), 290–294.

(received 09.10.2018; in revised form 22.04.2019; available online 08.10.2019)

DST-Centre for Interdisciplinary Mathematical Sciences, Institute of Science, Banaras Hindu University, Varanasi-221005, India

E-mail: banktesht@gmail.com

DST-Centre for Interdisciplinary Mathematical Sciences, Institute of Science, Banaras Hindu University, Varanasi-221005, India

E-mail: mvermamath@gmail.com