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SG_δ-SELECTIVE SEPARABILITY

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Abstract. A topological space X is called G_δ-selectively (resp., SG_δ-selectively) separable if for every sequence $(D_n : n \in \omega)$ of dense G_δ subsets of X, one can pick finite subsets $F_n \subset D_n$ such that $\bigcup_{n\in\omega} F_n$ is dense (resp., dense and G_δ). In this paper we introduce and study these kinds of spaces.

1. Introduction

Let X be a topological space. We denote the families of dense or dense G_δ subspaces of X respectively by $\mathcal{D}X$ or $\mathcal{DG}X$. By ω, $\mathbb{S}$, and $\mathbb{R}$ we denote the set of nonnegative integers, the Sorgenfrey line, and the real line, respectively. A topological space X is called *selectively separable* (also called *M-separable*) [4, 5] if for every sequence $(O_n : n \in \omega)$ of elements of $\mathcal{D}X$ there is a sequence $(T_n : n \in \omega)$ such that for each n, T_n is a finite subset of O_n, and $\bigcup_{n\in\omega} T_n$ is an element of $\mathcal{D}X$. This notion was first introduced by Scheepers [14]. Also, X is called *R-separable* [4] if for any sequence $(D_n)_{n\in\omega}$ of $\mathcal{D}X$ one can pick one-point subsets $F_n \subseteq D_n$ such that $\cup_{n\in\omega} F_n$ is an element of $\mathcal{D}X$. A family B of open sets in X is called a *π-base* for X if every nonempty open set in X contains a nonempty element of B. The *π-weight* of a space X, $\pi w(X)$, is the smallest cardinal of any π-base for X. If X is a Tychonoff space, and Y is a dense subspace of X then $\pi w(Y) = \pi w(X)$ [12].

A space X has *countable fan tightness* [3], if whenever $x \in \overline{A_n}$ for all $n \in \omega$, one can choose finite subsets $F_n \subset A_n$ so that $x \in \overline{\bigcup\{F_n : n \in \omega\}}$. It is natural to say that X has *countable fan tightness with respect to dense and G_δ-sets* if this statement is true for $A_n \in \mathcal{DG}X$. A continuous mapping $f : X \longrightarrow Y$ which is onto is called *irreducible* if $f(A) \neq Y$ for every proper closed subset $A \subset X$. A *paratopological group* is a group G equipped with a topology such that the group operation $(x, y) \mapsto xy$ from $G \times G \to G$ is a continuous mapping. A paratopological group G in which the mapping $x \mapsto x^{-1}$ from G to G is continuous is called a *topological group*.

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Proposition 1.1. *Let G be a Hausdorff topological group. Then the following are equivalent.*
(i) G is second countable;

(ii) G is a first countable space and every dense subset of X is separable;

(iii) G is an R-separable space which is first countable.

Proof. Clearly (i)$\Rightarrow$(iii), (iii)$\Rightarrow$ (ii) and (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(i) Let G be a first countable space and every dense subset of X is separable. According to Birkoff-Kakutani's theorem, G is a metric space and so by hypothesis, it is second countable. Thus, we are done. $\square$

Corollary 1.2. *Let G be a countable Hausdorff topological group; then G is selectively separable if and only if it is first countable.*

G. Gruenhage and M. Sakai [11, Example 2.13] showed that there is a selective separable, countable and dense subset S of $\{0,1\}^c$ such that the group generated by S which is not first countable is not selectively separable.

Remark 1.3. The Sorgenfrey line $\mathbb{S}$ is an example of a paratopological additive group which is not a topological group. $\mathbb{S}$ is not second countable but it is first countable, every dense subset of $\mathbb{S}$ is separable and $\mathbb{S}$ is R-separable since a set is a dense subset of $\mathbb{S}$ if and only if it is dense in $\mathbb{R}$. The space $\mathbf{Q}$ of rational numbers with the Sorgenfrey topology is a metrizable paratopological non-topological group [13], and it satisfies in conditions (i), (ii) and (iii) the Proposition 1.1.

2. Main results

In this section, we will introduce and investigate G_δ-selectively separable spaces and SG_δ-selectively separable spaces.

Definition 2.1. A topological space X is called *G_δ-selectively separable* if for every sequence $(D_n : n \in \omega)$ of elements of $\mathcal{DG}X$, one can pick finite subsets $F_n \subset D_n$ such that $\bigcup_{n\in\omega} F_n$ is an element of $\mathcal{D}X$.

Definition 2.2. Let X be a topological space. If for every sequence $(D_n : n \in \omega)$ of elements of $\mathcal{DG}X$, one can pick finite subsets $F_n \subset D_n$ so that $\bigcup_{n\in\omega} F_n \in \mathcal{DG}X$, then X is called *SG_δ-selectively separable.*

Clearly, every selectively separable space is a G_δ-selectively separable space and every SG_δ-selectively separable space is a G_δ-selectively separable space. By [5, Proposition 2.3] every topological space of countable π-weight is selectively separable, so we have the following result.

Proposition 2.3. *Each space with countable π-weight is G_δ-selectively separable.*

Recall that a topological space X is a *Baire space* if the intersection of any sequence of dense open subsets of X is dense.

Proposition 2.4. *Let X be an SG_δ-selectively separable Baire space which is a T_1-space. Then, the set of isolated points of X is dense and countable.*

Proof. Since $X \in \mathcal{DG}X$, there exists a countable dense subset E of X which is a G_δ-set in X. Let $I(X)$ denote the set of all isolated points of X. If $A = E \setminus I(X)$ is nonempty, then the countable set $I(X) = E \cap (\bigcap_{a \in A} X \setminus \{a\})$ is dense in X since X is a Baire space and E is a G_δ-set in X. □

Example 2.5. By Proposition 2.4, every selectively separable space X which is a Baire space and the set of isolated points of X is not dense is an example of a G_δ-selectively separable space which is not an SG_δ-selectively separable space. $\mathbb{R}$ and $\mathbb{S}$ have these properties.

Remark 2.6. Following Bourbaki [6], we say that a subset A of a topological space X is *locally closed* in X if A is the intersection of an open subset of X and a closed subset of X. A countable intersection of locally closed sets is called *σ-locally closed* [2]. X is called DG_δ-space if every subset of X is σ-locally closed. From [2, Theorem 2.4] it follows that X is a DG_δ-space if and only if every dense subset of X is G_δ. Thus, we observe that in the class of DG_δ-spaces which are T_1-spaces the concepts of selective separability, G_δ-selective separability and SG_δ-selective separability coincide. Clearly, every countable T_1-space is a DG_δ-space. G. Gruenhage and M. Sakai [11, Example 3.2] showed that under CH, there are two countable R-separable spaces whose product is not selectively separable. Thus, this example shows that under CH, the product of two G_δ-selectively (resp., SG_δ-selectively) separable spaces need not be a G_δ-selectively (resp., an SG_δ-selectively) separable space.

Proposition 2.7. *Assume that X is G_δ-selectively (resp., SG_δ-selectively) separable; then every dense G_δ subspace of X is G_δ-selectively (resp., SG_δ-selectively) separable.*

Proof. Let Y be a dense G_δ-subspace of X and $(D_n : n \in \omega)$ be a sequence of dense G_δ-subspaces of Y. Thus, $(D_n : n \in \omega)$ is a sequence of elements of $\mathcal{DG}X$, so there are finite $F_n \subset D_n$ such that $D = \bigcup\{F_n : n \in \omega\}$ is dense (resp., dense and G_δ) in X, i.e., $D \in \mathcal{D}X$ ($D \in \mathcal{DG}X$). Thus, Y is G_δ-selectively (resp., SG_δ-selectively) separable. □

Let $F(X)$ denote the set of all functions from X to $\mathbb{R}$ and the set of points at which $f \in F(X)$ is continuous is denoted by $C(f)$. Recall that a topological space X is called *Volterra* [10] if for all $f, g \in F(X)$ such that $C(f), C(g) \in \mathcal{DG}X$ we have that $C(f) \cap C(g)$ is dense in X. An algebraic characterization of Volterra spaces is given in [1]. Now we show that in the class of Volterra spaces the converse of Proposition 2.7 hold.

Corollary 2.8. *Let X be a Volttera space and $D \in \mathcal{DG}(X)$. Then X is G_δ-selectively (resp., SG_δ-selectively) separable if and only if D is G_δ-selectively (resp., SG_δ-selectively) separable.*

Proof. Let D be G_δ-selectively (resp., SG_δ-selectively) separable and $(D_n : n \in \omega)$ be a sequence of dense G_δ-subspaces of X. Then for each $n \in \omega$, $D \cap D_n \in \mathcal{DG}(X)$ since by [9] a space X is Volterra if and only if the intersection of any two dense G_δ-sets in X is dense. Thus, there are finite $F_n \subset D_n \cap D$ such that $E = \bigcup\{F_n : n \in \omega\}$ is dense (resp., dense and G_δ) in D. Clearly $E \in \mathcal{D}(X)$ (resp., $E \in \mathcal{DG}(X)$), and so X is G_δ-selectively (resp., SG_δ-selectively) separable. The converse follows from Proposition 2.7. □

THEOREM 2.9. *Every space having a G_δ-selectively (resp., an SG_δ-selectively) separable, open and dense subspace is G_δ-selectively (resp., SG_δ-selectively) separable.*

Proof. Since every dense open subspace of X intersected with a dense (resp., dense and G_δ) subspace of X is still dense (resp., dense and G_δ) in X, it is straightforward. □

REMARK 2.10. It is well known that every open subset of a selectively separable space is selectively separable. It is easy to prove that every open subset of a G_δ-selectively (resp., an SG_δ-selectively) separable space is G_δ-selectively (resp., SG_δ-selectively) separable and by the following example we show that this is not true for G_δ-sets.

EXAMPLE 2.11. Since $\omega \in \mathcal{DG}(\beta\omega)$ is the set of all isolated points of $\beta\omega$, every dense and G_δ subset of $\beta\omega$ contains ω. Thus, $\beta\omega$ is SG_δ-selectively separable and so it is G_δ-selectively separable. The G_δ-set $\omega^* = \beta\omega \setminus \omega$ admits a family of $\mathfrak{c}$ disjoint open sets, where $\mathfrak{c}$ is the cardinality of the continuum. Thus, ω^* is not separable and so it is not G_δ-selectively separable.

LEMMA 2.12. *Let X be a DG_δ-space which is T_1. Then, X is G_δ-selectively separable if and only if for every decreasing sequence $(D_n : n \in \omega)$ of elements of $\mathcal{DG}X$, there exist finite sets $F_n \subset D_n$ such that $\bigcup_{n\in\omega} F_n$ is dense in X.*

Proof. By Remark 2.6, in the class of DG_δ-spaces which are T_1-spaces the concepts of selective separability and G_δ-selective separability coincide. Thus, the result follows from [11, Lemma 2.1]. □

By slight changes in the proof of Lemma 2.12, we have the following result.

LEMMA 2.13. *Let X be a DG_δ-space which is T_1. Then, X is SG_δ-selectively separable if for every decreasing sequence $(D_n : n \in \omega)$ of dense G_δ-subspaces of X there are finite sets $F_n \subset D_n$ such that $\bigcup_{n\in\omega} F_n$ is dense and G_δ in X.*

THEOREM 2.14. *Let Y be a dense and open (resp., G_δ) subspace of X (resp., where X is a Volterra space). If Y has a countable open cover consisting of G_δ-selectively (resp., SG_δ-selectively) separable subsets, then X is G_δ-selectively (resp., SG_δ-selectively) separable.*

Proof. For each $n \in \omega$, let V_n be an open subset of Y which is a G_δ-selectively (resp., an SG_δ-selectively) separable subset of Y and $Y = \bigcup_{n\in\omega} V_n$. For each $n \in \omega$, let $W_n = V_n \setminus \overline{\bigcup_{i\leq n-1} V_i}$. Then, $\{W_n : n \in \omega\}$ is a disjoint family of G_δ-selectively (resp., SG_δ-selectively) separable open subsets of Y by Remark 2.10, and so it is easily seen that $W = \bigcup_{n\in\omega} W_n$ is G_δ-selectively (resp., SG_δ-selectively) separable. Thus, Y is G_δ-selectively (resp., SG_δ-selectively) separable since W is open and dense in Y. Therefore by Theorem 2.9 (resp., Corollary 2.8) X is G_δ-selectively (resp., SG_δ-selectively) separable since Y is a dense and open (resp., G_δ) subspace of X. □

Corollary 2.15. *X is a G_δ-selectively (resp,. an SG_δ-selectively) separable space if and only if the set $I(X)$ of isolated points of X is countable and $X \setminus \overline{I(X)}$ is G_δ-selectively (resp., SG_δ-selectively) separable.*

A map $f : X \to Y$ is called *feebly open* if for every nonempty open subset U of X, there is a nonempty open subset V of Y such that $V \subseteq f(U)$. It seems that the idea of a feebly open map was first introduced in [8].

Proposition 2.16. *Let X be a G_δ-selectively separable space. Then,*
(i) every closed irreducible continuous image of X is G_δ-selectively separable;

(ii) every feebly open continuous image of X is G_δ-selectively separable.

Proof. Let $f : X \to Y$ be a continuous onto function. If f is either feebly open or closed irreducible, then the inverse image of any dense subset of Y is dense in X. Thus, for any sequence $(D_n : n \in \omega)$ of elements of $\mathcal{DG}Y$, we have $E_n = f^{-1}(D_n) \in \mathcal{DG}X$ for all $n \in \omega$, and so we can find, for every $n \in \omega$, a finite $F_n \subseteq E_n$ such that $\bigcup_{n\in\omega} F_n \in \mathcal{D}X$. Hence $G_n = f(F_n)$ is a finite subset of D_n for every $n \in \omega$ and $\bigcup_{n\in\omega} G_n$ is a dense subspace of Y. □

Proposition 2.17. *Let Y be a separable, dense G_δ-subspace of a Volterra space X. Then X is G_δ-selectively separable if and only if Y has countable fan tightness with respect to dense G_δ-sets.*

Proof. Necessity. By Corollary 2.8, Y is G_δ-selectively separable and so Y has countable fan tightness with respect to dense G_δ-sets. Sufficiency. Let $S = \{s_n : n \in \omega\}$ be a dense subset of Y and $(D_n : n \in \omega)$ be a sequence of elements of $\mathcal{DG}X$. Thus, for each $n \in \omega$ $D_n \cap Y \in \mathcal{DG}(Y)$ since X is a Volterra space [9]. Pick a disjoint family $\mathcal{T} = \{T_n : n \in \omega\}$ of infinite subset of ω such that $\bigcup \mathcal{T} = \omega$. For any $n \in \omega$, we have $s_n \in Y \cap (\bigcap_{m\in T_n} \overline{D_m \cap Y})$ and so there is a finite subset F_m of $D_m \cap Y$ for every $m \in T_n$ such that $s_n \in Y \cap (\overline{\bigcup_{m\in T_m} F_m})$ since Y has countable fan tightness with respect to dense and G_δ-sets. Thus, F_n is a finite subset of D_n for every $n \in \omega$ and $\bigcup_{n\in\omega} F_n$ is dense in X and so X is G_δ-selectively separable. □

By slight changes in the proof of Proposition 2.17, the following result is obtained.

Proposition 2.18. *A separable space X is G_δ-selectively separable if and only if X has countable fan tightness with respect to dense G_δ-sets.*

Proposition 2.19. *For a T_1-space X, the following statements are equivalent.*
(i) X is hereditarily selectively separable;

(ii) X is hereditarily separable and all countable subspaces of X are selectively separable;

(iii) X is hereditarily G_δ-selectively separable,

Proof. (i)⇔(i) See [4, Proposition 16]. (i)⇒(iii) It is obvious.

(iii)⇒(ii) Let Y be a subspace of X. Then Y is separable since Y is G_δ-selectively separable. Clearly, every countable T_1-space is a DG_δ-space and by Remark 2.6, in the class of DG_δ-spaces which are T_1-spaces the concepts of selective separability and G_δ-selective separability coincide and so every countable subspace of X is selectively separable. □

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