MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
76, 4 (2024), 288–302
December 2024

research paper
оригинални научни рад
DOI: 10.57016/MV-BLXV3088

SOME IDENTITIES FOR GENERALIZED HARMONIC NUMBERS

S. Koparal, N. Ömür and K. N. Südemen

Abstract. In this paper, we derive some nonlinear differential equations from generating function of generalized harmonic numbers and give some identities involving generalized harmonic numbers and special numbers by using these differential equations. For example, for any positive integers N, n, r, α and any integer $m \geq 2$,

$$\frac{S_1(n+N,r+1)}{n!} = \sum_{j=0}^{n}\sum_{i=0}^{n}\sum_{l=0}^{i}\sum_{z=0}^{l}\sum_{k=0}^{r}(-1)^{l-z-i}\binom{m}{l-z}\binom{i-l+m-2}{i-l}\frac{N^j\alpha^i}{j!\,(n-i)!} \times S_1(N,r-k+1)S_1(n-i,k)\,H(z,j-1,\alpha)$$

where $S_1(n,k)$ is Stirling number of the first kind.

1. Introduction

The harmonic numbers are defined by $H_0 = 0$ and $H_n = \sum_{i=1}^{n} \frac{1}{i}$ for $n \geq 1$. Recently, harmonic numbers and generalized harmonic numbers have been studied by many mathematicians [1–3, 6, 14, 15, 18].

In [6], for any $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the generalized harmonic numbers $H_n(\alpha)$ are defined by $H_0(\alpha) = 0$ and $H_n(\alpha) = \sum_{i=1}^{n} \frac{1}{i\alpha^i}$. For $\alpha = 1$, the usual harmonic numbers are $H_n(1) = H_n$ and the generating function of $H_n(\alpha)$ is

$$-\frac{\ln\left(1-\frac{x}{\alpha}\right)}{1-x} = \sum_{n=1}^{\infty} H_n(\alpha)x^n.$$

In [13], for the generalized harmonic numbers $H_n(\alpha)$, Ömür et al. defined the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ as follows: For $r < 0$ or $n \leq 0$, $H_n^r(\alpha) = 0$ and for $n \geq 1$, the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ are defined by

$$H_n^r(\alpha) = \sum_{i=1}^{n} H_i^{r-1}(\alpha), \quad \text{for } r \geq 1,$$

2020 Mathematics Subject Classification: 05A15, 05A19, 11B73

Keywords and phrases: Generalized hyperharmonic numbers of order r; Daehee numbers; Stirling numbers of the first kind and the second kind; generating function.

where $H_n^0(\alpha) = \frac{1}{n\alpha^n}$. For $\alpha = 1$, $H_n^r(1) = H_n^r$ are the hyperharmonic numbers of order r. The generating function of the generalized hyperharmonic numbers of order r is

$$-\frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r} = \sum_{n=1}^{\infty} H_n^r(\alpha)x^n. \quad (1)$$

In [7,18], the generalized harmonic numbers $H(n,r)$ of rank r are defined as for $n \geq 1$ and $r \geq 0$,

$$H(n,r) = \sum_{1\leq n_0+n_1+\cdots+n_r\leq n} \frac{1}{n_0 n_1 \cdots n_r}$$

or, equivalently, as
$$H(n,r) = \frac{(-1)^{r+1}}{n!}\left(\frac{d^n}{dx^n}\frac{[\ln(1-x)]^{r+1}}{1-x}\right)\Bigg|_{x=0}.$$

It is clear that $H(n,0) = H_n$.

In [5], $H(n,r,\alpha)$ are defined as for $n \geq 1$ and $r \geq 0$,

$$H(n,r,\alpha) = \sum_{1\leq n_0+n_1+\cdots+n_r\leq n} \frac{1}{n_0 n_1 \cdots n_r \alpha^{n_0+n_1+\cdots+n_r}}$$

or, equivalently, as
$$H(n,r,\alpha) = \frac{(-1)^{r+1}}{n!}\left(\frac{d^n}{dx^n}\frac{\left[\ln\left(1-\frac{x}{\alpha}\right)\right]^{r+1}}{1-x}\right)\Bigg|_{x=0}.$$

For $\alpha = 1$, $H(n,r,1) = H(n,r)$. The generating function of the generalized harmonic numbers of rank r, $H(n,r,\alpha)$ is given by

$$\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x} = \sum_{n=0}^{\infty} H(n,r,\alpha)x^n. \quad (2)$$

The Daehee numbers of order r, D_n^r, are defined by the generating functions to be

$$\left(\frac{\ln(1+x)}{x}\right)^r = \sum_{n=0}^{\infty} D_n^r \frac{x^n}{n!}. \quad (3)$$

For $r = 1$, $D_n^1 = D_n$ are called Daehee numbers.

The Cauchy numbers of order r, C_n^r, are defined by the generating functions to be

$$\left(\frac{x}{\ln(1+x)}\right)^r = \sum_{n=0}^{\infty} C_n^r \frac{x^n}{n!}. \quad (4)$$

The Stirling numbers of the first kind $S_1(n,k)$ are defined by

$$x^{\underline{n}} = \sum_{k=0}^{n} S_1(n,k)x^k,$$

and the Stirling numbers of the second kind $S_2(n,k)$ are defined by

$$x^n = \sum_{k=0}^{n} S_2(n,k)x^{\underline{k}},$$

where $x^{\underline{n}}$ stands for the falling factorial defined by $x^{\underline{0}} = 1$ and $x^{\underline{n}} = x(x-1)\ldots(x-n+1)$.

The generating function of the Stirling numbers of the first kind $S_1(n,k)$ is given by

$$\frac{(\ln(1+x))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{x^n}{n!}, \quad \text{for } k \geq 0, \tag{5}$$

and the generating function of the Stirling numbers of the second kind $S_2(n,k)$ is given by

$$\frac{(e^x-1)^k}{k!} = \sum_{n=k}^{\infty} S_2(n,k)\frac{x^n}{n!}, \quad \text{for } k \geq 0. \tag{6}$$

The generalized geometric series are given by for any positive integer a,

$$\frac{1}{(1-x)^{a+1}} = \sum_{n=0}^{\infty}\binom{n+a}{n}x^n. \tag{7}$$

In [12], Kwon et al. investigated some explicit identities of Daehee numbers, using differential equations arising the generating function of Daehee numbers. For example, for positive integer N and nonnegative integer n,

$$D_{n+N-1} = \frac{(-1)^{N-1}(N-1)!}{n+N}\sum_{k=0}^{n}(-1)^k N^k S_1(n,k).$$

In [17], Rim et al. gave some identities involving hyperharmonic numbers, the Stirling numbers of the second kind and Daehee number as follows: for any positive integer N and nonnegative integer n,

$$D_{n+N-1} = (n+N-1)^{\underline{N-1}}\sum_{k=0}^{n+N}\binom{r}{n+N-k}(-1)^{k+1}H_k^r,$$

$$(-1)^n(N-1)!N^n = \sum_{i=0}^{n}\sum_{k=0}^{i+N}\binom{r}{i+N-k}(-1)^{N-k}(i+N)^{\underline{N}}i!S_2(n,i)H_k^r.$$

In [5], Duran et al. obtained sums including generalized hyperharmonic numbers and special numbers. For example, for any positive integers n, r, m and α,

$$D_n^{r+1} = n!\alpha^{n+1}\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{m}{n-i}\frac{(-1)^j}{\alpha^{i-j}(i-j)!}D_{i-j}^r H_{j+1}^m(\alpha).$$

It is known that for an ordinary series $f(x) = \sum_{n\geq 0} f_n x^n$,

$$\frac{d^m}{dx^m}f(x) = \sum_{n=0}^{\infty}\binom{m+n}{n}m!f_{n+m}x^n. \tag{8}$$

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and $G(x) = \sum_{n=0}^{\infty} b_n x^n$ be two generating functions. The product of these functions is given as follows:

$$F(x)G(x) = \left(\sum_{n=0}^{\infty}a_n x^n\right)\left(\sum_{n=0}^{\infty}b_n x^n\right) = \sum_{n=0}^{\infty}c_n x^n, \tag{9}$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Let $H(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$ and $K(x) = \sum_{n=0}^{\infty} k_n \frac{x^n}{n!}$ be two exponential generating functions. The product of these functions is given by

$$H(x)K(x) = \left(\sum_{n=0}^{\infty} h_n \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} k_n \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} l_n \frac{x^n}{n!}, \tag{10}$$

where $l_n = \sum_{i=0}^{n} \binom{n}{i} h_i k_{n-i}$.

Harmonic numbers and generalized harmonic numbers have been studied since the distant past and are involved in a wide range of diverse fields such as analysis, computer science and various in a wide range of diverse fields such as analysis, computer science and various branches of number theory [3–5, 13, 14, 19, 21].

Recently, many famous mathematicians have studied the combinatorial properties of special numbers and polynomials by using differential equations associated with the generating function [8, 9, 11]. There are some works including various special numbers arising from the differential equations [8, 10, 12, 16, 17].

2. Some identities arising from nonlinear differential equations

In this section, inspired by studies in [12, 17], we set for any positive integer α and variable x $G := G(x) = \ln\left(1 - \frac{x}{\alpha}\right)$, and from here, for every integer $r \geq 0$, $F := F(x) = (G(x))^{r+1}$.

In this paper, we denote the $N-$times product and the Nth derivative of F, respectively, by F^N and $F^{(N)}$. From the definitions of G and F, by differentiating these functions according to x, we then obtain

$$G' = -\frac{1}{\alpha}e^{-G} \quad \text{and} \quad F' = -\frac{r+1}{\alpha}G^r e^{-G},$$

$$G^{''} = -\frac{1}{\alpha^2}e^{-2G} \quad \text{and} \quad F'' = \frac{r+1}{\alpha^2}e^{-2G}\left(rG^{r-1} - G^r\right),$$

$$G^{(3)} = -\frac{2}{\alpha^3}e^{-3G} \quad \text{and} \quad F^{(3)} = -\frac{r+1}{\alpha^3}e^{-3G}\left(r(r-1)G^{r-2} - 3rG^{r-1} + 2G^r\right).$$

By repeating this process, we easily have

$$e^{NG}G^{(N)} = -\frac{(N-1)!}{\alpha^N},$$

and

$$e^{NG}F^{(N)} = (-1)^N \frac{(r+1)!}{\alpha^N} \sum_{i=0}^{r} S_1(N, r-i+1)\frac{G^i}{i!}. \tag{11}$$

It is clearly known that

$$\frac{G^i}{i!} = \sum_{n=0}^{\infty} (-1)^n \alpha^{-n} S_1(n, i) \frac{x^n}{n!}, \tag{12}$$

$$e^{NG} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (-1)^n \alpha^{-n} N^i S_1(n,i) \frac{x^n}{n!}, \tag{13}$$

and

$$G^{(N)} = \frac{N!}{\alpha^N} \sum_{n=0}^{\infty} \sum_{i=0}^{n} (-1)^{i+n+1} \alpha^{-n} N^{i-1} S_1(n,i) \frac{x^n}{n!}. \tag{14}$$

Now we can give some identities concerning the generalized hyperharmonic numbers of order r, the Daehee numbers of order r and the Stirling numbers of the first and second kind.

Theorem 2.1. *For any positive integers N, n, r and α, we have*

$$\sum_{i=0}^{n} (-1)^i N^{i-1} S_1(n,i) = (-1)^N \alpha^{N+n} \frac{(n+N)!}{N!} \sum_{j=0}^{n+N} (-1)^j \binom{r}{n+N-j} H_j^r(\alpha).$$

Proof. By (1), (2) and (8), we have

$$\begin{aligned}
G^{(N)} &= \frac{d^N}{dx^N}\left(\frac{\ln(1-\frac{x}{\alpha})}{(1-x)^r}(1-x)^r\right) = \frac{d^N}{dx^N}\left(\sum_{i=0}^{\infty} H_i^r(\alpha) x^i \sum_{j=0}^{\infty} (-1)^{j+1}\binom{r}{j} x^j\right) \\
&= \frac{d^N}{dx^N}\left(\sum_{n=0}^{\infty}\sum_{i=0}^{n} (-1)^{n-i+1}\binom{r}{n-i} H_i^r(\alpha) x^n\right) \\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n} \frac{d^N}{dx^N}\binom{r}{n-i}(-1)^{n-i+1} H_i^r(\alpha) x^n \\
&= N! \sum_{n=N}^{\infty}\sum_{i=0}^{n} (-1)^{n-i+1}\binom{r}{n-i}\binom{n}{N} H_i^r(\alpha) x^{n-N} \\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n+N} (-1)^{n+N-i+1}\binom{r}{n+N-i}\binom{n+N}{N} N! H_i^r(\alpha) x^n. \qquad (15)
\end{aligned}$$

By (14) and (15), comparing the coefficients on both sides, we have the proof. □

Theorem 2.2. *For any positive integers N, n, r and α, we have*

$$\alpha^{n+N}(n+N-1)! \sum_{i=0}^{n+N} (-1)^{i+1} \binom{r}{n+N-i} H_i^r(\alpha) = D_{n+N-1}.$$

Proof. By (8), we have

$$\begin{aligned}
G^{(N)} &= \frac{d^N}{dx^N}\left(\frac{\ln(1-\frac{x}{\alpha})}{\frac{x}{\alpha}}\frac{x}{\alpha}\right) = \frac{d^N}{dx^N}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha^n (n-1)!} D_{n-1} x^n\right) \\
&= \sum_{n=N}^{\infty} (-1)^n \binom{n}{N} \frac{N!}{\alpha^n (n-1)!} D_{n-1} x^{n-N} \\
&= \sum_{n=0}^{\infty} (-1)^{n+N} \binom{n+N}{N} \frac{N!}{\alpha^{n+N}(n+N-1)!} D_{n+N-1} x^n. \qquad (16)
\end{aligned}$$

Thus, from (15) and (16), the proof is complete. □

THEOREM 2.3. *For any positive integers N, n, r and α, we have*

$$\sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^i N^{i-1} S_1(k,i) S_2(n,k)$$
$$= (-1)^N \alpha^N \sum_{k=0}^{n}\sum_{j=0}^{k+N}(-1)^j \alpha^k k! \binom{r}{k+N-j}\binom{k+N}{N} H_j^r(\alpha) S_2(n,k).$$

Proof. Substituting $\alpha(1-e^x)$ instead of x in (14) and (15), respectively, we have

$$\begin{aligned} G^{(N)}(\alpha(1-e^x)) &= -\frac{N!}{\alpha^N}\sum_{k=0}^{\infty}\sum_{i=0}^{k}(-1)^i N^{i-1} S_1(k,i)\frac{(e^x-1)^k}{k!} \\ &= -\frac{N!}{\alpha^N}\sum_{k=0}^{\infty}\sum_{i=0}^{k}(-1)^i N^{i-1} S_1(k,i)\sum_{n=0}^{\infty} S_2(n,k)\frac{x^n}{n!} \\ &= -\frac{N!}{\alpha^N}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^i N^{i-1} S_1(k,i) S_2(n,k)\frac{x^n}{n!}, \end{aligned}$$

and

$$\begin{aligned} &G^{(N)}(\alpha(1-e^x)) \\ &= -\sum_{k=0}^{\infty}\sum_{j=0}^{k+N}(-1)^{k+N-j}\binom{r}{k+N-j}\binom{k+N}{N} N! H_j^r(\alpha)(-\alpha)^k (e^x-1)^k \\ &= -\sum_{k=0}^{\infty}\sum_{j=0}^{k+N}(-1)^{k+N-j}\binom{r}{k+N-j}\binom{k+N}{N} N! H_j^r(\alpha)(-\alpha)^k k! \times \sum_{n=0}^{\infty} S_2(n,k)\frac{x^n}{n!} \\ &= -\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k+N}(-1)^{N-j}\alpha^k k! N! \binom{r}{k+N-j}\binom{k+N}{N} H_j^r(\alpha) S_2(n,k)\frac{x^n}{n!}. \end{aligned}$$

Comparing the coefficients of x^n in the first and last series, the proof is complete. □

THEOREM 2.4. *For any positive integers N, n, m and r, we have*

$$S_1(n+N, r+1)\binom{n+N+m}{m}\binom{r+m+1}{m}^{-1}$$
$$= \sum_{i=0}^{n+N}\binom{n+N+m}{i} C_i^m S_1(n+N+m-i, r+m+1).$$

Proof. By (5), we have

$$F = \sum_{n=0}^{\infty}(-1)^n \alpha^{-n} S_1(n, r+1)(r+1)!\frac{x^n}{n!},$$

and from here, by (8)

$$F^{(N)}=\sum_{n=0}^{\infty}\frac{(-1)^{n+N}S_1(n+N,r+1)(r+1)!}{\alpha^{n+N}(n+N)!}N!\binom{n+N}{N}x^n. \tag{17}$$

(4) and (5) yield that

$$\begin{aligned}F&=(-1)^m\left(\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+m+1}\frac{(-x/\alpha)^m}{\left(\ln\left(1-\frac{x}{\alpha}\right)\right)^m}\frac{\alpha^m}{x^m}\\&=\sum_{n=0}^{\infty}\frac{(-1)^{n+m}S_1(n,r+m+1)(r+m+1)!}{\alpha^n n!}x^{n-m}\sum_{n=0}^{\infty}(-1)^n\alpha^{m-n}C_n^m\frac{x^n}{n!}\\&=\sum_{n=0}^{\infty}\frac{(-1)^nS_1(n+m,r+m+1)(r+m+1)!}{\alpha^{n+m}(n+m)!}x^n\sum_{n=0}^{\infty}(-1)^n\alpha^{m-n}C_n^m\frac{x^n}{n!}.\end{aligned}$$

By (8) and (9), we have

$$\begin{aligned}F^{(N)}&=\sum_{n=N}^{\infty}\sum_{i=0}^{n}(-1)^n\binom{n}{N}N!\frac{C_i^mS_1(n+m-i,r+m+1)(r+m+1)!}{\alpha^n i!(n+m-i)!}x^{n-N}\\&=\sum_{n=0}^{\infty}\sum_{i=0}^{n+N}(-1)^{n+N}\binom{n+N}{N}N!C_i^m\times\frac{S_1(n+N+m-i,r+m+1)(r+m+1)!}{\alpha^{n+N}(n+N+m-i)!}\frac{x^n}{i!}. \end{aligned}\tag{18}$$

Thus, comparing the coefficients on right side of (17) and (18), we have the result. □

LEMMA 2.5 ([20]). *Let n and m be any positive integers. For $0\le m\le n-1$, then*

$$\sum_{k=0}^{n}\binom{m-k}{n-k}(1-x)^k=(1-x)^{m+1}(-x)^{n-m-1}.$$

THEOREM 2.6. *Let m,t be any integers such that $0\le m\le t-1$. For positive integers n, r and α we have:*

$$\sum_{j=0}^{n}\sum_{k=0}^{t}(-1)^{t+m+1}\binom{m-k}{t-k}H(j+t-m-1,r-1,\alpha)H_{n-j}^{m-k+1}(\alpha)=H(n,r,\alpha).$$

Proof. With the help of Lemma 2.5, we have

$$\begin{aligned}\frac{\left(\ln(1-\frac{x}{\alpha})\right)^{r+1}}{1-x}&=\sum_{k=0}^{t}\binom{m-k}{t-k}(1-x)^k\frac{\left(\ln(1-\frac{x}{\alpha})\right)^{r+1}}{1-x}(1-x)^{-m-1}(-x)^{-t+m+1}\\&=\sum_{k=0}^{t}\binom{m-k}{t-k}\frac{\left(\ln(1-\frac{x}{\alpha})\right)^{r}}{(1-x)}\frac{\ln(1-\frac{x}{\alpha})}{(1-x)^{m-k+1}}(-x)^{-t+m+1},\end{aligned}$$

and from (2) and (9),

$$\begin{aligned}&\frac{\left(\ln(1-\frac{x}{\alpha})\right)^{r+1}}{1-x}\\&=\sum_{k=0}^{t}\binom{m-k}{t-k}\left((-1)^r\sum_{n=0}^{\infty}H(n,r-1,\alpha)x^n\right)\times\left(-\sum_{j=0}^{\infty}H_j^{m-k+1}(\alpha)x^j\right)(-x)^{-t+m+1}\end{aligned}$$

$$= \sum_{k=0}^{t}(-1)^{r-t+m}\binom{m-k}{t-k}\sum_{n=0}^{\infty}H(n,r-1,\alpha)x^{n-t+m+1} \times \sum_{j=0}^{\infty}H_j^{m-k+1}(\alpha)x^j$$

$$= \sum_{k=0}^{t}(-1)^{r-t+m}\binom{m-k}{t-k}\sum_{n=-t+m+1}^{\infty}H(t+n-m-1,r-1,\alpha)x^{n} \times \sum_{j=0}^{\infty}H_j^{m-k+1}(\alpha)x^j$$

$$= \sum_{k=0}^{t}(-1)^{r-t+m}\binom{m-k}{t-k}\sum_{n=0}^{\infty}\sum_{j=0}^{n}H(j+t-m-1,r-1,\alpha)H_{n-j}^{m-k+1}(\alpha)x^n$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{t}(-1)^{r-t+m}\binom{m-k}{t-k}H(j+t-m-1,r-1,\alpha) \times H_{n-j}^{m-k+1}(\alpha)x^n. \tag{19}$$

Thus, comparing the coefficients on right side of (2) and (19), we have the proof. □

The proof of the following lemma is easily obtained.

Lemma 2.7. *For $n \geq 0$, then $\sum_{k=0}^{n} k\binom{n}{k}x^k(1-x)^{n-k} = nx$.*

Theorem 2.8. *For any positive integers n, m, r and α, we have*

$$H(n,r,\alpha) = \sum_{j=0}^{n}\sum_{k=1}^{m}\binom{m-1}{k-1}H(j-k+1,r-1,\alpha)H_{n-j}^{k-m}(\alpha).$$

Proof. By (1), (2) and Lemma 2.7, the proof is similar to the proof of Theorem 2.6. □

Theorem 2.9. *Let N, n, r be any positive integers. Then*

$$n!\sum_{k=0}^{r}(-1)^k\binom{r}{k}\binom{Nk}{n} = \sum_{k=0}^{n}\sum_{i=0}^{r}(-1)^i\binom{r}{i}\binom{n}{k}i^kN^kD_{n-k}^k.$$

Proof. From definition of e^{NG}, we have $e^{NG}-1 = \left(1-\frac{x}{\alpha}\right)^N - 1$. From here, we write

$$\left(\left(1-\frac{x}{\alpha}\right)^N - 1\right)^r = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\left(1-\frac{x}{\alpha}\right)^{kN} = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\sum_{n=0}^{\infty}(-1)^n\binom{kN}{n}\frac{x^n}{\alpha^n}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{r}(-1)^{r-k+n}\binom{r}{k}\binom{kN}{n}\frac{x^n}{\alpha^n}$$

and by (3),

$$\left(e^{NG}-1\right)^r = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}e^{NGi} = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}\sum_{k=0}^{\infty}\frac{i^kN^k}{k!}\frac{\left(\ln\left(1-\frac{x}{\alpha}\right)\right)^k}{\left(-\frac{x}{\alpha}\right)^k}\left(-\frac{x}{\alpha}\right)^k$$

$$= \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}\sum_{k=0}^{\infty}\frac{i^kN^k}{k!}x^k\sum_{n=0}^{\infty}(-1)^{n+k}\frac{D_n^k}{\alpha^{n+k}}\frac{x^n}{n!}$$

$$= \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{i^kN^k}{k!}\frac{(-1)^n}{(n-k)!\alpha^n}D_{n-k}^k x^n$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{r}(-1)^{n+r-i}\binom{r}{i}\frac{i^k N^k}{k!(n-k)!\alpha^n}D_{n-k}^{k}x^n.$$

If we compare the coefficients of x^n in the first and last series, we have the proof. □

THEOREM 2.10. *For any positive integers N, n and r, we have*

$$\sum_{k=0}^{r} S_1(N, r+1-k)S_1(n,k)$$
$$= \frac{N!n!}{(r+1)!}\sum_{j=0}^{n}\sum_{i=0}^{j}\frac{N^i}{j!\,(n-j+N-r-1)!}\binom{n-j+N}{N}S_1(j,i)\,D_{n+N-j-r-1}^{r+1}.$$

Proof. By (5), (11) and (12), we write

$$e^{NG}F^{(N)} = (-1)^N\frac{(r+1)!}{\alpha^N}\sum_{k=0}^{r}S_1(N, r-k+1)\frac{G^k}{k!}$$
$$= (-1)^N\frac{(r+1)!}{\alpha^N}\sum_{k=0}^{r}S_1(N, r-k+1)\sum_{i=k}^{\infty}(-1)^i\alpha^{-i}S_1(i,k)\frac{x^i}{i!}$$
$$= (-1)^N\frac{(r+1)!}{\alpha^N}\sum_{i=0}^{\infty}\sum_{k=0}^{r}\frac{(-1)^i}{\alpha^i}S_1(N, r-k+1)S_1(i,k)\frac{x^i}{i!}. \qquad (20)$$

With the help of (3), we have

$$F = \frac{(\ln(1-\frac{x}{\alpha}))^{r+1}}{(-\frac{x}{\alpha})^{r+1}}\left(\frac{-x}{\alpha}\right)^{r+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha^n(n-r-1)!}D_{n-r-1}^{r+1}x^n,$$

and then taking the Nth derivative of function F, by (8),

$$F^{(N)} = \sum_{n=0}^{\infty}(-1)^{n+N}\binom{n+N}{N}\frac{N!}{\alpha^{n+N}(n+N-r-1)!}D_{n+N-r-1}^{r+1}x^n. \qquad (21)$$

From here, (9) and (13) yield that

$$e^{NG}F^{(N)} = (-1)^N\frac{N!}{\alpha^N}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}\frac{(-1)^n N^i}{j!(n-j+N-r-1)!\alpha^n}$$
$$\times\binom{n-j+N}{N}S_1(j,i)\,D_{n+N-j-r-1}^{r+1}x^n.$$

From here, by (20), the comparison of the coefficients on both sides, the proof is obtained. □

THEOREM 2.11. *For any positive integers N, n and r, we have*

$$D_{n+N-r-1}^{r+1}\binom{n+N}{r+1} = \sum_{j=0}^{n}\sum_{k=0}^{r}\sum_{i=0}^{n-j}(-1)^i\binom{n}{j}N^i S_1(j,k)\,S_1(N, r-k+1)S_1(n-j,i).$$

Proof. By (5), (11) and (12), we write

$$F^{(N)} = (-1)^N \frac{(r+1)!}{\alpha^N} e^{-NG} \sum_{k=0}^{r} S_1(N, r-k+1) \frac{G^k}{k!}$$

$$= (-1)^N \frac{(r+1)!}{\alpha^N} \sum_{n=0}^{\infty} \sum_{i=0}^{n} (-1)^{n+i} \frac{N^i S_1(n,i)}{\alpha^n} \frac{x^n}{n!} \times \sum_{i=0}^{\infty} \sum_{k=0}^{r} (-1)^i \frac{S_1(i,k)}{\alpha^i} S_1(N, r-k+1) \frac{x^i}{i!}.$$

(10) yields that

$$F^{(N)} = (-1)^N \frac{(r+1)!}{\alpha^N} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{r} \sum_{i=0}^{n-j} (-1)^{n+i} \frac{N^i}{n!\alpha^n} \times \binom{n}{j} S_1(j,k) S_1(N, r-k+1) S_1(n-j, i) x^n. \tag{22}$$

Also,

$$F^{(N)} = \frac{d^N}{dx^N} \left(\left(\frac{\ln\left(1-\frac{x}{\alpha}\right)}{-\frac{x}{\alpha}} \right)^{r+1} \left(-\frac{x}{\alpha}\right)^{r+1} \right) = \frac{d^N}{dx^N} \left(\sum_{n=0}^{\infty} (-1)^n \frac{D_{n-r-1}^{r+1}}{\alpha^n (n-r-1)!} x^n \right)$$

$$= \sum_{n=0}^{\infty} (-1)^{n+N} \binom{n+N}{N} \frac{N!}{\alpha^{n+N}(n+N-r-1)!} D_{n+N-r-1}^{r+1} x^n. \tag{23}$$

Thus, comparing the coefficients on right side of (22) and (23), we have the proof. □

Theorem 2.12. *For any positive integers N, n, m and r, we have*

$$\sum_{i=0}^{n} \sum_{k=0}^{r} (-1)^i S_1(N, r-k+1) S_1(i,k) S_2(n,i)$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{i=0}^{k} \sum_{m=0}^{n-j} (-1)^{m+k} N^i \binom{m+N}{r+1} \binom{n}{j} \times S_1(k,i) S_2(j,k) S_2(n-j,m) D_{m+N-r-1}^{r+1}.$$

Proof. Substituting $\alpha(e^x-1)$ instead of x in (13) and (21), respectively, we have

$$e^{NG} F^{(N)} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (-1)^n N^i S_1(n,i) \frac{(e^x-1)^n}{n!} \sum_{n=0}^{\infty} (-1)^{n+N} N! D_{n+N-r-1}^{r+1} \binom{n+N}{N} \times \frac{(e^x-1)^n}{\alpha^N (n+N-r-1)!}.$$

(6) yields that

$$e^{NG} F^{(N)} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (-1)^n S_1(n,i) N^i \sum_{k=n}^{\infty} S_2(k,n) \frac{x^k}{k!} \sum_{n=0}^{\infty} (-1)^{n+N} D_{n+N-r-1}^{r+1} \times \frac{(n+N)!}{\alpha^N (n+N-r-1)!} \sum_{k=n}^{\infty} S_2(k,n) \frac{x^k}{k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^k N^i S_1(k,i)\, S_2(n,k)\frac{x^n}{n!}$$
$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^{m+N}\frac{(m+N)!}{\alpha^N(m+N-r-1)!}S_2(n,m)D_{m+N-r-1}^{r+1}\frac{x^n}{n!}.$$

By (10), we have

$$e^{NG}F^{(N)} = \frac{(r+1)!}{\alpha^N}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{i=0}^{k}\sum_{m=0}^{n-j}(-1)^{m+k+N} N^i\binom{m+N}{r+1}\binom{n}{j}$$
$$\times S_1(k,i)\, S_2(j,k)S_2(n-j,m)D_{m+N-r-1}^{r+1}\frac{x^n}{n!}. \tag{24}$$

Similarly, substituting $\alpha\,(e^x-1)$ instead of x in (11), by (6) and (12),

$$e^{NG}F^{(N)} = (-1)^N\frac{(r+1)!}{\alpha^N}\sum_{j=0}^{r}S_1(N,r-j+1)\frac{G^j}{j!}$$
$$= (-1)^N\frac{(r+1)!}{\alpha^N}\sum_{j=0}^{r}S_1(N,r-j+1)\sum_{i=0}^{\infty}(-1)^i S_1(i,j)\,\frac{(\alpha\,(e^x-1))^i}{\alpha^i i!}$$
$$= (-1)^N\frac{(r+1)!}{\alpha^N}\sum_{i=0}^{\infty}\sum_{j=0}^{r}(-1)^i S_1(N,r-j+1)S_1(i,j)\,\frac{(e^x-1)^i}{i!}$$
$$= (-1)^N\frac{(r+1)!}{\alpha^N}\sum_{i=0}^{\infty}\sum_{j=0}^{r}(-1)^i S_1(N,r-j+1)S_1(i,j)\sum_{j=i}^{\infty}S_2(j,i)\frac{x^j}{j!}$$
$$= (-1)^N\frac{(r+1)!}{\alpha^N}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{r}(-1)^i S_1(N,r-j+1)S_1(i,j)\,S_2(n,i)\frac{x^n}{n!}. \tag{25}$$

Thus, comparing the coefficients on right side of (24) and (25), we have the proof. □

THEOREM 2.13. *For any positive integers N, n and r, we have*

$$(r+1)\sum_{k=0}^{r}S_1(N,r-k+1)S_1(n,k)$$
$$= \sum_{j=0}^{n}\sum_{k=0}^{j+N}\sum_{i=0}^{n-j}\binom{n}{j}\binom{j+N}{k+1}(k+1)\,N^i S_1(k,r)S_1(n-j,i)D_{j+N-k-1}.$$

Proof. From (3) and (5), we have

$$F = \sum_{n=0}^{\infty}(-1)^n\alpha^{-n}r!S_1(n,r)\frac{\ln\left(1-\frac{x}{\alpha}\right)}{-\frac{x}{\alpha}}\left(-\frac{x}{\alpha}\right)\frac{x^n}{n!}$$
$$= \sum_{n=0}^{\infty}(-1)^n\alpha^{-n}r!S_1(n,r)\frac{x^n}{n!}\sum_{k=0}^{\infty}(-1)^k\,\frac{D_{k-1}}{\alpha^k(k-1)!}x^k.$$

By (9),

$$F=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n}\frac{r!}{\alpha^{n}k!\,(n-k-1)!}S_{1}(k,r)D_{n-k-1}x^{n}.$$

Thus, from (8), we get

$$F^{(N)}=N!\sum_{n=0}^{\infty}\sum_{k=0}^{n+N}(-1)^{n+N}\binom{n+N}{N}\frac{r!S_{1}(k,r)}{\alpha^{n+N}k!\,(n+N-k-1)!}D_{n+N-k-1}x^{n}.$$

Notice that from (11),

$$e^{NG}F^{(N)}=$$

$$N!r!\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j+N}\sum_{i=0}^{n-j}\binom{j+N}{N}N^{i}\frac{(-1)^{n+N}S_{1}(k,r)S_{1}\left(n-j,i\right)}{\alpha^{n+N}k!(n-j)!\,(j+N-k-1)!}\times D_{j+N-k-1}x^{n}. \quad (26)$$

Thus, comparing the coefficients on right side of (20) and (26) yield the desired result. □

We also give the following identities with the generalized harmonic numbers of rank r, $H\left(n,r,\alpha\right)$ and the Stirling number of the first kind.

Theorem 2.14. *Let N, n, r and α be any positive integers. For any integer $m\geq 2$,*

$$\sum_{j=0}^{n+N}\sum_{i=0}^{j}(-1)^{j-i}\binom{m}{j-i}\binom{n+N-j+m-2}{n+N-j}H(i,r,\alpha)$$

$$=\frac{(-1)^{N+n+r+1}(r+1)!}{\alpha^{N+n}(n+N)!}\sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{j=0}^{r}(-1)^{k}N^{k}\binom{n}{i}S_{1}(N,r+1-j)\times S_{1}\left(n-i,j\right)S_{1}\left(i,k\right).$$

Proof. By (7), (8) and (9), we have

$$\begin{aligned}
F^{(N)}&=\frac{d^{N}}{dx^{N}}\left(\frac{(-1)^{r+1}\left(-\ln(1-\frac{x}{\alpha})\right)^{r+1}}{1-x}\left(1-x\right)^{m}\frac{1}{\left(1-x\right)^{m-1}}\right)\\
&=\frac{d^{N}}{dx^{N}}\left((-1)^{r+1}\sum_{n=0}^{\infty}H(n,r,\alpha)x^{n}\sum_{i=0}^{\infty}(-1)^{i}\binom{m}{i}x^{i}\sum_{j=0}^{\infty}\binom{j+m-2}{j}x^{j}\right)\\
&=\frac{d^{N}}{dx^{N}}\left((-1)^{r+1}\sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^{n-i}\binom{m}{n-i}H(i,r,\alpha)x^{n}\sum_{j=0}^{\infty}\binom{j+m-2}{j}x^{j}\right)\\
&=\frac{d^{N}}{dx^{N}}\left((-1)^{r+1}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{j-i}\binom{m}{j-i}\binom{n-j+m-2}{n-j}H(i,r,\alpha)x^{n}\right)\\
&=(-1)^{r+1}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}\frac{d^{N}}{dx^{N}}\left((-1)^{j-i}\binom{m}{j-i}\binom{n-j+m-2}{n-j}H(i,r,\alpha)x^{n}\right)
\end{aligned}$$

$$= (-1)^{r+1} \sum_{n=0}^{\infty} \sum_{j=0}^{n+N} \sum_{i=0}^{j} (-1)^{j-i} N! \binom{m}{j-i} \binom{n+N-j+m-2}{n+N-j} \binom{n+N}{N} \times H(i,r,\alpha)x^n. \quad (27)$$

Thus, comparing the coefficients on right side of (22) and (27), we have the proof. □

THEOREM 2.15. *Let N, n, r and α be any positive integers. For any integer $m \geq 2$,*

$$(-1)^{N+n+r+1} \frac{(r+1)!}{N!n!\alpha^{N+n}} \sum_{k=0}^{r} S_1(N, r-k+1) S_1(n,k)$$
$$= \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{j=0}^{n-k+N} \sum_{i=0}^{j} \frac{N^l (-1)^{k+j-i}}{\alpha^k k!} S_1(k,l) H(i,r,\alpha) \times \binom{m}{j-i} \binom{n-k+N-j+m-2}{m-2} \binom{n+N-k}{N}.$$

Proof. By (13) and (27), we write

$$e^{NG} F^{(N)} = (-1)^{r+1} \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{N^l S_1(n,l)(-1)^n}{\alpha^n} \right) \frac{x^n}{n!}$$
$$\times \sum_{n=0}^{\infty} \sum_{j=0}^{n+N} \sum_{i=0}^{j} (-1)^{j-i} \binom{m}{j-i} \binom{n+N-j+m-2}{n+N-j} \binom{n+N}{N} N! H(i,r,\alpha) x^n$$
$$= (-1)^{r+1} N! \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{j=0}^{n-k+N} \sum_{i=0}^{j} (-1)^{k+j-i} \frac{N^l}{\alpha^k k!} \binom{m}{j-i}$$
$$\times \binom{n-k+N-j+m-2}{n-k+N-j} \binom{n-k+N}{N} S_1(k,l) H(i,r,\alpha) x^n. \quad (28)$$

Comparing the coefficients on right side of (20) and (28), the proof is complete. □

THEOREM 2.16. *Let N, n, r and α be any positive integers. For any integer $m \geq 2$,*

$$\frac{S_1(n+N, r+1)}{n!}$$
$$= \sum_{j=0}^{n} \sum_{i=0}^{n} \sum_{l=0}^{i} \sum_{z=0}^{l} \sum_{k=0}^{r} (-1)^{l-z-i} \binom{m}{l-z} \binom{i-l+m-2}{i-l} \frac{N^j \alpha^i}{j!(n-i)!}$$
$$\times S_1(N, r-k+1) S_1(n-i,k) H(z, j-1, \alpha).$$

Proof. By (11) and (12), we have

$$F^{(N)} = (-1)^N \frac{(r+1)!}{\alpha^N} \sum_{i=0}^{\infty} \sum_{k=0}^{r} (-1)^i \alpha^{-i} S_1(i,k) S_1(N, r-k+1) \frac{x^i}{i!} e^{-NG},$$

and from the generating function of exponential function,

$$F^{(N)} = (-1)^N \frac{(r+1)!}{\alpha^N} \sum_{i=0}^{\infty} \sum_{k=0}^{r} (-1)^i \alpha^{-i} S_1(i,k) S_1(N, r-k+1) \frac{x^i}{i!}$$

$$\times \sum_{j=0}^{\infty} \frac{N^j}{j!} \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^j}{1-x} (1-x)^m \frac{1}{(1-x)^{m-1}}.$$

From here, for $m \geq 2$, by (7) and (9), we have

$$\begin{aligned}
F^{(N)} &= (-1)^N \frac{(r+1)!}{\alpha^N} \sum_{i=0}^{\infty} \sum_{k=0}^{r} (-1)^i \alpha^{-i} S_1(i,k)\, S_1(N, r-k+1) \frac{x^i}{i!} \\
&\quad \times \sum_{j=0}^{\infty} \frac{N^j}{j!} \sum_{n=0}^{\infty} H(n, j-1, \alpha) x^n \sum_{z=0}^{\infty} (-1)^z \binom{m}{z} x^z \sum_{l=0}^{\infty} \binom{l+m-2}{l} x^l \\
&= (-1)^N \frac{(r+1)!}{\alpha^N} \sum_{i=0}^{\infty} \sum_{k=0}^{r} (-1)^i \alpha^{-i} S_1(i,k)\, S_1(N, r-k+1) \frac{x^i}{i!} \\
&\quad \times \sum_{j=0}^{\infty} \frac{N^j}{j!} \sum_{n=0}^{\infty} \sum_{z=0}^{n} (-1)^{n-z} \binom{m}{n-z} H(z, j-1, \alpha) x^n \sum_{l=0}^{\infty} \binom{l+m-2}{l} x^l \\
&= (-1)^N \frac{(r+1)!}{\alpha^N} \sum_{i=0}^{\infty} \sum_{k=0}^{r} (-1)^i \alpha^{-i} S_1(i,k)\, S_1(N, r-k+1) \frac{x^i}{i!} \\
&\quad \times \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{l=0}^{n} \sum_{z=0}^{l} (-1)^{l-z} \binom{m}{l-z} \binom{n-l+m-2}{n-l} \frac{N^j}{j!} H(z, j-1, \alpha) x^n \\
&= (-1)^N \frac{(r+1)!}{\alpha^N} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{i=0}^{j} \sum_{l=0}^{i} \sum_{z=0}^{l} \sum_{k=0}^{r} (-1)^{l-z+n-i} \binom{m}{l-z} \binom{i-l+m-2}{i-l} \\
&\quad \times H(z, j-1, \alpha) S_1(N, r-k+1) S_1(n-i, k) \frac{N^j}{(n-i)!j!} \frac{x^n}{\alpha^{n-i}}. \qquad (29)
\end{aligned}$$

Comparing the coefficients on right side of (17) and (29), the proof is complete. □

References

[1] A. T. Benjamin, D. Gaebler, R. Gaebler, *A combinatorial approach to hyperharmonic numbers*, Integers,**3(A15)** (2003), 1–9.

[2] A. T. Benjamin, G. O. Preston, J. J. Quinn, *A Stirling encounter with harmonic numbers*, Math. Mag., **75(2)** (2002), 95–103.

[3] G.-S. Cheon, M. El-Mikkawy, *Generalized harmonic numbers with Riordan arrays*, J. Number Theory, **128(2)** (2008), 413–425.

[4] G. Dattoli, H. M. Srivastava, *A note on harmonic numbers, umbral calculus and generating functions*, Appl. Math. Lett., **21(7)** (2008), 686–693.

[5] Ö. Duran, N. Ömür, S. Koparal,*On sums with generalized harmonic, hyperharmonic and special numbers*, Miskolc Math. Notes, **21(2)** (2020), 791–803.

[6] M. Genčev, *Binomial sums involving harmonic numbers*, Math. Slovaca,**61(2)** (2011), 215–226.

[7] A. Gertsch, *Generalized harmonic numbers*, C. R. Acad. Sci. Paris, Sér. I Math., **324(1)** (1997), 7–10.

[8] G.-W. Jang, J. Kwon, J. G. Lee, *Some identities of degenerate Deehee numbers arising from nonlinear differential equation*, Adv. Differential Equations, **2017** (2017), Article ID 206.

[9] T. Kim, D. S. Kim, *Identities involving harmonic and hyperharmonic numbers*, Adv. Differential Equations, **2013** (2013), Article ID 235.

[10] T. Kim, D. S. Kim, *Some identities for Bernoulli numbers of the second kind arising from a non-linear differential equation*, Bull. Korean Math. Soc., **52** (2015), 2001–2010.

[11] T. Kim, D. S. Kim, *Some identities of degenerate Daehee numbers arising from certain differential equations*, J. Nonlinear Sci. Appl., **10** (2017), 744–751.

[12] H. I. Kwon, T. Kim, J. J. Seo, *A note on Daehee numbers arising from differential equations*, Global J. Pure Appl. Math., **12(3)** (2016), 2349–2354.

[13] N. Ömür, G. Bilgin, *Some applications of the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$*, Adv. Appl. Math. Sci, **17(9)** (2018), 617–627.

[14] N. Ömür, S. Koparal, *On the matrices with the generalized hyperharmonic numbers of order r*, Asian-Eur. J. Math., **11(3)** (2018), Article ID 1850045.

[15] N. Ömür, S. Koparal, *Sums involving generalized harmonic and Daehee numbers*, Notes on Number Theory and Discrete Math., **28(1)** (2022), 92–99.

[16] S. S. Pyo, T. Kim, S. H. Rim, *Identities of the degenerate Daehee numbers with the Bernoulli numbers of the second kind arising from nonlinear differential equation*, J. Nonlinear Sci. Appl., **10** (2017), 6219–6228.

[17] S.-H. Rim, T. Kim, S.-S. Pyo, *Identities between harmonic, hyperharmonic and Daehee numbers*, J. Inequal. Appl., **2018** (2018), Article ID 168.

[18] J. M. Santmyer, *A Stirling like sequence of rational numbers*, Discrete Math., **171(1-3)** (1997), 229–235.

[19] A. Sofo, H. M. Srivastava, *Identities for the harmonic numbers and binomial coefficients*, Ramanujan J., **25(1)** (2011), 93–113.

[20] R. Sprugnoli, *Riordan array proofs of identities in Gould's book*, University of Florence, Italy, 2006.

[21] T.-C. Wu, S.-T. Tu, H. M. Srivastava, *Some combinatorial series identities associated with the digamma function and harmonic numbers*, Appl. Math. Lett., **13(3)** (2000), 101–106.

(received 27.11.2022; in revised form 16.05.2023; available online 18.01.2024)

Bursa Uludağ University, Department of Mathematics, Bursa, Turkey
E-mail: sibelkoparal@uludag.edu.tr
ORCID iD: https://orcid.org/0000-0001-9574-9652

Kocaeli University, Department of Mathematics, Kocaeli, Turkey
E-mail: neseomur@gmail.com
ORCID iD: https://orcid.org/0000-0002-3972-9910

Kocaeli University, Department of Mathematics, Kocaeli, Turkey
E-mail: kubranursudemen@hotmail.com
ORCID iD: https://orcid.org/0000-0003-4695-7855