ON JET LIKE BUNDLES OF VECTOR BUNDLES

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Abstract. We describe completely the so called jet like functors of a vector bundle E over an m-dimensional manifold M, i.e. bundles FE over M canonically depending on E such that $F(E_1 \times_M E_2) = FE_1 \times_M FE_2$ for any vector bundles E_1 and E_2 over M. Then we study how a linear vector field on E can induce canonically a vector field on FE.

1. Introduction

This introduction contains rather intuitive presentation of the main results of the paper. The strict presentation of all these results will start from the next section.

From now on, let $\mathcal{M}f$ be the category of smooth manifolds and all smooth maps, $\mathcal{M}f_m$ be the category of m-dimensional manifolds and local diffeomorphisms, $\mathcal{FM}$ the category of fibered manifolds and fibered maps, $\mathcal{FM}_m$ the category of fibered manifolds with m-dimensional bases and their fibered maps local diffeomorphisms as base maps, $\mathcal{VB}_{m,n}$ the category of vector bundles with m-dimensional bases and n-dimensional fibers and vector bundle local isomorphisms, and $\mathcal{VB}_m$ the category of vector bundles with m-dimensional bases and vector bundle maps with local diffeomorphisms as base maps.

We point out that our research is a continuation of the general study of product preserving bundle functors defined on different categories. The first fundamental result of this type is the complete description of product preserving bundle functors on $\mathcal{M}f$, see [4]. Next, I. Kolář and the third author [5] proved similar characterization of fiber product preserving bundle functors on the category $\mathcal{FM}_m$. The present paper deals with fiber product preserving bundle functors on the category of vector bundles $\mathcal{VB}_m$ instead of $\mathcal{FM}_m$.

In this paper we use the notation and terminology from the book [4]. In particular, geometric objects are precisely interpreted as (gauge) bundle functors and geometric constructions have the role of (gauge) natural operators. Using such a point of view,

the concept "canonical" means natural and "complete description" means all (gauge) natural operators of specific type. For example, bundle GM canonically depending on M from General construction 1 below means a natural bundle (bundle functor) $G : \mathcal{M}f_m \to \mathcal{FM}$ in the sense of Nijenhuis, see [4].

Any manifold considered in the paper is assumed to be Hausdorff, second countable, finite dimensional, without boundary and smooth (i.e. of class C^∞). All maps between manifolds are assumed to be smooth.

Let $E = (p : E \to M)$ be a vector bundle over an m-manifold M and r be a non-negative integer. The usual r-jet prolongation of E is the vector bundle $J^r E$ over M of all r-jets $j_x^r \sigma$ at points $x \in M$ of local sections σ of E. Clearly, $J^r E$ depends canonically on E and $J^r(E_1 \times_M E_2) = J^r E_1 \times_M J^r E_2$ for any vector bundles E_1 and E_2 over M.

In the present paper we study the so called jet like functors of E, i.e. bundles FE over M canonically (functorially) depending on E and such that $F(E_1 \times_M E_2) = FE_1 \times_M FE_2$ for any vector bundles E_1 and E_2 over M. Clearly, $J^r E$ is a jet like bundle of E. We have the following general construction of jet like bundles of E.

General construction 1. *Suppose that we have a bundle GM over M canonically depending on M such that the fiber $G_x M$ of GM at x is canonically a $J_x^r(M, \mathbb{R})$-module for any $x \in M$, where $J_x^r(M, \mathbb{R})$ is the usual commutative ring with unity of r-jets at $x \in M$ of maps $M \to \mathbb{R}$. The fiber $J_x^r E$ of $J^r E$ over $x \in M$ is a $J_x^r(M, \mathbb{R})$-module in obvious way. Then we have tensor product $J_x^r E \otimes_{J^r(M, \mathbb{R})} G_x M$ over $J_x^r(M, \mathbb{R})$ of $J_x^r E$ and $G_x M$, and we define $F^G E := \bigcup_{x \in M} J_x^r E \otimes_{J_x^r(M, \mathbb{R})} G_x M$. Then $F^G E$ is a jet like bundle of E.*

The first main result of the paper is following.

Main result 1. *Any jet like bundle of E is $F^G E$ as above for some GM.*

REMARK 1.1. Many jet like functors are presented in Remarks 5.1 and 5.2 below, the most well known one is the classical r-jet prolongation functor J^r. This justifies our notation "jet like functor" from Definition 2.4 below. The full characterization of all jet like functors will be given in Theorem 2.14.

Let X be a vector field on E. Then X is called linear if the flow of X is formed by (locally defined) vector bundle maps of E. Then there exists a (unique) vector field $\underline{X}$ on M such that $Tp \circ X = \underline{X} \circ p$.

Any vertical linear vector field X on E is complete. Indeed, X is vertical and linear iff $B := pr_2 \circ X : E \to E$ is a base preserving vector bundle map, where $pr_2 : VE = E \times_M E \to E$ is the fiber projection onto the essential factor, and then the (global) flow $\{f_t\}$ of X is given by $(f_t)_x = e^{tB_x} : E_x \to E_x$, $x \in X$, $t \in \mathbb{R}$.

Let Y be a vertical linear vector field on GM, where GM is as above, and let $\{\psi_t\}$ be the (global) flow of Y. Then Y is called vertical $J^r(M, \mathbb{R})$-linear if $(\psi_t)_x : G_x M \to G_x M$ is a $J_x^r(M, \mathbb{R})$-module isomorphism for any $t \in \mathbb{R}$ and $x \in M$.

Next, we fix a jet like bundle FE of E and we study the problem how a linear vector field X on E with the underlying vector field $\underline{X}$ on M can induce canonically a vector field $A(X)$ on FE. We may assume that $FE = F^G E$.

Let $\{f_t\}$ be the (local) flow of X. Then we have the (local) flow $\{J^r f_t\}$ on $J^r E$, and consequently we have the corresponding vector field $\mathcal{J}^r X$ on $J^r E$. Replacing $J^r E$ by FE we obtain the vector field $\mathcal{F} X$ on FE. Clearly, $\mathcal{F} X$ is an example of such $A(X)$.

If $FE = J^r E$, an example of such $A(X)$ is the Liouville vector field L on $J^r E$. The flow of L is $\{e^t \mathrm{id}_{J^r E}\}$.

General construction 2. *Suppose that there exists a vertical $J^r(M, \mathbb{R})$-linear vector field $B(\underline{X})$ on GM canonically depending on $\underline{X}$. (For example, we can take $B(\underline{X}) :=$ the Liouville vector field on GM.) Then the flow $\{\psi_t\}$ of $B(\underline{X})$ is global and $(\psi_t)_x :$ $G_x M \to G_x M$ is a $J^r_x(M, \mathbb{R})$-module isomorphism over $\mathrm{id}_{J^r_x(M,\mathbb{R})}$ for any $x \in M$ and $t \in \mathbb{R}$. Let $(\Psi_t)_x := e^t \mathrm{id}_{J^r_x E} \otimes (\psi_t)_x : J^r_x E \otimes_{J^r_x(M,\mathbb{R})} G_x M \to J^r_x E \otimes_{J^r_x(M,\mathbb{R})} G_x M$ be the module isomorphism obtained from the ones $e^t \mathrm{id}_{J^r_x E} : J^r_x E \to J^r_x E$ and $(\psi_t)_x :$ $G_x M \to G_x M$ in obvious way. Then we have the resulting flow $\{\Psi_t\}$ on FE. Let $L \otimes_{J^r(M,\mathbb{R})} B(\underline{X})$ denote the vector field on FE given by $\{\Psi_t\}$.*

The second main result of the paper is following.

Main result 2. *Any vector field $A(X)$ on FE canonically depending on a linear vector field X on E with the underlying vector field $\underline{X}$ on M is of the form*

$$A(X) = \lambda \mathcal{F} X + L \otimes_{J^r(M,\mathbb{R})} B(\underline{X})$$

for some $\lambda \in \mathbb{R}$ and $B(\underline{X})$ as above.

In the rest of the paper, the above results will be precisely presented and proved.

2. The jet like functors on $\mathcal{VB}_m$

DEFINITION 2.1. A *gauge bundle functor* (gb-functor) on $\mathcal{VB}_m$ is a covariant functor $F : \mathcal{VB}_m \to \mathcal{FM}$ such that the conditions (a)–(d) are satisfied.
(a) every $\mathcal{VB}_m$-object $E = (E \to M)$ is transformed into a fibered manifold $\pi_E :$ $FE \to M$,

(b) every $\mathcal{VB}_m$-morphism $f : E \to E_1$ with the base map $\underline{f} : M \to M_1$ is transformed into a fibered morphism $Ff : FE \to FE_1$ over $\underline{f} : M \to M_1$,

(c) for every $\mathcal{VB}_m$-object $E = (E \to M)$ and every open subset $U \subset M$ the inclusion $i : E_{|U} \to E$ induces diffeomorphism $Fi : F(E_{|U}) \to \pi_E^{-1}(U)$,

(d) F transforms smoothly parameterized families of $\mathcal{VB}_m$-maps into smoothly parameterized families.

DEFINITION 2.2. Given gb-functors F_1 and F_2 on $\mathcal{VB}_m$, a natural transformation $\eta : F_1 \to F_2$ is a system of base preserving fibered maps $\eta_E : F_1 E \to F_2 E$ for every $\mathcal{VB}_m$-object E satisfying $F_2 f \circ \eta_E = \eta_{E'} \circ F_1 f$ for every $\mathcal{VB}_m$-map $f : E \to E'$.

DEFINITION 2.3. A gb-functor F on $\mathcal{VB}_m$ is of order r if for any $\mathcal{VB}_m$-maps $f, g :$ $E \to E_1$ with the base maps $\underline{f}, \underline{g} : M \to M_1$ and any point $x \in M$ from $j^r_x f = j^r_x g$ (i.e.

from $j_z^r f = j_z^r g$ for any $z \in E_x$) it follows $F_x f = F_x g$, where $F_x f : F_x E \to F_{\underline{f}(x)} E_1$ is the restriction of $Ff : FE \to FE_1$ to respective fibers.

DEFINITION 2.4. A gb-functor F on $\mathcal{VB}_m$ is called a *jet like functor* if it is fiber product preserving, i.e. if $F(E_1 \times_M E_2) = FE_1 \times_M FE_2$ modulo (Fpr_1, Fpr_2) for every $\mathcal{VB}_m$ objects E_1 and E_2 with base M, where $pr_i : E_1 \times_M E_2 \to E_i$ is the fibered projection.

We recall that in [9] it is proved that any jet like gb-functor on $\mathcal{VB}_m$ is of finite order.

EXAMPLE 2.5. Let r be a non-negative integer. The usual r-jet prolongation functor J^r sending any $\mathcal{VB}_m$-object E with base M into the space (vector bundle) $J^r E$ of r-jets $j_x^r \sigma$ at points $x \in M$ of local sections $\sigma : M \to E$ of E and any $\mathcal{VB}_m$-map $f : E \to E_1$ with the base map $\underline{f} : M \to M_1$ into the induced map (vector bundle map) $J^r f : J^r E \to J^r E_1$ given (correctly) by $J^r f(\eta) := j_{\underline{f}(x)}^r (f \circ \sigma \circ \underline{f}^{-1})$ for all $\eta \in J_x^r E$ and $x \in M$, where σ is a local section of E with $\eta = j_x^r \sigma$, is a classical example of a jet like functor of order r.

DEFINITION 2.6. Replacing $\mathcal{VB}_m$ by $\mathcal{Mf}_m$, we obtain the concept of bundle functors of order r on $\mathcal{Mf}_m$ and their natural transformations.

EXAMPLE 2.7. An important (for us) bundle functor on $\mathcal{Mf}_m$ of finite order r is the usual r-jet functor $J^r(-, \mathbb{R}) : \mathcal{Mf}_m \to \mathcal{FM}$ sending any $\mathcal{Mf}_m$-object M into the bundle $J^r(M, \mathbb{R})$ over M of r-jets $j_x^r \gamma$ at points $x \in M$ of maps $\gamma : M \to \mathbb{R}$, and any $\mathcal{Mf}_m$-map $\varphi : M \to M'$ into the induced map $J^r(\varphi, \mathrm{id}_\mathbb{R}) : J^r(M, \mathbb{R}) \to J^r(M', \mathbb{R})$ given (correctly) by $J^r(\varphi, \mathrm{id}_\mathbb{R})(g) := j_{\varphi(x)}^r (\gamma \circ \varphi^{-1})$ for all $g \in J_x^r(M, \mathbb{R})$ and $x \in M$, where $\gamma : M \to \mathbb{R}$ is a map with $g = j_x^r \gamma$. We see that the fiber $J_x^r(M, \mathbb{R})$ is (in obvious way) a commutative ring with unity (even it is a real algebra) for any $\mathcal{Mf}_m$-object M and any $x \in M$, and $J_x^r(\varphi, \mathrm{id}_\mathbb{R}) : J_x^r(M, \mathbb{R}) \to J_{\varphi(x)}^r(M', \mathbb{R})$ is a ring isomorphism for any $\mathcal{Mf}_m$-map $\varphi : M \to M'$ and any $x \in M$.

DEFINITION 2.8. Let r be a non-negative integer. A $J^r(-, \mathbb{R})$-*module bundle functor* on $\mathcal{Mf}_m$ is a bundle functor G on $\mathcal{Mf}_m$ of order r such that the following conditions (i)–(ii) are satisfied.
(i) $G_x M$ is a $J_x^r(M, \mathbb{R})$-module for any m-manifold M and any $x \in M$ such that the resulting maps $+ : GM \times_M GM \to GM$ and $\cdot : J^r(M, \mathbb{R}) \times_M GM \to GM$ are of class C^∞.

(ii) $G_x \varphi : G_x M \to G_{\varphi(x)} M'$ is a module isomorphism over $J_x^r(\varphi, \mathrm{id}_\mathbb{R}) : J_x^r(M, \mathbb{R}) \to J_{\varphi(x)}^r(M', \mathbb{R})$ for any $\mathcal{Mf}_m$-map $\varphi : M \to M'$ and any $x \in M$.

DEFINITION 2.9. If G' is another $J^r(-, \mathbb{R})$-module bundle functor on $\mathcal{Mf}_m$, a natural transformation $G \to G'$ of $J^r(-, \mathbb{R})$-module bundle functors is a natural transformation $\mu : G \to G'$ of bundle functors such that $\mu_x : G_x M \to G'_x M$ is a module homomorphism over $\mathrm{id}_{J_x^r(M, \mathbb{R})}$ for any m-manifold M and any point $x \in M$.

EXAMPLE 2.10. A simple example of a $J^r(-, \mathbb{R})$-module bundle functor on $\mathcal{M}f_m$ is the r-jet functor $J^r(-, \mathbb{R})$. Indeed, $J^r_x(M, \mathbb{R})$ is the $J^r_x(M, \mathbb{R})$-module (in obvious way).

PROPOSITION 2.11. *Let r be a non-negative integer. Let $F : \mathcal{VB}_m \to \mathcal{FM}$ be a jet like functor of order r. Then $FE \to M$ is (canonically) a $J^r(M, \mathbb{R})$-module bundle for any $\mathcal{VB}_m$-object $E = (E \to M)$. More precisely, we have the following conditions: (i) F_xE is a $J^r_x(M, \mathbb{R})$-module for any $\mathcal{VB}_m$-object $E = (E \to M)$ and any $x \in M$ such that the resulting maps $+ : FE \times_M FE \to FE$ and $\cdot : J^r(M, \mathbb{R}) \times_M FE \to FE$ are smooth.*

(ii) $F_xf : F_xE \to F_{\underline{f}(x)}E^1$ is a module map over $J^r_x(\underline{f}, \mathrm{id}_{\mathbb{R}}) : J^r_x(M, \mathbb{R}) \to J^r_{\underline{f}(x)}(M^1, \mathbb{R})$ for any $\mathcal{VB}_m$-map $f : E \to E^1$ with the base map $\underline{f} : M \to M^1$ and any $x \in M$.

Proof. We define the fiber sum map of FE to be

$$F(+) : FE \times_M FE = F(E \times_M E) \to FE,$$

where $+ : E \times_M E \to E$ is the fiber sum map of E treated as a $\mathcal{VB}_m$-map. We define the fiber multiplication map $\cdot : J^r(M, \mathbb{R}) \times_M FE \to FE$ by $\rho \cdot v := F\tilde{\gamma}(v)$, for all $\rho \in J^r_x(M, \mathbb{R})$, $x \in M$ and $v \in F_xE$, where $\gamma : M \to \mathbb{R}$ is a map with $j^r_x\gamma = \rho$, and where $\tilde{\gamma} : E \to E$ is the base preserving $\mathcal{VB}_m$-map given by $\tilde{\gamma}(w) := \gamma(x)w$ for all $w \in E_x$, $x \in M$. Since F is of order r, this definition is correct, i.e. independent of the choice of such γ. Then one can easily show conditions (i) and (ii) of the proposition. For example, since F is a functor, from $\widetilde{\gamma_1\gamma_2} = \tilde{\gamma}_1 \circ \tilde{\gamma}_2$ it follows $(j^r_x\gamma_1 j^r_x\gamma_2) \cdot v = j^r_x\gamma_1 \cdot (j^r_x\gamma_2 \cdot v)$, or more detailed, $(j^r_x\gamma_1 j^r_x\gamma_2) \cdot v = j^r_x(\gamma_1\gamma_2) \cdot v = F_x\widetilde{\gamma_1\gamma_2}(v) = F_x(\tilde{\gamma}_1 \circ \tilde{\gamma}_2)(v) = F_x\tilde{\gamma}_1 \circ F_x\tilde{\gamma}_2(v) \underbrace{= F_x\tilde{\gamma}_1}(F_x\tilde{\gamma}_2(v)) = j^r_x\gamma_1 \cdot (j^r_x\gamma_2 \cdot v)$. Similarly, since F is a functor, from $f \circ \tilde{\gamma} = (\gamma \circ \underline{f}^{-1}) \circ f$ it follows $F_xf(j^r_x\gamma \cdot v) = j^r_{\underline{f}(x)}(\gamma \circ \underline{f}^{-1}) \cdot F_xf(v)$. The other module and module map axioms can be similarly verified. $\square$

Consequently, any jet like functor on $\mathcal{VB}_m$ of finite order r induces a $J^r(-, \mathbb{R})$-module bundle functor on $\mathcal{M}f_m$. More precisely, we have the following example.

EXAMPLE 2.12. Let r be a non-negative integer. Let F be a jet like functor on $\mathcal{VB}_m$ of order r. We put $G^FM := F(M \times \mathbb{R})$, $G^F\varphi := F(\varphi \times \mathrm{id}_{\mathbb{R}})$ for any $\mathcal{M}f_m$-object M and any $\mathcal{M}f_m$-map $\varphi : M \to M'$. Applying Proposition 2.11, we see that G^F is a $J^r(-, \mathbb{R})$-module bundle functor on $\mathcal{M}f_m$.

Conversely, any $J^r(-, \mathbb{R})$-module bundle functor on $\mathcal{M}f_m$ (with finite r) induces a jet like functor on $\mathcal{VB}_m$ of order r. More precisely, we have the following example.

EXAMPLE 2.13. Let r be a non-negative integer. Let G be a $J^r(-, \mathbb{R})$-module bundle functor on $\mathcal{M}f_m$. If $E = (E \to M)$ is a $\mathcal{VB}_m$-object and $x \in M$, we put $F^G_xE := J^r_xE \otimes_{J^r_x(M, \mathbb{R})} G_xM$, the tensor product over $J^r_x(M, \mathbb{R})$ of the $J^r_x(M, \mathbb{R})$-modules J^r_xE and G_xM. For any $\mathcal{VB}_m$-map $f : E \to E^1$ with the base map $\underline{f} : M \to M^1$ and $x \in M$, we define $F^G_xf : F^G_xE \to F^G_{\underline{f}(x)}E^1$ to be the module map over $J^r_x(\underline{f}, \mathrm{id}_{\mathbb{R}}) : J^r_x(M, \mathbb{R}) \to J^r_{\underline{f}(x)}(M^1, \mathbb{R})$ satisfying $F^Gf(\epsilon \otimes_{J^r_x(M, \mathbb{R})} v) := J^r_xf(\epsilon) \otimes_{J^r_{\underline{f}(x)}(M^1, \mathbb{R})} G_x\underline{f}(v)$

for any $\epsilon \in J_x^r E$ and any $v \in G_x M$. Clearly, the functor F^G such that $F^G E :=$ $\bigcup_{x \in M} F_x^G E$ and $F^G f := \bigcup_{x \in M} F_x^G f$ is a jet like functor on $\mathcal{VB}_m$ of order r.

The main result of this section is the following

THEOREM 2.14. *Let F be a jet like functor on $\mathcal{VB}_m$ of finite order r. Then $F = F^{G^F}$ modulo $\mathcal{VB}_m$-natural isomorphism of gauge bundle functors. If $F = F^G$ then $G = G^F$ modulo isomorphism of $J^r(-, \mathbb{R})$-module bundle functors on $\mathcal{M}f_m$.*

Proof. We define a $\mathcal{VB}_m$-natural isomorphism $F^{G^F} \to F$ as follows. Let $E = (E \to M)$ be a $\mathcal{VB}_m$-object and $x \in M$ be a point. We have a correctly defined (as F is of order r) a $J_x^r(M, \mathbb{R})$-bilinear map $\alpha : J_x^r E \times G_x^F M \to F_x E$ given by

$$\alpha(\eta, v) := F_x \widehat{\sigma}(v)$$

for any $\eta \in J_x^r E$ and any $v \in G_x^F M = F_x(M \times \mathbb{R})$, where σ is a local section of E with $\eta = j_x^r \sigma$, and where $\widehat{\sigma} : M \times \mathbb{R} \to E$ is the base preserving $\mathcal{VB}_m$-map given by $\widehat{\sigma}(u, t) = t\sigma(u)$, $(u, t) \in M \times \mathbb{R}$. For example, since $\widehat{\gamma\sigma} = \tilde{\gamma} \circ \widehat{\sigma}$ for any section σ of $E \to M$ and any $\gamma : M \to \mathbb{R}$, we have $\alpha(j_x^r \gamma \cdot j_x^r \sigma, v) = \alpha(j_x^r(\gamma\sigma), v) = F_x \widehat{\gamma\sigma}(v) = F_x(\tilde{\gamma} \circ \widehat{\sigma})(v) = F_x \tilde{\gamma} \circ F_x \widehat{\sigma}(v) = F_x \tilde{\gamma}(\alpha(j_x^r \sigma, v)) = j_x^r \gamma \cdot \alpha(j_x^r \sigma, v)$. That is why α is $J_x^r(M, \mathbb{R})$-linear in first factor. Further, since $F_x \widehat{\sigma} : F_x(M \times \mathbb{R}) \to F_x E$ is a $J_x^r(M, \mathbb{R})$-module map (by Proposition 2.11), we obtain the $J_x^r(M, \mathbb{R})$-linearity of α in the second factor. Equivalently, we have the $J_x^r(M, \mathbb{R})$-linear map

$$\alpha : J_x^r E \otimes_{J_x^r(M, \mathbb{R})} G_x^F M \to F_x E.$$

So, we have the $\mathcal{VB}_m$-natural transformation $F^{G^F} \to F$. It is an isomorphism as it is an isomorphism for $E = M \times \mathbb{R}$ and both F^{G^F} and F are fiber product preserving. That is why, $F = F^{G^F}$ modulo isomorphism. Moreover, if $F = F^G$ then

$$G_x^F M = F_x(M \times \mathbb{R}) = F_x^G(M \times \mathbb{R}) = J_x^r(M, \mathbb{R}) \otimes_{J_x^r(M, \mathbb{R})} G_x M = G_x M. \qquad \square$$

By Theorem 2.14 we have the following expression of rth order jet like functors $FE = J^r E \otimes_{J^r(M, \mathbb{R})} GM$ for any $\mathcal{VB}_m$-object $E \to M$, where G is some $J^r(-, \mathbb{R})$-module bundle functor on $\mathcal{M}f_m$, namely, $G = G^F$. Other description (but more complicated) of jet like functors on $\mathcal{VB}_m$ can be found in [9].

REMARK 2.15. A jet like functor F on $\mathcal{VB}_m$ is called vertical if for any $\mathcal{VB}_m$-objects $E \to M$ and $E_1 \to M$ with the same basis M, any $x \in M$ and any base preserving $\mathcal{VB}_m$-map $\varphi : E \to E_1$, $F_x \varphi : F_x E \to F_x E_1$ depends only on $\varphi_x : E_x \to (E_1)_x$. An example of such F is the vertical r-jet prolongation $(J_v^r)E = \bigcup_{x \in M} J_x^r(M, E_x)$. Given a vertical jet like functor F on $\mathcal{VB}_m$, by the proof of Proposition 2.11 and Example 2.12, the $J_x^r(M, \mathbb{R})$-module multiplication of $J_x^r(M, \mathbb{R})$ on $G_x^F M$ satisfies $j_x^r \gamma \cdot v = \gamma(x) \cdot v$. Then $J_x^r E \otimes_{J_x^r(M, \mathbb{R})} G_x^F M = E_x \otimes_{\mathbb{R}} G_x^F M$, where we identify $j_x^r \sigma \otimes_{J_x^r(M, \mathbb{R})} v$ with $\sigma(x) \otimes_{\mathbb{R}} v$. Thus we have the following expression of rth order vertical jet like functors $FE = E \otimes_{\mathbb{R}} GM$ for any $\mathcal{VB}_m$-object $E \to M$, where G is some vector bundle functor on $\mathcal{M}f_m$ of order r, namely, $G = G^F$. Of course, $F_x f(e \otimes_{\mathbb{R}} v) = f_x(e) \otimes_{\mathbb{R}} G_x f(v)$ for any $(e \otimes_{\mathbb{R}} v) \in E_x \otimes_{\mathbb{R}} G_x M$, $x \in M$. Such description was observed in [7] in another way (by using the results of [9]).

3. The iteration

Let $F : \mathcal{VB}_m \to \mathcal{FM}$ be a jet like functor. Using Proposition 2.11, we can see that if $E \to M$ is a vector bundle, then $FE \to M$ is also a vector bundle. This means that F has values in $\mathcal{VB}_m$, so that one can iterate jet like functors. Clearly, the composition of jet like functors is a jet like functor as well. For the iteration of two jet like functors we prove

PROPOSITION 3.1. *Let $F^1 = F^{G^1}$ and $F^2 = F^{G^2}$ be two jet like functors of order r^1 and r^2, respectively. Then the composition $F = F^1 \circ F^2$ is a jet like functor of order $r = r^1 + r^2$. We can write $F = F^G$, where $G = G^{F^1 \circ F^2}$. Then*

$$GM = J^{r^1} G^2 M \otimes_{J^{r^1}(M,\mathbb{R})} G^1 M$$

for any m-manifold M, and $G\varphi : GM \to GM_1$ is naturally induced by $J^{r^1} G^2 \varphi : J^{r^1} G^2 M \to J^{r^1} G^2 M_1$ and $G^1 \varphi : G^1 M \to G^1 M_1$ for any $\mathcal{M}f_m$-map $\varphi : M \to M_1$. Moreover, given a point $x_o \in M$ of an m-manifold M, the module multiplication $\cdot : J^r_{x_o}(M,\mathbb{R}) \times G_{x_o} M \to G_{x_o} M$ satisfies

$$j^r_{x_o} \gamma \cdot (j^{r^1}_{x_o} \sigma \otimes_{J^{r^1}_{x_o}(M,\mathbb{R})} v) = j^{r^1}_{x_o}(x \rightarrowtail j^{r^2}_x \gamma \cdot \sigma(x)) \otimes_{J^{r^1}_{x_o}(M,\mathbb{R})} v$$

for any section $\sigma : M \to G^2 M$ of $G^2 M \to M$, any map $\gamma : M \to \mathbb{R}$ and any $v \in G^1_{x_o} M$, where $\cdot$ (on the right of the equality) is the module bundle multiplication on $G^2 M$.

Proof. We have $GM = F^{G^1}(F^{G^2}(M \times \mathbb{R})) = F^{G^1}(G^2 M) = J^{r^1} G^2 M \otimes_{J^{r^1}(M,\mathbb{R})} G^1 M$ and $G\varphi = F^{G^1} F^{G^2}(\varphi \times \mathrm{id}_{\mathbb{R}}) = F^{G^1} G^2 \varphi = J^{r^1} G^2 \varphi \otimes_{J^{r^1}(M,\mathbb{R})} G^1 \varphi$. Moreover, let $\tilde{\gamma} : M \times \mathbb{R} \to M \times \mathbb{R}$ be defined as in the proof of Proposition 2.11. We have $j^r_{x_o} \gamma \cdot (j^{r^1}_{x_o} \sigma \otimes_{J^{r^1}_{x_o}(M,\mathbb{R})} v) = F^{G^1} F^{G^2} \tilde{\gamma}(j^{r^1}_{x_o} \sigma \otimes_{J^{r^1}_{x_o}(M,\mathbb{R})} v) = J^{r^1} F^{G^2} \tilde{\gamma}(j^{r^1}_{x_o} \sigma) \otimes_{J^{r^1}_{x_o}(M,\mathbb{R})} v = j^{r^1}_{x_o}(x \rightarrowtail F^{G^2} \tilde{\gamma}(\sigma(x))) \otimes_{J^{r^1}_{x_o}(M,\mathbb{R})} v = j^{r^1}_{x_o}(x \rightarrowtail j^{r^2}_x \gamma \cdot \sigma(x)) \otimes_{J^{r^1}_{x_o}(M,\mathbb{R})} v$. $\square$

By this proposition, jet like functors (which are defined on $\mathcal{VB}_m$) are very often not commuting. On the other hand, the iteration of fiber product preserving bundle functors defined on other categories was studied in [1] and [2].

4. Lifting linear vector fields to jet like functors on $\mathcal{VB}_m$

We fix a regular fiber product preserving gauge bundle functor F on $\mathcal{VB}_m$ of finite order r. By Theorem 2.14, we may assume that $F = F^G$ for some $J^r(-,\mathbb{R})$-module bundle functor G on $\mathcal{M}f_m$, i.e. $F_x E = J^r_x E \otimes_{J^r_x(M,\mathbb{R})} G_x M$ for any $\mathcal{VB}_m$-object $E = (E \to M)$ and any $x \in M$.

DEFINITION 4.1. A vector field X on a vector bundle E is called linear, if the flow of X is formed by (locally defined) vector bundle maps of E.

Then $X : E \to TE$ is a homomorphism of vector bundles. An example of a

linear vector field is the classical Liouville (or Euler) vector field L on E defined by $L_{|y} = \frac{d}{dt}_{|t=0}(y + ty)$, $y \in E$. Then the flow of L is $\{e^t \mathrm{id}_E\}$.

We will use the following well known concept of (gauge) natural operators.

DEFINITION 4.2. A $\mathcal{VB}_{m,n}$-gauge natural operator $A : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF$ (lifting linear vector fields X on $\mathcal{VB}_{m,n}$-objects $E = (E \to M)$ into vector fields $A(X)$ on FE) is a $\mathcal{VB}_{m,n}$-invariant family of regular operators (functions) $A : \mathcal{X}_{\mathrm{lin}}(E) \to \mathcal{X}(FE)$ for all $\mathcal{VB}_{m,n}$-objects $E = (E \to M)$, where $\mathcal{X}_{\mathrm{lin}}(E)$ is the space of all linear vector fields X on E (i.e. with flows formed by $\mathcal{VB}_{m,n}$-maps) and $\mathcal{X}(FE)$ is the space of all vector fields on FE. The invariance of A means that if $X \in \mathcal{X}_{\mathrm{lin}}(E)$ and $X_1 \in \mathcal{X}_{\mathrm{lin}}(E_1)$ are f-related by a $\mathcal{VB}_{m,n}$-map $f : E \to E_1$ (i.e. $Tf \circ X = X_1 \circ f$), then $A(X) \in \mathcal{X}(FE)$ and $A(X_1) \in \mathcal{X}(FE_1)$ are Ff-related. The regularity means that A transforms smoothly parameterized families (of linear vector fields) into smoothly parameterized families.

Because of the locality condition (c) of the definition of gb-functors, any $\mathcal{VB}_{m,n}$-natural operator A as above is local, i.e. for any $\mathcal{VB}_{m,n}$-object $E = (p : E \to M)$, any $X, X_1 \in \mathcal{X}_{\mathrm{lin}}(E)$ and any open $U \subset M$ from $X_{|p^{-1}(U)} = X_{1|p^{-1}(U)}$ it follows $A(X)_{|\pi_E^{-1}(U)} = A(X_1)_{|\pi_E^{-1}(U)}$.

EXAMPLE 4.3. If $F = J^r$, an example of such A is given by $A(X) :=$ the Liouville vector field L on $J^r E$ for any linear vector field X on a $\mathcal{VB}_{m,n}$-object $E = (p : E \to M)$. The flow of L is $\{e^t \mathrm{id}_{J^r E}\}$.

EXAMPLE 4.4. An example of such A is also the flow operator $\mathcal{F} : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF$ sending any $X \in \mathcal{X}_{\mathrm{lin}}(E)$ into $\mathcal{F}X \in \mathcal{X}(FE)$ such that if $\{f_t\}$ is the (local) flow of X then $\{F(f_t)\}$ is the (local) flow of $\mathcal{F}X$. We can apply F to f_t because f_t is a $\mathcal{VB}_{m,n}$-map as X is linear.

DEFINITION 4.5. Let G be the fixed $J^r(-, \mathbb{R})$-module bundle functor on $\mathcal{M}f_m$. Let M be an m-manifold. A vertical linear vector field Y on GM with the flow $\{\psi_t\}$ (being global as remarked in Introduction) is called $J^r(M, \mathbb{R})$-linear if $(\psi_t)_x : G_x M \to G_x M$ is a $J_x^r(M, \mathbb{R})$-module isomorphism for any $t \in \mathbb{R}$ and $x \in M$.

Equivalently, a vertical vector field Y on GM is $J^r(M, \mathbb{R})$-linear iff the corresponding base preserving vector bundle homomorphism $\tilde{Y} : GM \to GM$ (such that $Y_{|y} = \frac{d}{dt}_{|t=0}(y + t\tilde{Y}(y))$ for any $y \in G_x M$ and $x \in M$) is $J_x^r(M, \mathbb{R})$-linear on $G_x M$ for any $x \in M$. (Indeed, the (global) flow $\{\psi_t\}$ of Y satisfies $(\psi_t)_x = e^{t\tilde{Y}_x}$ for any $x \in M$ and $t \in \mathbb{R}$.) Consequently, we have the vector space of all vertical $J^r(M, \mathbb{R})$-linear vector fields on GM.

DEFINITION 4.6. A $\mathcal{M}f_m$-natural operator $B : T_{|\mathcal{M}f_m} \rightsquigarrow V_{J^r(-, \mathbb{R})-\mathrm{lin}}G$ (lifting vector fields $\underline{X}$ on $\mathcal{M}f_m$-objects M into $J^r(M, \mathbb{R})$-linear vertical vector fields $B(\underline{X})$ on GM) is an $\mathcal{M}f_m$-invariant family of regular operators $B : \mathcal{X}(M) \to \mathcal{X}_{J^r(M,\mathbb{R})-\mathrm{lin}}^{\mathrm{vert}}(GM)$ for all $\mathcal{M}f_m$-objects M, where $\mathcal{X}_{J^r(M,\mathbb{R})-\mathrm{lin}}^{\mathrm{vert}}(GM)$ is the space of vertical $J^r(M, \mathbb{R})$-linear vector fields on GM.

EXAMPLE 4.7. An example of such B can be given by $B(\underline{X})$:=the Liouville vector field on GM for any vector field $\underline{X}$ on an m-manifold M. (The flow of the Liouville vector field on GM is $\{e^t \mathrm{id}_{GM}\}$.)

EXAMPLE 4.8. Consider an $\mathcal{M}f_m$-natural operator $B : T_{|\mathcal{M}f_m} \rightsquigarrow V_{J^r(-,\mathbb{R})-\mathtt{lin}}G$. Let $E = (p : E \to M)$ be a $\mathcal{VB}_{m,n}$-object and let $\underline{X}$ be a vector field on M. Then $B(\underline{X})$ is vertical $J^r(M,\mathbb{R})$-linear on GM. Then the flow $\{\psi_t\}$ of $B(\underline{X})$ is global and $(\psi_t)_x : G_x M \to G_x M$ is a $J^r_x(M,\mathbb{R})$-module isomorphism over $\mathrm{id}_{J^r_x(M,\mathbb{R})}$ for any $x \in M$ and $t \in \mathbb{R}$. Let $(\Psi_t)_x := e^t \mathrm{id}_{J^r_x E} \otimes (\psi_t)_x : J^r_x E \otimes_{J^r_x(M,\mathbb{R})} G_x M \to J^r_x E \otimes_{J^r_x(M,\mathbb{R})} G_x M$ be the module isomorphism obtained from the ones $e^t \mathrm{id}_{J^r_x E} : J^r_x E \to J^r_x E$ and $(\psi_t)_x : G_x M \to G_x M$ in obvious way. Then we have the resulting flow $\{\Psi_t\}$ on FE. Let $L \otimes_{J^r(M,\mathbb{R})} B(\underline{X})$ denotes the vector field on FE given by $\{\Psi_t\}$.

REMARK 4.9. We underline that $L \otimes_{J^r(M,\mathbb{R})} B(\underline{X})$ is not linear in B. Indeed, the flow $\Psi^{<\tau>}$ of $\tau(L \otimes_{J^r(M,\mathbb{R})} B(\underline{X}))$ satisfies $(\Psi_t^{<\tau>})_x = e^{\tau t} \mathrm{id}_{J^r_x E} \otimes (\psi_{\tau t})_x$ and the flow $\Psi^{(\tau)}$ of $L \otimes_{J^r(M,\mathbb{R})} \tau B(\underline{X})$ satisfies $(\Psi_t^{(\tau)})_x = e^t \mathrm{id}_{J^r_x E} \otimes (\psi_{\tau t})_x$.

THEOREM 4.10. *Let F and G be the fixed data. Given a $\mathcal{VB}_{m,n}$-gauge natural operator $A : T_{\mathtt{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF$, there is a (uniquely determined by A) real number λ and a (uniquely determined by A) $\mathcal{M}f_m$-natural operator $B : T_{|\mathcal{M}f_m} \rightsquigarrow V_{J^r(-,\mathbb{R})-\mathrm{lin}}G$ such that $A(X) = \lambda \mathcal{F}X + L \otimes_{J^r(M,\mathbb{R})} B(\underline{X})$ for any linear vector field $X \in \mathcal{X}_{\mathrm{lin}}(E)$ on a $\mathcal{VB}_{m,n}$-object $E = (E \to M)$ with the underlying vector field $\underline{X} \in \mathcal{X}(M)$.*

Proof. We extend (and correct a little) the proof of the main theorem of [7] as follows. Let $\mathbb{R}^{m,n} = \mathbb{R}^m \times \mathbb{R}^n$ be the trivial $\mathcal{VB}_{m,n}$-object over $\mathbb{R}^m$ and $(x^1, \ldots, x^m, y^1, \ldots, y^n)$ be the usual coordinates on $\mathbb{R}^{m,n}$. Consider a $\mathcal{VB}_{m,n}$-gauge natural operator $A : T_{\mathtt{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TF$. Such A is uniquely determined by $A(\frac{\partial}{\partial x^1})$ over $0 \in \mathbb{R}^m$ as any linear vector field X with the non-vanishing underlying vector field is $\frac{\partial}{\partial x^1}$ in some $\mathcal{VB}_{m,n}$-trivialization.

By the invariance of A with respect to fiber homotheties, $T\pi \circ A(\frac{\partial}{\partial x^1})(u) = T\pi \circ A(\frac{\partial}{\partial x^1})(0) \in T_0 \mathbb{R}^m = \mathbb{R}^m$ for any $u \in F_0 \mathbb{R}^{m,n}$, where $\pi : FE \to M$ is the bundle projection. Then by the invariance of A with respect to $\mathcal{VB}_{m,n}$-maps $(x^1, tx^2, \ldots, tx^m, y^1, \ldots, y^n)$ we derive that $T\pi \circ A(\frac{\partial}{\partial x^1})(u) = \lambda e_1$ for some $\lambda \in \mathbb{R}$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^m$. Replacing A by $A - \lambda \mathcal{F}$, where $(A - \lambda \mathcal{F})(X) := A(X) - \lambda \mathcal{F}X$ for all linear vector fields X on $\mathcal{VB}_{m,n}$-objects E, we can assume $T\pi \circ A(\frac{\partial}{\partial x^1})(u) = 0$ for any $u \in F_0 \mathbb{R}^{m,n}$. Then $A(X)$ is vertical for any $X \in \mathcal{X}_{\mathtt{lin}}(E)$ and any $\mathcal{VB}_{m,n}$-object E, because A is determined by $A(\frac{\partial}{\partial x^1})_{|F_0 \mathbb{R}^{m,n}}$.

Let $\underline{X} \in \mathcal{X}(M)$ be a vector field on an m-manifold M. We consider $\underline{X}$ as the linear vector field on $M \times \mathbb{R}^n$. Then we have a base preserving fibered map $\tilde{A}(\underline{X}) : GM \to GM$ given by $\tilde{A}(\underline{X})(v) = pr_1 \circ I \circ Pr_2 \circ A(\underline{X})(j^r_x e_1 \otimes_{J^r_x(M,\mathbb{R})} v) \in G_x M$ for all $v \in G_x M$ and $x \in M$, where

$$Pr_2 : V(F(M \times \mathbb{R}^n)) \tilde{=} F(M \times \mathbb{R}^n) \times_M F(M \times \mathbb{R}^n) \to F(M \times \mathbb{R}^n)$$

is the projection on the second essential factor, $VE \tilde{=} E \times_N E$ is the standard identification for vector bundles $E \to N$ (i.e. given by $\frac{d}{dt}_{|t=0}(\xi + t\eta) \tilde{=} (\xi, \eta)$ for $\xi, \eta \in E_\rho$ and $\rho \in N$), $I : F(M \times \mathbb{R}^n) \tilde{=} GM \times_M \ldots \times_M GM$ (n times) is the fiber product preserving

identification (as $GM \tilde{=} F(M \times \mathbb{R})$), $pr_1 : GM \times_M \ldots \times_M GM \to GM$ is the projection on the first factor and $e_1, \ldots, e_n$ is the usual basis of sections of $M \times \mathbb{R}^n$.

Thus we have a vertical vector field $B(\underline{X})$ on GM given by

$$B(\underline{X})(v) = \frac{d}{dt}|_{t=0}(v + t(\tilde{A}(\underline{X})(v) - v)) \text{ for all } v \in G_x M \text{ and } x \in M .$$

We are going to show that $B(\underline{X})$ is $J^r(M, \mathbb{R})$-linear, i.e. that the flow $\{\psi_t\}$ of $B(\underline{X})$ is such that $(\psi_t)_x : G_x M \to G_x M$ is a $J^r_x(M, \mathbb{R})$-module isomorphism for any $(t, x) \in \mathbb{R} \times M$.

For this we firstly show that $\tilde{A}(\underline{X})_x : G_x M \to G_x M$ is an endomorphism of $J^r_x(M, \mathbb{R})$-modules for any $x \in M$ or (more) that

$$pr_1 \circ I \circ Pr_2 \circ A(\underline{X}) \circ F\tilde{\gamma}(w) = F\tilde{\gamma} \circ pr_1 \circ I \circ Pr_2 \circ A(\underline{X})(w)$$

for any $w \in F_x(M \times \mathbb{R}^n)$, $x \in M$, $\gamma : M \to \mathbb{R}$, where $\tilde{\gamma} : M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ or $\tilde{\gamma} : M \times \mathbb{R} \to M \times \mathbb{R}$ are defined from γ as in the proof of Proposition 2.11 for $E = M \times \mathbb{R}^n$ or $E = M \times \mathbb{R}$.

To prove the last equality, we consider $w \in F_x(M \times \mathbb{R}^n)$, $x \in M$, $\gamma : M \to \mathbb{R}$. By the regularity, we may assume $\gamma(x) \neq 0$. Then the map $\tilde{\gamma} : M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is a $\mathcal{VB}_{m,n}$-map over some neighborhood of x. Then $F\tilde{\gamma}$ and $Pr_2 \circ A(\underline{X})$ are commuting because of the $\mathcal{VB}_{m,n}$-invariance of A and of the fact that the base map of $\tilde{\gamma}$ preserves $\underline{X}$ (this base map is the identity map of M). Further, since F is fiber product preserving gb-functor and $\tilde{\gamma} \circ p_1 = p_1 \circ \tilde{\gamma}$, where $p_1 : M \times \mathbb{R}^n \to M \times \mathbb{R}$ is the fiber-projection, then $F\tilde{\gamma} \circ pr_1 \circ I = pr_1 \circ I \circ F\tilde{\gamma}$. These facts imply the equality in question.

However, the flow $\{\psi_t\}$ of $B(\underline{X})$ satisfies $(\psi_t)_x(v) = e^{t(\tilde{A}(\underline{X})_x - \mathrm{id}_{G_x M})}v$ for all $v \in G_x M$ and all $t \in \mathbb{R}$ and all $x \in M$. Then the flow $\{\psi_t\}$ is global and $(\psi_t)_x : G_x M \to G_x M$ is a $J^r_x(M, \mathbb{R})$-module isomorphism for any $(t, x) \in \mathbb{R} \times M$. So, $B(\underline{X})$ is vertical $J^r(M, \mathbb{R})$-linear.

Let $E = (p : E \to M)$ be a $\mathcal{VB}_{m,n}$-object and let X be a linear vector field on E with the underlying vector field $\underline{X}$ on M. It remains to show that $A(X) = L \otimes_{J^r(M, \mathbb{R})} B(\underline{X})$, i.e. that $A(X)(u) = (L \otimes_{J^r(M, \mathbb{R})} B(\underline{X}))(u)$ for any $u \in F_x E$ and $x \in M$. We may (of course) assume $E = \mathbb{R}^{m,n}$ and $X = \frac{\partial}{\partial x^1}$ and $x = 0 \in \mathbb{R}^m$.

For simplicity, we will write $\otimes$ instead of $\otimes_{J^r_x(M, \mathbb{R})}$. Then

$$pr_1 \circ I \circ Pr_2 \circ (L \otimes B(\frac{\partial}{\partial x^1}))(j^r_0 e_1 \otimes v) = pr_1 \circ I \circ Pr_2 \circ A(\frac{\partial}{\partial x^1})(j^r_0 e_1 \otimes v) \qquad (1)$$

for any $v \in G_0 \mathbb{R}^m$, where pr_1, I, Pr_2 are as above with $\mathbb{R}^m$ instead of M. Indeed, the flow Ψ_t of $L \otimes B(\frac{\partial}{\partial x^1})$ satisfies $(\Psi_t)_0(j^r_0 e_1 \otimes v) = e^t j^r_0 e_1 \otimes e^{t(\tilde{A}(\frac{\partial}{\partial x^1})_0 - \mathrm{id}_{G_0 \mathbb{R}^m})}v$, where $v \in G_0 \mathbb{R}^m$ and $t \in \mathbb{R}$. Then $Pr_2 \circ (L \otimes B(\frac{\partial}{\partial x^1}))(j^1_0 e_1 \otimes v) = \frac{d}{dt}|_{t=0}(\Psi_t(j^r_0 e_1 \otimes v)) = j^r_0 e_1 \otimes v + j^r_0 e_1 \otimes (\tilde{A}(\frac{\partial}{\partial x^1})(v) - v) = j^r_0 e_1 \otimes \tilde{A}(\frac{\partial}{\partial x^1})(v)$. Then we have (1), as well.

Next, if $i = 2, \ldots, n$, then by the invariance of the operators A and $L \otimes B$ with respect to the $\mathcal{VB}_{m,n}$-map $(x^1, \ldots, x^m, y^1, \ldots, y^i + y^1, \ldots, y^n)$ preserving $\frac{\partial}{\partial x^1}$ and $v \in G_0 \mathbb{R}^m$ and sending e_1 into $e_1 + e_i$, we obtain from (1) the equality (1) with $j^r_0 e_i$ instead of $j^r_0 e_1$.

Then, since the elements $j^r_0 e_i \otimes v$ generate $J^r_0(\mathbb{R}^m, \mathbb{R})$-module $F_0 \mathbb{R}^{m,n}$, we have (1)

for any $u \in F_0\mathbb{R}^{m,n}$ instead of $j_0^r e_1 \otimes v$. Then by the invariance with respect to $\mathcal{VB}_{m,n}$-maps $\mathrm{id}_{\mathbb{R}^m} \times \psi$ for linear isomorphisms $\psi : \mathbb{R}^n \to \mathbb{R}^n$ permuting fibered coordinates we get $Pr_2 \circ A(\frac{\partial}{\partial x^1})(u) = Pr_2 \circ (L \otimes B(\frac{\partial}{\partial x^1}))(u)$ for any $u \in F_0\mathbb{R}^{m,n}$. That is why, $A(\frac{\partial}{\partial x^1}) = L \otimes B(\frac{\partial}{\partial x^1})$ over $0 \in \mathbb{R}^m$. $\qquad\square$

COROLLARY 4.11 ([10]). *Given a $\mathcal{VB}_{m,n}$-gauge natural operator $A : T_{\mathrm{lin}|\mathcal{VB}_{m,n}} \rightsquigarrow TJ^r$, there exist (determined by A) real numbers λ and α such that $A(X) = \lambda \mathcal{J}^r X + \alpha L$ for any linear vector field X on a $\mathcal{VB}_{m,n}$-object $E = (E \to M)$, where L is the Liouville vector field on $J^r E$.*

Proof. Let $G = J^r(-,\mathbb{R})$ be the usual $J^r(-,\mathbb{R})$-module bundle functor on $\mathcal{Mf}_m$. Clearly, any $\mathcal{Mf}_m$-natural operator $B : T_{|\mathcal{Mf}_m} \rightsquigarrow V_{J^r(-,\mathbb{R})-\mathrm{lin}}G$ is determined by $Pr_2 \circ B(\frac{\partial}{\partial x^1})(j_0^r 1)$, where $Pr_2 : VG\mathbb{R}^m \tilde{=} G\mathbb{R}^m \times_{\mathbb{R}^m} G\mathbb{R}^m \to G\mathbb{R}^m$ is the projection on the second (essential) factor. (Indeed, B is determined by module morphism $(Pr_2 \circ B(\frac{\partial}{\partial x^1}))_0 : G_0\mathbb{R}^m \to G_0\mathbb{R}^m$ and the module $G_0\mathbb{R}^m$ is generated by $j_0^r 1$.) Using the $\mathcal{Mf}_m$-invariance of B with respect to the homotheties, we deduce that $Pr_2 \circ B(\frac{\partial}{\partial x^1})(j_0^r 1) = \beta j_0^r 1$ for the real number β. Then $B(\underline{X}) = \beta L$ for any vector field $\underline{X}$ on an $\mathcal{Mf}_m$-object M, where L is the Liouville vector field on GM. On the other hand, the jet like functor F^G corresponding to $G = J^r(-,\mathbb{R})$ is the usual r-jet prolongation functor J^r on $\mathcal{VB}_m$. $\qquad\square$

REMARK 4.12. We recall that the prolongation of vector fields was studied by many authors. For example, I. Kolář classified all natural operators transforming vector fields on a manifold M into vector fields on FM, where F is any Weil (equivalently product preserving) bundle functor on $\mathcal{Mf}_m$, see [4]. In the case of fibered manifolds, D. Krupka [6] defined the flow operator (also called complete lift) of projectable vector fields from $Y \to M$ to its r-jet prolongation $J^r Y$. Next, I. Kolář and J. Slovák proved that all natural operators of this type are constant multiples of the flow operator $\mathcal{J}^r$, [4]. On the other hand, the present paper deals with vector bundles instead of general fibered manifolds and with linear vector fields instead of projectable ones. In particular, we lift linear vector fields to jet like functors (not only to J^r) and classified all such operators. We point out that in the case of vector bundles, linear vector fields are projectable, but not vice versa. So in the case of $F = J^r$ and a vector bundle E, our flow lift $\mathcal{J}^r$ of a linear vector field X on E to $J^r E$ coincides with the complete lift of X (treated as a projectable vector field on fibered manifold E) in the sense of D. Krupka [6].

5. Remarks

REMARK 5.1. There are many classical examples of jet like functors on $\mathcal{VB}_m$. For example, for every $\mathcal{VB}_m$-object $E \to M$ we have its classical r-jet prolongation $J^r E \to M$, which is called holonomic. Further, the functor $\tilde{J}^r$ of nonholonomic jets is defined by iteration $\tilde{J}^r = J^1 \circ \cdots \circ J^1$ (r times), see [3,8]. Moreover, the rth semiholonomic prolongation $\overline{J}^r E \subset \tilde{J}^r E$ is defined by the following induction. Write $\overline{J}^0 E = E$,

$\overline{J}^1 E = J^1 E$ and assume we have defined $\overline{J}^{r-1} E \subset \tilde{J}^{r-1} E$ such that the restriction of the projection $\beta_{r-1} : \tilde{J}^{r-1} E \to \tilde{J}^{r-2} E$ maps $\overline{J}^{r-1} E$ into $\overline{J}^{r-2} E$. Then we have an induced map $J^1 \beta_{r-1} : J^1 \overline{J}^{r-1} E \to J^1 \overline{J}^{r-2} E$ and we can define

$$\overline{J}^r E = \{ U \in J^1 \overline{J}^{r-1} E \mid \beta_r(U) = J^1 \beta_{r-1}(U) \in \overline{J}^{r-1} E \}.$$

One can also define other kinds of subspaces in $\tilde{J}^r E$. For example, the rth sesqui-holonomic prolongation $\hat{J}^r E = J^1(J^{r-1}E) \bigcap \overline{J}^r E$, see [8]. Of course, $\hat{J}^2 E$ coincides with $\overline{J}^2 E$ and for $r > 2$ we have $J^r E \subset \hat{J}^r E \subset \overline{J}^r E \subset \tilde{J}^r E$. Clearly, all the functors J^r, $\hat{J}^r$, $\overline{J}^r$ and $\tilde{J}^r$ preserve fiber products. Many other new jet like functors can be obtained as the kernel of the jet projection $(J^r E \to J^s E)$ for $s < r$.

REMARK 5.2. Many other examples of jet like functors on $\mathcal{VB}_m$ can be obtained by applying fiber product preserving bundle functors F on $\mathcal{FM}_m$ (=the category of fibered manifolds with m-dimensional bases and fibered maps with local diffeomorphisms as base maps) to $\mathcal{VB}_m$-objects and $\mathcal{VB}_m$-morphisms. Fiber product preserving bundle functors F on $\mathcal{FM}_m$ are recently described in [2] by means of modified vertical Weil functors, see also [5].

REMARK 5.3. There are many examples of $J^r(-, \mathbb{R})$-module bundle functors on $\mathcal{M}f_m$. For example, given an $\mathcal{M}f_m$-object M and $\alpha \in \mathbb{R}$, the vector bundle $J^r(M, \mathbb{R})$ is a $J^r(M, \mathbb{R})$-module bundle with respect to the multiplication $\star^\alpha : J^r(M, \mathbb{R}) \times_M J^r(M, \mathbb{R}) \to J^r(M, \mathbb{R})$ given by

$$j_x^r \gamma \star^\alpha j_x^r \rho = j_x^r(\gamma \rho) + \alpha \rho(x) j_x^r \gamma - \alpha \gamma(x) \rho(x) j_x^r 1.$$

Given an $\mathcal{M}f_m$-object M, the vector bundle $J^r(M, \mathbb{R})$ is a $J^r(M, \mathbb{R})$-module bundle with respect to the multiplication $\odot : J^r(M, \mathbb{R}) \times_M J^r(M, \mathbb{R}) \to J^r(M, \mathbb{R})$ given by

$$j_x^r \gamma \odot j_x^r \rho = \gamma(x) j_x^r \rho.$$

For any $\mathcal{M}f_m$-object M, the vector bundle $J^2(M, \mathbb{R})$ is a $J^2(M, \mathbb{R})$-module bundle with respect to the multiplication $*^\beta : J^2(M, \mathbb{R}) \times_M J^2(M, \mathbb{R}) \to J^2(M, \mathbb{R})$ given by

$$j_x^2 \gamma *^\beta j_x^2 \rho = (\beta + 1) j_x^2(\gamma \rho) - (\beta + 1)\rho(x) j_x^2 \gamma - \beta \gamma(x) j_x^2 \rho + (\beta + 1)\gamma(x)\rho(x) j_x^2 1 .$$

REMARK 5.4. Let F be a regular fiber product preserving bundle functor on $\mathcal{FM}_m$ of finite order r. The main result of [2] says that (modulo isomorphism) $F = V^{A,\sigma}$ for some Weil algebra bundle functor A on $\mathcal{M}f_m$ of order r and some $\mathcal{M}f_m$-canonical section σ of $T^A : \mathcal{M}f_m \to \mathcal{FM}$. By Remark 1 in [2], σ can be treated as the $\mathcal{M}f_m$-natural transformation $t : J^r(-, \mathbb{R}) \to A$ of Weil algebra bundle functors by $(t_M)_x = \sigma(x) \in T_x^A M = T_x^{A_x M} M = \mathrm{Hom}(J_x^r(M, \mathbb{R}), A_x M)$ for any $\mathcal{M}f_m$-object M and any $x \in M$. So, the Weil algebra bundle functor A is also the $J^r(-, \mathbb{R})$-module bundle functor on $\mathcal{M}f_m$ with respect to the module bundle multiplication $\odot : J^r(M, \mathbb{R}) \times_M AM \to AM$ given by $j_x^r \gamma \odot v := (t_M)_x(j_x^r \gamma) \cdot v$, $j_x^r \gamma \in J_x^r(M, \mathbb{R})$, $v \in A_x M$, $x \in M$, where $\cdot$ is the multiplication of the Weil algebra $A_x M$. We will denote this $J^r(-, \mathbb{R})$-module bundle functor by A^σ. One can see that (modulo isomorphism) A^σ is the $J^r(-, \mathbb{R})$-module bundle functor corresponding (in the sense of Example 2.12) to the jet like functor F on $\mathcal{VB}_m$ of order r obtained from F by

treating $\mathcal{VB}_m$-morphisms as $\mathcal{FM}_m$-ones in obvious way. Then (by Theorem 2.14) we have the expression $FE = J^r E \otimes_{J^r(M,\mathbb{R})} A^\sigma M$ for any $\mathcal{VB}_m$-object $E \to M$.

REFERENCES

[1] M. Doupovec, I. Kolář: *Iteration of fiber product preserving bundle functors*, Monatsh. Math. **134** (2001), 39–50.

[2] M. Doupovec, J. Kurek, W. M. Mikulski: *Vertical functors on fibered manifolds*, Differ. Geom. Appl. **54** (2017), 100–110.

[3] C. Ehresmann: *Extension du calcul des jets non holonomes*, C. R. A. S. Paris **239** (1954), 1762–1764.

[4] I. Kolář, P. W. Michor, J. Slovák: *Natural Operations in Differential Geometry*, Springer Verlag, 1993.

[5] I. Kolář, W. M. Mikulski: *On the fiber product preserving bundle functors*, Differ. Geom. Appl. **11** (1999), 105–115.

[6] D. Krupka: *Lagrange theory in fibered manifolds*, Rep. Math. Phys. **2** (1971), 121–133.

[7] J. Kurek, W. M. Mikulski: *Lifting of linear vector fields to fiber product preserving vertical gauge bundles*, Colloq. Math. **103** (2005), 33–40.

[8] P. Libermann: *Introduction to the theory of semi-holonomic jets*, Arch. Math. (Brno) **33** (1997), 173–189.

[9] W. M. Mikulski: *On the fiber product preserving gauge bundle functors on vector bundles*, Ann. Polon. Math. **82** (2003), 251–264.

[10] W. M. Mikulski: *The natural operators lifting linear vector fields from a vector bundle into its rth prolongations*, Ann. Polon. Math. **82** (2003), 155–170.

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