Abstract. A topological space $X$ is called $G_δ$-selectively (resp., $SG_δ$-selectively) separable if for every sequence $(D_n : n ∈ ω)$ of dense $G_δ$ subsets of $X$, one can pick finite subsets $F_n ⊂ D_n$ such that $S_n ∈ ω F_n$ is dense (resp., dense and $G_δ$). In this paper we introduce and study these kinds of spaces.

1. Introduction

Let $X$ be a topological space. We denote the families of dense or dense $G_δ$ subspaces of $X$ respectively by $DX$ or $DGX$. By $ω$, $S$, and $R$ we denote the set of nonnegative integers, the Sorgenfrey line, and the real line, respectively. A topological space $X$ is called selectively separable (also called $M$-separable) [4, 5] if for every sequence $(O_n : n ∈ ω)$ of elements of $DX$ there is a sequence $(T_n : n ∈ ω)$ such that for each $n$, $T_n$ is a finite subset of $O_n$, and $∪_{n∈ω} T_n$ is an element of $DX$. This notion was first introduced by Scheepers [14]. Also, $X$ is called $R$-separable [4] if for any sequence $(D_n)_{n∈ω}$ of $DX$ one can pick one-point subsets $F_n ⊆ D_n$ such that $∪_{n∈ω} F_n$ is an element of $DX$. A family $B$ of open sets in $X$ is called a $π$-base for $X$ if every nonempty open set in $X$ contains a nonempty element of $B$. The $π$-weight of a space $X$, $πw(X)$, is the smallest cardinal of any $π$-base for $X$. If $X$ is a Tychonoff space, and $Y$ is a dense subspace of $X$ then $πw(Y) = πw(X)$ [12].

A space $X$ has countable fan tightness [3], if whenever $x ∈ A_n$ for all $n ∈ ω$, one can choose finite subsets $F_n ⊂ A_n$ so that $x ∈ ∪ \{F_n : n ∈ ω\}$. It is natural to say that $X$ has countable fan tightness with respect to dense and $G_δ$-sets if this statement is true for $A_n ∈ DGX$. A continuous mapping $f : X → Y$ which is onto is called irreducible if $f(A) ≠ Y$ for every proper closed subset $A ⊂ X$. A paratopological group is a group $G$ equipped with a topology such that the group operation $(x, y) ↦ xy$ from $G × G → G$ is a continuous mapping. A paratopological group $G$ in which the mapping $x ↦ x^{-1}$ from $G$ to $G$ is continuous is called a topological group.

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Proposition 1.1. Let $G$ be a Hausdorff topological group. Then the following are equivalent.

(i) $G$ is second countable;

(ii) $G$ is a first countable space and every dense subset of $X$ is separable;

(iii) $G$ is an $R$-separable space which is first countable.

Proof. Clearly (i)$\Rightarrow$(iii), (iii)$\Rightarrow$(ii) and (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(i) Let $G$ be a first countable space and every dense subset of $X$ is separable. According to Birkhoff-Kakutani’s theorem, $G$ is a metric space and so by hypothesis, it is second countable. Thus, we are done. □

Corollary 1.2. Let $G$ be a countable Hausdorff topological group; then $G$ is selectively separable if and only if it is first countable.

G. Gruenhage and M. Sakai [11, Example 2.13] showed that there is a selectively separable, countable and dense subset $S$ of $\{0,1\}^\mathbb{C}$ such that the group generated by $S$ which is not first countable is not selectively separable.

Remark 1.3. The Sorgenfrey line $\mathbb{S}$ is an example of a paratopological additive group which is not a topological group. $\mathbb{S}$ is not second countable but it is first countable, every dense subset of $\mathbb{S}$ is separable and $\mathbb{S}$ is $R$-separable since a set is a dense subset of $\mathbb{S}$ if and only if it is dense in $\mathbb{R}$. The space $\mathbb{Q}$ of rational numbers with the Sorgenfrey topology is a metrizable paratopological non-topological group [13], and it satisfies in conditions (i), (ii) and (iii) the Proposition 1.1.

2. Main results

In this section, we will introduce and investigate $G_\delta$-selectively separable spaces and $SG_\delta$-selectively separable spaces.

Definition 2.1. A topological space $X$ is called $G_\delta$-selectively separable if for every sequence $(D_n: n \in \omega)$ of elements of $\mathcal{D}(G^X)$, one can pick finite subsets $F_n \subset D_n$ such that $\bigcup_{n \in \omega} F_n$ is an element of $\mathcal{D}(X)$.

Definition 2.2. Let $X$ be a topological space. If for every sequence $(D_n: n \in \omega)$ of elements of $\mathcal{D}(G^X)$, one can pick finite subsets $F_n \subset D_n$ so that $\bigcup_{n \in \omega} F_n \in \mathcal{D}(G^X)$, then $X$ is called $SG_\delta$-selectively separable.

Clearly, every selectively separable space is a $G_\delta$-selectively separable space and every $SG_\delta$-selectively separable space is a $G_\delta$-selectively separable space. By [5, Proposition 2.3] every topological space of countable $\pi$-weight is selectively separable, so we have the following result.

Proposition 2.3. Each space with countable $\pi$-weight is $G_\delta$-selectively separable.
Recall that a topological space \( X \) is a **Baire space** if the intersection of any sequence of dense open subsets of \( X \) is dense.

**Proposition 2.4.** Let \( X \) be an \( SG_3 \)-selectively separable Baire space which is a \( T_1 \)-space. Then, the set of isolated points of \( X \) is dense and countable.

**Proof.** Since \( X \in DGX \), there exists a countable dense subset \( E \) of \( X \) which is a \( G_\delta \)-set in \( X \). Let \( I(X) \) denote the set of all isolated points of \( X \). If \( A = E \setminus I(X) \) is nonempty, then the countable set \( I(X) = E \cap (\bigcap_{a \in A} X \setminus \{a\}) \) is dense in \( X \) since \( X \) is a Baire space and \( E \) is a \( G_\delta \)-set in \( X \).

**Example 2.5.** By Proposition 2.4, every selectively separable space \( X \) which is a Baire space and the set of isolated points of \( X \) is not dense is an example of a \( G_\delta \)-selectively separable space which is not an \( SG_3 \)-selectively separable space. \( \mathbb{R} \) and \( \mathbb{S} \) have these properties.

**Remark 2.6.** Following Bourbaki [6], we say that a subset \( A \) of a topological space \( X \) is **locally closed** in \( X \) if \( A \) is the intersection of an open subset of \( X \) and a closed subset of \( X \). A countable intersection of locally closed sets is called **\( \sigma \)-locally closed** [2]. \( X \) is called \( DG_\delta \)-space if every subset of \( X \) is \( \sigma \)-locally closed. From [2, Theorem 2.4] it follows that \( X \) is a \( DG_\delta \)-space if and only if every dense subset of \( X \) is \( G_\delta \). Thus, we observe that in the class of \( DG_\delta \)-spaces which are \( T_1 \)-spaces the concepts of selective separability, \( G_\delta \)-selective separability and \( SG_3 \)-selective separability coincide. Clearly, every countable \( T_1 \)-space is a \( DG_\delta \)-space. G. Gruenhage and M. Sakai [11, Example 3.2] showed that under CH, there are two countable \( R \)-separable spaces whose product is not selectively separable. Thus, this example shows that under CH, the product of two \( G_\delta \)-selectively (resp., \( SG_3 \)-selectively) separable spaces need not be a \( G_\delta \)-selectively (resp., an \( SG_3 \)-selectively) separable space.

**Proposition 2.7.** Assume that \( X \) is \( G_\delta \)-selectively (resp., \( SG_3 \)-selectively) separable; then every dense \( G_\delta \)-subspace of \( X \) is \( G_\delta \)-selectively (resp., \( SG_3 \)-selectively) separable.

**Proof.** Let \( Y \) be a dense \( G_\delta \)-subspace of \( X \) and \( (D_n : n \in \omega) \) be a sequence of dense \( G_\delta \)-subspaces of \( Y \). Thus, \( (D_n : n \in \omega) \) is a sequence of elements of \( DGX \), so there are finite \( F_n \subset D_n \) such that \( D = \bigcup \{ F_n : n \in \omega \} \) is dense (resp., dense and \( G_\delta \)) in \( X \), i.e., \( D \in DGX \). Thus, \( Y \) is a \( G_\delta \)-selectively (resp., \( SG_3 \)-selectively) separable.

Let \( F(X) \) denote the set of all functions from \( X \) to \( \mathbb{R} \) and the set of points at which \( f \in F(X) \) is continuous is denoted by \( C(f) \). Recall that a topological space \( X \) is called Volterra [10] if for all \( f, g \in F(X) \) such that \( C(f), C(g) \in DGX \) we have that \( C(f) \cap C(g) \) is dense in \( X \). An algebraic characterization of Volterra spaces is given in [1]. Now we show that in the class of Volterra spaces the converse of Proposition 2.7 hold.

**Corollary 2.8.** Let \( X \) be a Volterra space and \( D \in DG(X) \). Then \( X \) is \( G_\delta \)-selectively (resp., \( SG_3 \)-selectively) separable if and only if \( D \) is \( G_\delta \)-selectively (resp., \( SG_3 \)-selectively) separable.
Proof. Let \( D \) be \( G_\delta \)-selectively (resp., \( SG_\delta \)-selectively) separable and \( (D_n : n \in \omega) \) be a sequence of dense \( G_\delta \)-subspaces of \( X \). Then for each \( n \in \omega \), \( D \cap D_n \in \mathcal{DG}(X) \) since by [9] a space \( X \) is Volterra if and only if the intersection of any two dense \( G_\delta \)-sets in \( X \) is dense. Thus, there are finite \( F_n \subset D_n \cap D \) such that \( E = \bigcup \{ F_n : n \in \omega \} \) is dense (resp., dense and \( G_\delta \)) in \( D \). Clearly \( E \in \mathcal{D}(X) \) (resp., \( E \in \mathcal{DG}(X) \)), and so \( X \) is \( G_\delta \)-selectively (resp., \( SG_\delta \)-selectively) separable. The converse follows from Proposition 2.7. \( \square \)

**Theorem 2.9.** Every space having a \( G_\delta \)-selectively (resp., an \( SG_\delta \)-selectively) separable, open and dense subspace is \( G_\delta \)-selectively (resp., \( SG_\delta \)-selectively) separable.

Proof. Since every dense open subspace of \( X \) intersected with a dense (resp., dense and \( G_\delta \)) subspace of \( X \) is still dense (resp., dense and \( G_\delta \)) in \( X \), it is straightforward. \( \square \)

**Remark 2.10.** It is well known that every open subset of a selectively separable space is selectively separable. It is easy to prove that every open subset of a \( G_\delta \)-selectively (resp., an \( SG_\delta \)-selectively) separable space is \( G_\delta \)-selectively (resp., \( SG_\delta \)-selectively) separable and by the following example we show that this is not true for \( G_\delta \)-sets.

**Example 2.11.** Since \( \omega \in \mathcal{DG}(\beta \omega) \) is the set of all isolated points of \( \beta \omega \), every dense and \( G_\delta \) subset of \( \beta \omega \) contains \( \omega \). Thus, \( \beta \omega \) is \( SG_\delta \)-selectively separable and so it is \( G_\delta \)-selectively separable. The \( G_\delta \)-set \( \omega^* = \beta \omega \setminus \omega \) admits a family of \( \epsilon \) disjoint open sets, where \( \epsilon \) is the cardinality of the continuum. Thus, \( \omega^* \) is not separable and so it is not \( G_\delta \)-selectively separable.

**Lemma 2.12.** Let \( X \) be a \( DG_\delta \)-space which is \( T_1 \). Then, \( X \) is \( G_\delta \)-selectively separable if and only if for every decreasing sequence \( (D_n : n \in \omega) \) of elements of \( \mathcal{DG}X \), there exist finite sets \( F_n \subset D_n \) such that \( \bigcup_{n \in \omega} F_n \) is dense in \( X \).

Proof. By Remark 2.6, in the class of \( DG_\delta \)-spaces which are \( T_1 \)-spaces the concepts of selective separability and \( G_\delta \)-selective separability coincide. Thus, the result follows from [11, Lemma 2.1]. \( \square \)

By slight changes in the proof of Lemma 2.12, we have the following result.

**Lemma 2.13.** Let \( X \) be a \( DG_\delta \)-space which is \( T_1 \). Then, \( X \) is \( SG_\delta \)-selectively separable if for every decreasing sequence \( (D_n : n \in \omega) \) of dense \( G_\delta \)-subspaces of \( X \) there are finite sets \( F_n \subset D_n \) such that \( \bigcup_{n \in \omega} F_n \) is dense and \( G_\delta \) in \( X \).

**Theorem 2.14.** Let \( Y \) be a dense and open (resp., \( G_\delta \)) subspace of \( X \) (resp., where \( X \) is a Volterra space). If \( Y \) has a countable open cover consisting of \( G_\delta \)-selectively (resp., \( SG_\delta \)-selectively) separable subsets, then \( X \) is \( G_\delta \)-selectively (resp., \( SG_\delta \)-selectively) separable.

Proof. For each \( n \in \omega \), let \( V_n \) be an open subset of \( Y \) which is a \( G_\delta \)-selectively (resp., an \( SG_\delta \)-selectively) separable subset of \( Y \) and \( Y = \bigcup_{n \in \omega} V_n \). For each \( n \in \omega \), let \( W_n = V_n \setminus \bigcup_{i \leq n - 1} V_i \). Then, \( \{ W_n : n \in \omega \} \) is a disjoint family of \( G_\delta \)-selectively (resp., \( SG_\delta \)-selectively) separable open subsets of \( Y \) by Remark 2.10, and so it is
easily seen that $W = \bigcup_{n \in \omega} W_n$ is $G_{\delta}$-selectively (resp., $SG_{\delta}$-selectively) separable. Thus, $Y$ is $G_{\delta}$-selectively (resp., $SG_{\delta}$-selectively) separable since $W$ is open and dense in $Y$. Therefore by Theorem 2.9 (resp., Corollary 2.8) $X$ is $G_{\delta}$-selectively (resp., $SG_{\delta}$-selectively) separable since $Y$ is a dense and open (resp., $G_{\delta}$) subspace of $X$. 

\textbf{Corollary 2.15.} $X$ is a $G_{\delta}$-selectively (resp., an $SG_{\delta}$-selectively) separable space if and only if the set $I(X)$ of isolated points of $X$ is countable and $X \setminus \overline{I(X)}$ is $G_{\delta}$-selectively (resp., $SG_{\delta}$-selectively) separable.

A map $f : X \rightarrow Y$ is called \textit{feebly open} if for every nonempty open subset $U$ of $X$, there is a nonempty open subset $V$ of $Y$ such that $V \subseteq f(U)$. It seems that the idea of a feebly open map was first introduced in [8].

\textbf{Proposition 2.16.} Let $X$ be a $G_{\delta}$-selectively separable space. Then,

(i) every closed irreducible continuous image of $X$ is $G_{\delta}$-selectively separable;

(ii) every feebly open continuous image of $X$ is $G_{\delta}$-selectively separable.

\textbf{Proof.} Let $f : X \rightarrow Y$ be a continuous onto function. If $f$ is either feebly open or closed irreducible, then the inverse image of any dense subset of $Y$ is dense in $X$. Thus, for any sequence $(D_n : n \in \omega)$ of elements of $\mathcal{DG}Y$, we have $E_n = f^{-1}(D_n) \in \mathcal{DG}X$ for all $n \in \omega$, and so we can find, for every $n \in \omega$, a finite $F_n \subseteq E_n$ such that $\bigcup_{n \in \omega} F_n \in \mathcal{D}X$. Hence $G_n = f(F_n)$ is a finite subset of $D_n$ for every $n \in \omega$ and $\bigcup_{n \in \omega} G_n$ is a dense subspace of $Y$. 

\textbf{Proposition 2.17.} Let $Y$ be a separable, dense $G_{\delta}$-subspace of a Volterra space $X$. Then $X$ is $G_{\delta}$-selectively separable if and only if $Y$ has countable fan tightness with respect to dense $G_{\delta}$-sets.

\textbf{Proof.} Necessity. By Corollary 2.8, $Y$ is $G_{\delta}$-selectively separable and so $Y$ has countable fan tightness with respect to dense $G_{\delta}$-sets. Sufficiency. Let $S = \{s_n : n \in \omega\}$ be a dense subset of $Y$ and $(D_n : n \in \omega)$ be a sequence of elements of $\mathcal{DG}X$. Thus, for each $n \in \omega$, $D_n \cap Y \in \mathcal{DG}(Y)$ since $X$ is a Volterra space [9]. Pick a disjoint family $\mathcal{T} = \{T_n : n \in \omega\}$ of infinite subsets of $\omega$ such that $\bigcup \mathcal{T} = \omega$. For any $n \in \omega$, we have $s_n \in Y \cap (\bigcap_{m \in T_n} D_m \cap Y)$ and so there is a finite subset $F_m$ of $D_m \cap Y$ for every $m \in T_n$ such that $s_n \in Y \cap (\bigcup_{m \in T_n} F_m)$ since $Y$ has countable fan tightness with respect to dense and $G_{\delta}$-sets. Thus, $F_n$ is a finite subset of $D_n$ for every $n \in \omega$ and $\bigcup_{n \in \omega} F_n$ is dense in $X$ and so $X$ is $G_{\delta}$-selectively separable. 

By slight changes in the proof of Proposition 2.17, the following result is obtained.

\textbf{Proposition 2.18.} A separable space $X$ is $G_{\delta}$-selectively separable if and only if $X$ has countable fan tightness with respect to dense $G_{\delta}$-sets.

\textbf{Proposition 2.19.} For a $T_1$-space $X$, the following statements are equivalent.

(i) $X$ is hereditarily selectively separable;
(ii) \( X \) is hereditarily separable and all countable subspaces of \( X \) are selectively separable;

(iii) \( X \) is hereditarily \( G_\delta \)-selectively separable,

**Proof.** (i)\(\Leftrightarrow\)(i) See [4, Proposition 16]. (i)\(\Rightarrow\)(iii) It is obvious.

(iii)\(\Rightarrow\)(ii) Let \( Y \) be a subspace of \( X \). Then \( Y \) is separable since \( Y \) is \( G_\delta \)-selectively separable. Clearly, every countable \( T_1 \)-space is a \( DG_\delta \)-space and by Remark 2.6, in the class of \( DG_\delta \)-spaces which are \( T_1 \)-spaces the concepts of selective separability and \( G_\delta \)-selective separability coincide and so every countable subspace of \( X \) is selectively separable. \( \square \)

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Department of Mathematics, Yazd University, P. O. Box 89195741, Yazd, Iran

_E-mail:_ mahmadi@yazd.ac.ir

Department of Mathematics, Yazd University, P. O. Box 89195741, Yazd, Iran

_E-mail:_ fmnhmn18@yahoo.com