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ITERATIVE SCHEMES FOR VARIATIONAL INEQUALITIES WITH BOUNDEDLY LIPSCHITZIAN AND STRONGLY MONOTONE MAPPINGS

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Abstract. In this paper, for solving a variational inequality problem governed by a boundedly Lipschitzian and strongly monotone mapping over the set of common fixed points of a sufficiently large finite family of nonexpansive mappings on Hilbert spaces, we propose a new strongly convergent self-adaptive hybrid steepest-descent block-iterative scheme. The strong convergence of any sequence, generated by this scheme, is proved under weaker conditions on iterative parameters without any additional assumption on the family of fixed point sets as well as the dimension of the spaces. An application to networked systems and a convex optimization problem over the intersection of a finite family of closed convex subsets with numerical experiments are given for illustration.

1. Introduction

Let H be a Hilbert space equipped with the inner product $\langle .,. \rangle$ and with the corresponding norm $\|.\|$. Let C and F be a closed convex subset and a mapping in H, respectively. The variational inequality problem, considered in this paper, is formulated as finding a point

$$p_* \in C \text{ such that } \langle Fp_*, p_* - p\rangle \leq 0 \ \forall p \in C. \tag{1}$$

The variational inequality problem with a closed convex subset C and an operator F in a Hilbert space H has been firstly introduced and investigated in [7, 14]. A fundamental method to solve (1), when F is l-Lipschitzian and η-strongly monotone, is the projected gradient iterative method

$$x^{k+1} = P_C(I - \mu_k F)x^k, \tag{2}$$

introduced by Goldstein [6], where P_C is the metric projection from H onto C, I is the identity of H, $\mu_k = \mu$, a constant in $(0, 2\eta/l^2)$, and the starting point x^0 is any point in C. Since then, two basic directions to develop method (2) were

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attracted a great number of mathematicians. In the first direction, one tried to relax the Lipschitzian and monotone properties of F and to choose μ_k at each iteration step without knowledge of the values η and l. He et al. [9] suggested a self-adaptive choice in their method as follows. Choose $x_0 \in C$ and set $k := 1$. Calculate x_1 by $x_1 = P_C(x_0 - Fx_0)$. If $x_1 = x_0$ then x_0 is a solution. If it is not the case, the value μ_k in (2) is chosen by $\mu_k = \eta_k/L_k^2$ where η_k and L_k are defined, respectively, by the following rules:

$$\eta_k = \begin{cases} \min\left\{\eta_{k-1}, \frac{\langle Fx_k - Fx_{k-1}, x_k - x_{k-1}\rangle}{\|x_k - x_{k-1}\|^2}, \frac{\langle Fx_k - Fx_0, x_k - x_0\rangle}{\|x_k - x_0\|^2}\right\}, & \text{if } x_k \neq x_0 \\ \min\left\{\eta_{k-1}, \frac{\langle Fx_k - Fx_{k-1}, x_k - x_{k-1}\rangle}{\|x_k - x_{k-1}\|^2}\right\}, & \text{otherwise} \end{cases}$$

and
$$L_k = \begin{cases} \max\left\{L_{k-1}, \frac{\|Fx_k - Fx_{k-1}\|}{\|x_k - x_{k-1}\|}, \frac{\|Fx^k - Fx_0\|}{\|x_k - x_0\|}\right\}, & \text{if } x_k \neq x_0 \\ \max\left\{L_{k-1}, \frac{\|Fx_k - Fx_{k-1}\|}{\|x_k - x_{k-1}\|}\right\}, & \text{otherwise,} \end{cases}$$

where $\eta_0 = \langle Fx_1 - Fx_0, x_1 - x_0\rangle/\|x_1 - x_0\|^2$ and $L_0 = \|Fx_1 - Fx_0\|/\|x_1 - x_0\|$. Some related results in this direction has been given in [5,8,13] where the Lipschitzian and monotone properties of F are replaced by the weaker ones such as non-Lipschitzian, boundedly Lipschitzian and pseudo-monotone. The second direction concerns the closed form expression of P_C, that unfortunately is not always known, and the choice of μ_k in independence on l and η. The first result in this direction has been belonged to Yamada [18]. By replacing P_C by T, a nonexpansive mapping on H, he introduced the hybrid steepest-descent method,

$$x^{k+1} = (I - t_k\mu F)Tx^k, \ k \geq 0, \tag{3}$$

and proved the strong convergence of a sequence $\{x^k\}$, generated by (3), to a point in $\mathrm{Fix}\,(T)$, the fixed point set of T, with the same $\mu \in (0, 2\eta/l^2)$ under the following two conditions:

(t) $t_k \in (0,1)$ for all $k \geq 1$, $\lim_{k\to\infty} t_k = 0$, $\sum_{k\geq 0} t_k = \infty$ and

(t') $\sum_{k=0}^{\infty} |t_k - t_{k+1}| < +\infty$ or $\lim_{k\to\infty}(t_k/t_{k+1}) = 1$.

Several results, related with method (3) in the case that C is the either intersection of fixed point sets of nonexpansive mappings or set of common zeros of a finite family of nonlinear mappings of monotone-type, were given in [2,3]. Buong et al. [2], for solving the accretive variational inequality problem $\langle Fp_*, j(p_* - p)\rangle \leq 0$, an extension of (1) to Banach spaces, where j is the normall duality mapping of a Banach spaces E, F is an η-strongly accretive and γ-strictly pseudocontractive mapping on E with $\eta + \gamma > 1$, and C is the intersection of an infinite family of nonexpancive mappings T_i on E, studied the iterative method

$$x^{k+1} = (I - t_kF)T^kx^k, \quad T^k = (1-\beta_k)I + \beta_kW^k\big((1-\alpha_k)I + \alpha_kW^k\big), \quad k \geq 1,$$

t_k satisfies condition (t), $\beta_k \in [a,b] \subset (0,1), \alpha_k \in [0,\overline{a}]$ with $\overline{a} < 1$, and W^k is a combination of T_i with $i = 1,\ldots,k$. Meantime, for finding a solution of the accretive variational inequality when C is the set of common zeros of a family of m-accretive mappings A_i on E with $i = 1, \dots, N$, Buong et al. [3] introduced the iterative method

$$x^{k+1} = (1-\gamma_k)(I - t_k F)x^k + \gamma_k P_k x^k + e^k, \quad x^1 \in E, \quad k \geq 1,$$

where the parameters

(a) t_k satisfies condition (t);

(b) γ_k satisfies a condition $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$;

(c) $r_k^i \geq \varepsilon > 0$ for all $i = 1, 2, \dots, N$;

(d) either $\sum_{k=1}^{\infty} \|e^k\| < \infty$ or $\lim_{k\to 0} \|e^k\|/t_k = 0$;

and $$P_k = J^{A_N}_{r_k^N} J^{A_{N-1}}_{r_k^{N-1}} \cdots J^{A_1}_{r_k^1}, J^{A_i}_{r_k^i} = (I + r_k^i A_i)^{-1}.$$

Up to now, there are a few works [10,11], where one combined the two basic directions. Recently, when C is the fixed point set of a nonexpansive mapping T and F is boundedly Lipschitzian and η-strongly monotone, He et al. [11] proposed a self-adaptive hybrid steepest-descent method,

$$x^{k+1} = T(I - t_k\mu_k F)x^k, \; k \geq 1, \tag{4}$$

where μ_k is selected through a self-adaptive way: $\mu_k = \eta_k / L_k^2$ with η_k and L_k defined, respectively, by

$$\eta_k = \begin{cases} \min\{\eta_{k-1}, \frac{\langle Fx_k - FT^{k-1}x_0, x_k - T^{k-1}x_0\rangle}{\|x_k - T^{k-1}x_0\|^2}\}, & \text{if } x_k \neq T^{k-1}x_0 \\ \eta_{k-1}, & \text{otherwise,} \end{cases}$$

and $$L_k = \begin{cases} \max\left\{L_{k-1}, \frac{\|Fx_k - FT^{k-1}x_0\|}{\|x_k - T^{k-1}x_0\|}\right\}, & \text{if } x_k \neq T^{k-1}x_0 \\ L_{k-1}, & \text{otherwise,} \end{cases}$$

for arbitrary two initial points x_0 and x_1 such that $x_1 \neq x_0$. They proved a strong convergence result with two conditions (t) and (t').

Let $L = \{1, \dots, m\}$. When m is sufficiently large, for finding a point $p \in C = \cap_{i\in L} C_i$, where C_i is a closed convex subset in H, Butnariu and Censor [4] proved the strong convergence of the almost simultaneous block-iterative projection scheme

$$x^{k+1} = x^k + \alpha_k\big(T^k x^k - x^k\big), \; T^k = \sum_{i\in L} \omega_i^k P_{C_i}, \tag{5}$$

where α_k is a relaxation parameter, satisfying $0 < \underline{\alpha} \leq \alpha_k \leq \overline{\alpha} < 2$, $\omega_i^k \geq 0$ with $\sum_{i\in L} \omega_i^k = 1$, if one of the following conditions conditions are satisfied:

(c_1) there exists $i_0 \in L$ such that $C_{i_0} \cap \mathrm{Int}[\cap_{i\neq i_0} C_i] \neq \emptyset$;

(c_2) all, except for possibly one of the sets C_i, are uniformly convex;

(c_3) each C_i is a halfspace;

(c_4) at least one set C_i is boundedly compact;

(c_5) H is finite-dimensional.

In this paper, we consider problem (1) with $C = \cap_{i\in L} C_i \neq \emptyset$ and the set $C_i = \mathrm{Fix}\,(T_i)$, where T_i is a nonexpansive mapping on H and the number of T_i, m, is sufficiently large. Based on methods (2)-(5), we propose a new self-adaptive hybrid steepest-descent iterative scheme for solving (1) when F is boundedly Lipschitzian and η-strongly monotone. The strong convergence of our scheme is proved without any condition on $C_i = \mathrm{Fix}\,(T_i)$ from (c_1)-(c_5) and (t') above. Moreover, our choice of μ_k is different from that in (4). Note that in this situation, problem (1) possesses a unique solution p_* (see [8]).

The rest of this paper is organized as follows. In Section 2, we list some terminologies, using in this paper, and related facts, that will be used in the proof of our results. In Section 3, we prove a strongly convergent theorem for the introduced scheme. Applications to some networked systems and a convex optimization problem over the intersection of a finite family of closed convex subsets with numerical experiments are given for illustration.

2. Preliminaries

It is well known that in any real Hilbert space H, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$$
$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2,$$

for any $x, y \in H$ and $\alpha \in [0,1]$. An operator T in H is called

• *nonexpansive* if $\|Tx - Ty\| \leq a\|x-y\|$ for all $x, y \in \mathcal{D}(T)$, the domain of T, with $a = 1$ and a contraction if $a \in [0,1)$.

• *l-Lipschitzian and η-strongly monotone*, if it satisfies, respectively, the conditions $\|Tx - Ty\| \leq l\|x-y\|$ and $\langle Tx - Ty, x-y\rangle \geq \eta\|x-y\|^2$ with $l \geq \eta > 0$.

• *boundedly Lipschitzian* on $\mathcal{D}(T)$ if it is Lipschitzian on each bounded subset of $\mathcal{D}(T)$; namely, for each bounded non-empty subset B of $\mathcal{D}(T)$, there exists a positive constant l^B depending only on the set B such that $\|Tx - Ty\| \leq l^B\|x-y\|$ for all $x, y \in B$.

• *demiclosed* if for any sequence $\{x^k\} \subset H$ the following implication holds

$$(x^k \text{converges weakly to } x \text{ and } \|(T-I)x^k\| \to 0) \Longrightarrow x \in \mathrm{Fix}(T),$$

where $\mathrm{Fix}\,(T) = \{p \in H : p = Tp\}$.

Lemma 2.1 ([18]). *Let H be a real Hilbert space and let F be an l-Lipschitz continuous and η-strongly monotone mapping in H with some positive constants $l \geq \eta > 0$. Let $T^\mu = I - \mu F$ and let $T^{t,\mu} = I - t\mu F$. Then, for a fixed number $\mu \in (0, 2\eta/l^2)$ and any $t \in (0,1)$, $I - \mu F$ and $I - t\mu F$ are all contractions with coefficients $1-\tau$ and $1 - t\tau$, respectively, where $\tau = (1/2)\mu(2\eta - \mu l^2)$.*

Lemma 2.2 ([17]). *Let $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ be sequences of real numbers such that, for all $k \geq 0$, $a_{k+1} \leq (1-b_k)a_k + b_k c_k$; $a_k \geq 0$; b_k satisfies a condition of type (t); and either $\sum_{k=1}^{\infty} b_k|c_k| < \infty$ or $\limsup_{k\to\infty} c_k \leq 0$. Then, we have $lim_{k\to\infty} a_k = 0$.*

Lemma 2.3 ([15]). *Let $\{a_k\}$ be a sequence of real numbers with a subsequence $\{n_k\}$ of $\{k\}$ such that $a_{n_k} < a_{n_k+1}$. Then, there exists a non-decreasing sequence $\{m_k\} \subseteq \{k\}$ such that $m_k \to \infty, a_{m_k} \le a_{m_k+1}$ and $a_k \le a_{m_k+1}$ for all (sufficiently large) numbers $k \ge 0$. In fact, $m_k = \max\{n \le k : a_l \le a_{n+1}\}$.*

3. Main results

Our iterative scheme is constructed as follows.

Step 1 Let x^0, x^1 be any two points in H such that $x^1 \neq x^0$ and let s be any fixed positive integer. Set $k := 1$.

Step 2 For $t = 1, 2, \cdots, s$, let $L^{k,t}$ be a non-empty ordered subset of L, satisfying the condition $L = L^{k,1} \cup L^{k,2} \cup \cdots \cup L^{k,s}$, and let $\mathcal{L}_s^k$ and $\mathcal{P}_s^k$ be two mappings defined by $\mathcal{L}_s^k = \mathcal{T}_s^k \mathcal{T}_{s-1}^k \ldots \mathcal{T}_1^k$ and $\mathcal{P}_s^k = \mathcal{L}_s^k \mathcal{L}_s^{k-1} \ldots \mathcal{L}_s^1$, respectively, where

$$\mathcal{T}_t^k = I + \alpha_k\big(\tilde{T}_t^k - I\big), \tilde{T}_t^k = \sum_{i(t) \in L^{k,t}} \omega_{i(t)}^k T_{i(t)}$$

$\omega_{i(t)}^k$ is chosen so that $\omega_{i(t)}^k \ge \omega > 0$, $\sum_{i(t) \in L^{k,t}} \omega_{i(t)}^k = 1$, and the parameter $\alpha_k \in [\underline{\alpha}, \overline{\alpha}] \subset (0, 1)$.

Step 3 Given the iterate x^k. Calculate $z^k = (I - t_k \mu_k F)x^k$, where t_k satisfies only condition (t) and μ_k is chosen by

$$\mu_k = \begin{cases} \frac{\langle Fx^k - F\mathcal{P}_s^{k-1}x^0, x^k - \mathcal{P}_s^{k-1}x^0 \rangle}{\|Fx^k - F\mathcal{P}_s^{k-1}x^0\|^2}, & x^k \neq \mathcal{P}_s^{k-1}x^0, \\ \mu_{k-1}, & \text{otherwise.} \end{cases}$$

Step 4 Compute $x^{k+1} = \mathcal{L}_s^k z^k$. Set $x^k := x^{k+1}$ and $k := k + 1$. Return to **Step 2**.

First, we prove the following lemmas.

Lemma 3.1. *Let H be a real Hilbert space, let F be boundedly Lipschitzian and η-strongly monotone on H and, for each $i \in L := \{1, \ldots, m\}$, let T_i be a nonexpansive mapping such that $C := \cap_{i \in L} \mathrm{Fix}\,(T_i) \neq \emptyset$. Then, any sequence $\{x^k\}$, generated by the introduced iterative scheme, is bounded. Moreover, if $\lim_{k\to\infty} \|T_i x^k - x^k\| = 0$ for every $i \in L$ then*

$$\limsup_{k \to \infty} \langle Fp_*, p_* - x^k \rangle \le 0. \tag{6}$$

Proof. Obviously, $\mu_k = \eta_k / l_k^2$, where

$$\eta_k = \begin{cases} \frac{\langle Fx^k - F\mathcal{P}_s^{k-1}x^0, x^k - \mathcal{P}_s^{k-1}x^0 \rangle}{\|x^k - \mathcal{P}_s^{k-1}x^0\|^2}, & x^k \neq \mathcal{P}_s^{k-1}x^0, \\ \eta_{k-1}, & \text{otherwise,} \end{cases}$$

and

$$l_k = \begin{cases} \frac{\|Fx^k - F\mathcal{P}_s^{k-1}x^0\|}{\|x^k - \mathcal{P}_s^{k-1}x^0\|}, & x^k \neq \mathcal{P}_s^{k-1}x^0, \\ l_{k-1}, & \text{otherwise.} \end{cases}$$

Put $\mathcal{P}_s^0 x = x$ for any $x \in H$ and $y^{k+1} = \mathcal{L}_s^k(I - t_k\mu_k F)\mathcal{P}_s^{k-1}x^0$. Then, since $\mathcal{L}_s^k$ is nonexpansive, we have

$$\begin{aligned}
\|x^{k+1} - y^{k+1}\|^2 &= \|\mathcal{L}_s^k(I - t_k\mu_k F)x^k - \mathcal{L}_s^k(I - t_k\mu_k F)\mathcal{P}_s^{k-1}x^0\|^2 \\
&\le \|(I - t_k\mu_k F)x^k - (I - t_k\mu_k F)\mathcal{P}_s^{k-1}x^0\|^2 \\
&= \|x^k - \mathcal{P}_s^{k-1}x^0 - t_k\mu_k(Fx^k - F\mathcal{P}_s^{k-1}x^0)\|^2 \\
&= \|x^k - \mathcal{P}_s^{k-1}x^0\|^2 - 2t_k\mu_k\langle Fx^k - F\mathcal{P}_s^{k-1}x^0, x^k - \mathcal{P}_s^{k-1}x^0\rangle \\
&\quad + t_k^2\mu_k^2\|Fx^k - F\mathcal{P}_s^{k-1}x^0\|^2 \\
&= \|x^k - \mathcal{P}_s^{k-1}x^0\|^2 - 2t_k\mu_k\eta_k\|x^k - \mathcal{P}_s^{k-1}x^0\|^2 + t_k^2\mu_k^2 l_k^2\|x^k - \mathcal{P}_s^{k-1}x^0\|^2 \\
&= \left(1 - 2t_k\frac{\eta_k^2}{l_k^2} + t_k^2\frac{\eta_k^2}{l_k^2}\right)\|x^k - \mathcal{P}_s^{k-1}x^0\|^2.
\end{aligned}$$

Hence,

$$\|x^{k+1} - y^{k+1}\|^2 \le \left(1 - t_k\frac{\eta_k^2}{2l_k^2}(2 - t_k)\right)^2\|x^k - \mathcal{P}_s^{k-1}x^0\|^2.$$

Therefore,

$$\|x^{k+1} - y^{k+1}\| \le \left(1 - t_k\frac{\eta_k^2}{2l_k^2}(2 - t_k)\right)\|x^k - \mathcal{P}_s^{k-1}x^0\|. \tag{7}$$

On the other hand,

$$\|y^{k+1} - \mathcal{P}_s^k x^0\| = \|\mathcal{L}_s^k(I - t_k\mu_k F)\mathcal{P}_s^{k-1}x^0 - \mathcal{L}_s^k\mathcal{P}_s^{k-1}x^0\| \le t_k\mu_k\|F\mathcal{P}_s^{k-1}x^0\|.$$

As $T_i p = p$ for any $p \in C$ and for any $i \in L$, $\tilde{T}_t^k p = p$ for any $k \ge 1$ and any $t \in \{1, 2, \dots, s\}$. Hence, $\mathcal{T}_t^k p = p$ and $\mathcal{P}_s^k p = p$ for all $k \ge 1$ and $t = 1, \dots, s$. Consequently, $\|\mathcal{P}_s^{k-1}x^0 - p\| \le \|x^0 - p\|$. It means that $\{\mathcal{P}_s^{k-1}x^0\}$ is bounded. Since F is boundedly Lipschitzian, $\sup_{k\ge1}\|F\mathcal{P}_s^k x^0\| \le M_1$ for some positive constant M_1. Then, from (7) it follows that

$$\begin{aligned}
\|x^{k+1} - \mathcal{P}_s^k x^0\| &\le \|x^{k+1} - y^{k+1}\| + \|y^{k+1} - \mathcal{P}_s^k x^0\| \\
&\le \left(1 - t_k\frac{\eta_k^2}{2l_k^2}(2 - t_k)\right)\|x^k - \mathcal{P}_s^{k-1}x^0\| + t_k\frac{\eta_k^2}{2l_k^2}M_1/\eta_k \\
&\le \left(1 - t_k\frac{\eta_k^2}{2l_k^2}\right)\|x^k - \mathcal{P}_s^{k-1}x^0\| + t_k\frac{\eta_k^2}{2l_k^2}2M_1/\eta,
\end{aligned}$$

because of $t_k \in (0,1)$ and $\eta_k \ge \eta$. Hence, $\|x^{k+1} - \mathcal{P}_s^k x^0\| \le \max\{\|x^1 - x^0\|, 2M_1/\eta\}$. Therefore, $\{x^k\}$ is bounded.

Let $B := \overline{co}\{\{p_*\} \cup \{x^k\}_{k\ge0} \cup \{\mathcal{P}_s^k x^0\}_{k\ge1}\}$, the closed convex hull containing p_*, $\{x^k\}$ and $\{\mathcal{P}_s^k x^0\}$. Clearly, B is a bounded closed convex subset of H. Then, F is Lipschitzian on B, i.e., $\|Fx - Fy\| \le l^B\|x - y\|$ for any $x, y \in B$ where l^B is some positive constant and the intersection $D := B \cap C$ is also non-empty closed convex.

Now, if $\lim_{k\to\infty}\|T_i x^k - x^k\| = 0$ for every $i \in L$ then any weak cluster point $\tilde{p}$ of $\{x^k\}$, by the demiclosed property of T_i, belongs to D. Therefore,

$$\limsup_{k\to\infty}\langle Fp_*, p_* - x^k\rangle = \lim_{k\to\infty}\langle Fp_*, p_* - x^{n_k}\rangle = \langle Fp_*, p_* - \tilde{p}\rangle \le 0,$$

because p_* is a unique solution of (1) with C replaced by D. This completes the proof. □

Further, let $z^{k,0} = z^k$ and $z^{k,t} = \mathcal{T}_t^k z^{k,t-1}$ for $t = 1, \dots, s$. Then, $x^{k+1} = z^{k,s}$.

LEMMA 3.2. *Let all the assumptions in Lemma 3.1 be satisfied. Then, for any sequence* $\{x^k\}$ *generated by the introduced iterative scheme, we have*

$$\|x^{k+1} - p\|^2 \le (1 - \gamma_k\beta_k)\|x^k - p\|^2 + 2\gamma_k\big(\langle Fp, p - x^k\rangle + \gamma_k\langle Fp, Fx^k\rangle\big) - \tilde{c}\alpha\sum_{t=1}^{s}\sum_{i(t)\in L^{k,t}}\omega_{i(t)}^k\|(T_{i(t)} - I)z^{k,t-1}\|^2 \quad \forall p \in D \tag{8}$$

and for all $k \ge k^1$ *that is a positive integer, satisfying* $1 - \gamma_k\beta_k \ge \tilde{c}$, *some positive real number,* $\gamma_k \in (0, \eta/(l^B)^2)$ *and* $\beta_k \ge \eta/2$, *where* $\gamma_k = t_k\mu_k$, $\beta_k = (1/2)(2\eta - \gamma_k(l^B)^2)$, *and* $\alpha = \underline{\alpha}(1 - \overline{\alpha})$.

Proof. Using the definition of $z^{k,t}$, the properties of $\|.\|^2$, H and T_i, we obtain

$$\begin{aligned}\|z^{k,t} - p\|^2 &\le \sum_{i(t)\in L^{k,t}}\omega_{i(t)}^k\|(1-\alpha_k)(z^{k,t-1} - p) + \alpha_k(T_{i(t)}z^{k,t-1} - p)\|^2 \\ &= \sum_{i(t)\in L^{k,t}}\omega_{i(t)}^k\big((1-\alpha_k)\|z^{k,t-1} - p\|^2 + \alpha_k\|T_{i(t)}z^{k,t-1} - p\|^2 \\ &\quad - \alpha_k(1-\alpha_k)\|(T_{i(t)} - I)z^{k,t-1}\|^2\big) \\ &\le \|z^{k,t-1} - p\|^2 - \alpha\sum_{i(t)\in L^{k,t}}\omega_{i(t)}^k\|(T_{i(t)} - I)z^{k,t-1}\|^2,\end{aligned}$$

for all $t = 1, 2, \dots, s$. Summing the last inequality with $t = 1, \dots, s$ and replacing $z^{k,s}$ and $z^{k,0}$ by their values in the scheme, we get

$$\|x^{k+1} - p\|^2 \le \|(I - t_k\mu_kF)x^k - p\|^2 - \alpha\sum_{t=1}^{s}\sum_{i(t)\in L^{k,t}}\omega_{i(t)}^k\|(T_{i(t)} - I)z^{k,t-1}\|^2. \tag{9}$$

Evidently,

$$\eta \le \eta_k \le \frac{\|Fx^k - F\mathcal{P}_s^{k-1}x^0\|}{\|x^k - P_s^{k-1}x^0\|} = l_k \le l^B,$$

and hence,

$$\frac{\eta}{(l^B)^2} \le \mu_k = \frac{\eta_k}{l_k^2} \le \frac{1}{\eta_k} \le \frac{1}{\eta}. \tag{10}$$

Since $t_k \to 0$ as $k \to \infty$, by (10), $\gamma_k \in (0, \eta/(l^B)^2)$ and $\beta_k \ge \eta/2$ for all $k \ge k^1$, some positive integer. Moreover, we can chose an integer k^1 so that $1 - \gamma_k\beta_k \ge \tilde{c}$ for all $k \ge k^1$. Then, from Lemma 2.1 it follows that

$$\begin{aligned}\|(I - \gamma_kF)x^k - p\|^2 &= \|(I - \gamma_kF)x^k - (I - \gamma_kF)p - \gamma_kFp\|^2 \\ &\le (1 - \gamma_k\beta_k)\|x^k - p\|^2 + 2\gamma_k\big(\langle Fp, p - x^k\rangle + \gamma_k\langle Fp, Fx^k\rangle\big).\end{aligned}$$

This together with (9) implies (8). The proof is completed. □

Now, we are in the position to prove our main result.

THEOREM 3.3. *Let H, F and T_i be as in Lemma 3.1. Then, any sequence $\{x^k\}$, generated by our iterative scheme, as $k \to \infty$, converges strongly to the point p_*, solving* (1) *where $C = \cap_{i\in L}\mathrm{Fix}\,(T_i)$.*

Proof. We need only to consider two cases.
Case 1. $\|x^{k+1} - p_*\| \le \|x^k - p_*\|$ for all $k \ge k^1$.

Then, there exists $\lim_{k\to\infty} \|x^k - p_*\|$. From (8) we get

$$\tilde{c}\alpha \sum_{t=1}^{s} \sum_{i(t)\in L^{k,t}} \omega^k_{i(t)} \|(T_{i(t)} - I)z^{k,t-1}\|^2 \le \|x^k - p\|^2 - \|x^{k+1} - p\|^2 + d\gamma_k, \qquad (11)$$

where $d_k = 2\|Fp\|(r + (\eta/(l^B)^2)M_2)$ and $M_2 = \sup_{k\ge 1} \|Fx^k\|$. Next, we prove that

$$\lim_{k\to\infty} \sum_{t=1}^{s} \sum_{i(t)\in L^{k,t}} \omega^k_{i(t)} \|(T_{i(t)} - I)z^{k,t-1}\|^2 = 0. \qquad (12)$$

Clearly, if

$$\tilde{c}\alpha \sum_{t=1}^{s} \sum_{i(t)\in L^{k,t}} \omega^k_{i(t)} \|(T_{i(t)} - I)z^{k,t-1}\|^2 \le d\gamma_k$$

for all $k \ge k^1$, then (12) holds, since $\gamma_k = t_k\mu_k \le t_k/\eta$ and $t_k \to 0$. If

$$\tilde{c}\alpha \sum_{t=1}^{s} \sum_{i(t)\in L^{k,t}} \omega^k_{i(t)} \|(T_{i(t)} - I)z^{k,t-1}\|^2 > d\gamma_k$$

for all $k \ge k^1$, then from (11) it follows that

$$\sum_{k=k^1}^{M} \Big(\tilde{c}\alpha \sum_{t=1}^{s} \sum_{i(t)\in L^{k,t}} \omega^k_{i(t)} \|(T_{i(t)} - I)z^{k,t-1}\|^2 - d\gamma_k\Big) \le \|x^{k^1} - p\|^2 - \|x^{M+1} - p_*\|^2,$$

for any positive integer M. Thus,

$$\sum_{k=k^1}^{\infty} \Big(\tilde{c}\alpha \sum_{t=1}^{s} \sum_{i(t)\in L^{k,t}} \omega^k_{i(t)} \|(T_{i(t)} - I)z^{k,t-1}\|^2 - d\gamma_k\Big) \le \|x^{k^1} - p_*\|^2.$$

Therefore,

$$\lim_{k\to\infty} \big(\tilde{c}\alpha \sum_{t=1}^{s} \sum_{i(t)\in L^{k,t}} \omega^k_{i(t)} \|(T_{i(t)} - I)z^{k,t-1}\|^2 - d\gamma_k\big) = 0,$$

and hence, we have (12). Clearly, (12) is equivalent to

$$\lim_{k\to\infty} \sum_{i(t)\in L^{k,t}} \omega^k_{i(t)} \|(T_{i(t)} - I)z^{k,t-1}\|^2 = 0$$

for $t = 1, \dots, s$. Then,

$$\lim_{k\to\infty} \sum_{i(t)\in L^{k,t}} \omega^k_{i(t)} \|(T_{i(t)} - I)z^{k,t-1}\| = 0 \qquad (13)$$

since $\omega^k_{i(t)} \geq \omega$ for all $i(t) \in L^k_t$ and $t = 1, \ldots, s$. Next, by the definition of $z^{k,t}$ and the property of α_k,

$$\|z^{k,t} - z^{k,t-1}\| \leq \sum_{i(t)\in L^{k,t}} \omega^k_{i(t)} \|(T_{i(t)} - I)z^{k,t-1}\|.$$

This together with (13) implies that

$$\lim_{k\to\infty} \|z^{k,t} - z^{k,t-1}\| = 0, \tag{14}$$

for all $t = 1, \cdots, s$. Now, since $L^{k,1} \cup \cdots \cup L^{k,s} = L$, for any $i \in L$ there exist at least an integer t_k such that $i \in L^{k,t_k} := \{i_1(t_k), \ldots, i, \ldots, i_{|L^{k,t_k}|}(t_k)\}$. Then, from (13) with $t = t_k$, it deduces that

$$0 \leq \omega \lim_{k\to\infty} \|(T_i - I)z^{k,t_k-1}\| \leq \lim_{k\to\infty} \sum_{i(t_k)\in L^{k,t_k}} \omega^k_{i(t_k)} \|(T_{i(t_k)} - I)z^{k,t_k-1}\| = 0.$$

Therefore, $\lim_{k\to\infty} \|(T_i - I)z^{k,t_k-1}\| = 0$, from which and (14) follows that $\lim_{k\to\infty} \|(T_i - I)x^k\| = 0$ for every $i \in L$. So, (6) holds. Further, noting (8), the inequality

$$\|x^{k+1} - p_*\|^2 \leq (1 - \gamma_k\beta_k)\|x^k - p_*\|^2 + 2\gamma_k\big(\langle Fp_*, p_* - x^k\rangle + \gamma_k\|Fp_*\|M_2\big),$$

followed from (9) with $p = p_*$, and $\beta_k \geq \eta/2$,

$$\|x^{k+1} - p_*\|^2 \leq (1 - \gamma_k\beta_k)\|x^k - p_*\|^2 + \gamma_k\beta_k 2\big(\langle Fp_*, p_* - x^{k+1}\rangle + \gamma_k\|Fp\|M_2\big)/\eta.$$

From this, (8) and Lemma 2.2 we deduce that $\lim_{k\to\infty} \|x^k - p_*\| = 0$.

Case 2. There exists a subsequence $\{n_k\} \subset \{k\}$ such that $\|x^{n_k} - p_*\| \leq \|x^{n_k+1} - p_*\|$ for all $k \geq k^1$. Then, by Lemma 2.3, there exists a non-decreasing sequence $\{m_k\} \subseteq \{k : k \geq k^1\}$ such that $m_k \to \infty$ as $k \to \infty$,

$$\|x^{m_k} - p_*\| \leq \|x^{m_k+1} - p_*\| \quad \text{and} \quad \|x^k - p_*\| \leq \|x^{m_k+1} - p_*\|. \tag{15}$$

Hence, from (8) and the first inequality in (15), we know

$$\|x^{m_k} - p_*\|^2 \leq 4[\langle Fp_*, p_* - x^{m_k}\rangle + \gamma_{m_k}\|Fp_*\|M_2]/\eta \tag{16}$$

and

$$\tilde{c}\alpha \sum_{t=1}^{s} \sum_{i(t)\in L^{m_k,t}} \omega^{m_k}_{i(t)} \|(T_{i(t)} - I)z^{m_k,t-1}\|^2 \leq d\gamma_{m_k}.$$

By the similar argument as in the first case, $\lim_{k\to\infty} \|(T_i - I)x^{m_k}\| = 0$ for each $i \in L$. So, we have (6) with k replaced by m_k. This together with (16) and $\gamma_{m_k} \to 0$ implies that

$$\lim_{k\to\infty} \|x^{m_k} - p_*\| = 0. \tag{17}$$

Using (8) with k and p replaced, respectively, by m_k and p_*, we can write

$$\|x^{m_k+1} - p_*\|^2 \leq (1 - \gamma_{m_k}\beta_{m_k})\|x^{m_k} - p_*\|^2 + 2\gamma_{m_k}\big(\langle Fp_*, p_* - x^{m_k}\rangle + \gamma_{m_k}\|Fp_*\|M_2\big)/\eta.$$

From this, (17), and $\gamma_{m_k} \to 0$, we deduce that $\lim_{k\to\infty}\|x^{m_k+1} - p_*\|^2 = 0$. The last limit together with the second inequality in (15) implies that $\lim_{k\to\infty}\|x^k - p_*\| = 0$. □

Remark 3.4. 1. Consider the case when $m = 1$. Then, we have $s = 1$, $t = 1$, $\mathcal{L}^k_1 = \mathcal{T}^k_1$, the mappings $\mathcal{P}^k_1 = \mathcal{L}^k_1\mathcal{L}^{k-1}_1 \ldots \mathcal{L}^1_1$ and $\mathcal{T}^k_1 = I + \alpha_k\big(T_1 - I\big)$. By our scheme, x^{k+1} is calculated by

$$x^{k+1} = \big(I + \alpha_k(T_1 - I)\big)(I - t_k\mu_k F)x^k, \tag{18}$$

where μ_k is defined by the formula

$$\mu_k = \begin{cases} \frac{\langle Fx^k - F\mathcal{P}_1^{k-1}x^0, x^k - \mathcal{P}_1^{k-1}x^0\rangle}{\|Fx^k - F\mathcal{P}_1^{k-1}x^0\|^2}, & x^k \neq \mathcal{P}_1^{k-1}x^0, \\ \mu_{k-1}, & \text{otherwise.} \end{cases}$$

Obviously, method (18) is different from (4) and converges without condition (t'). Moreover, the choice μ_k in our scheme is simpler than that in (4). So, it is a improved modification of (4).

2. We want to say a few words regarding possible extensions of this work.

One possible extension is to consider the case when $\tilde{T}_t^k$ is a convex combination of products of the mappings T_i, i.e.,

$$\tilde{T}_t^k = \sum_{n=1}^{N_{k,t}} \omega_n^{k,t} \prod_{i(t)\in L_n^{k,t}} T_{i(t)},$$

where $L^{k,t} = L_1^{k,t} \cup \ldots \cup L_{N_{k,t}}^{k,t}$, $\omega_n^{k,t}$ is chosen so that $\omega_n^{k,t} \geq \omega > 0$ and $\sum_{n=1}^{N_{k,t}} \omega_n^{k,t} = 1$ for all $k \geq 0$ and $t = 1, \ldots, s$.

Another possible extension is to consider the obtained result in a Banach space as the accretive variational inequality investigated in [2, 3].

4. Applications and numerical experiments

4.1 Applications to networked systems

We show that our results can be applied to solving a networked system consisting of an operator, who manages the system, and a finite number $m-1$ of participating users. In the system the manage operator can be seen as a user m. We suppose that each user $i \in L$ has its own private objective function $f_{(i)}$ on $\mathbf{E^n}$, an n-dimensional Euclidian space, and a nonempty closed convex C_i in $\mathbf{E^n}$. Moreover, the following is assumed.

(a_1) T_i is nonexpansive on $\mathbf{E^n}$ with $\mathrm{Fix}\,(T_i) = C_i$ for each $i \in L$ and the intersection $\cap_{i\in L}\mathrm{Fix}\,(T_i) \neq \emptyset$.

(a_2) $f_{(i)}$ is concave and Fréchet differentiable on $\mathbf{E^n}$ and $-\bigtriangledown f^{(i)}$ is boundedly Lipschitzian and η_i-strongly monotone.

(a_3) User $i \in L$ can use its own private C_i and $f_{(i)}$.

(a_4) The operator can communicate with all users.

The considered problem can be formulated as finding a point p_* in $\mathbf{E^n}$ such that

$$f(p_*) = \max_{p\in C} f(p), f(x) = \sum_{i\in L} f_{(i)}(x), C = \cap_{i\in L}\mathrm{Fix}\,(T_i). \tag{19}$$

Problem (19) is closely related to network recourse allocation [1,16] which is a central issue in modern communication networks. The main objective of the allocation is to share the available resources among users in the network so as to maximize the sum of their utilities subject to the feasible regions for allocating the resources. Problem (19) is equivalent to the following one, $\tilde{f}(p_*) = \inf_{p\in C} \tilde{f}(p)$, where $\tilde{f} = -f$ is convex and Fréchet differentiable with boundedly Lipschitzian and η-strongly monotone $\bigtriangledown \tilde{f}$ and $\eta = \min_{i\in L} \eta_i$. To solve (19), when $- \bigtriangledown f_{(i)}$ is l_i-Lipschitzian and η_i-strongly monotone, Iiduka [12] introduced a parallel optimization algorithm, at each iteration steep of which the value μ is chosen in dependence of η_i and l_i. It is easy to see that the considered problem can be solved by our iterative scheme, where $F = \bigtriangledown \tilde{f}$ and C is given in (19). In the next subsection, we consider a convex optimization problem and an example for computations.

4.2 Convex optimization and numerical experiments

Clearly, the results in the previous section can be applied to finding a solution of the following convex optimization problem: Find a point

$$p_* \in C;\ \varphi(p_*) = \min_{p\in C} \varphi(p), C = \cap_{i\in L} C_i, \tag{20}$$

where C_i is a closed convex subset in a real Hilbert space H and φ', the gradient of φ is boundedly Lipschitzian and η-strongly monotone. Obviously, $C_i = \mathrm{Fix}\,(T_i)$, the fixed point set of $T_i = P_{C_i}$, the metric projection from H onto C_i. It is well known that T_i is nonexpansive. We also know that p_* is a solution of (20) if and only if it solves (1) where $F = \varphi'$ and $C = \cap_{i\in L}\mathrm{Fix}\,(T_i)$.

For computational illustration, we consider the case, when

$$\begin{aligned}
&Fz = (x_1 + x_2 + x_1^5, -x_1 + x_2 + x_2^7),\\
&C_i = \{x \in \boldsymbol{E}^2 : \|x - a^i\| \le 1\},\ i \in L = \{1, \dots, 8\},\\
&a^1 = (0, 1/2), \quad a^2 = (0, -1/2), \quad a^3 = (1/2, 0),\\
&a^4 = (-1/2, 0), \quad a^5 = (1/2, 1/2), \quad a^6 = (-1/2, -1/2),\\
&a^7 = (1/2, -1/2), \quad a^8 = (-1/2, -1/2).
\end{aligned}$$

It is easy to see that the real functions x^3 and x^7 are strictly increasing in $\mathbf{E^1}$. Therefore, F is 1-strongly monotone on $\mathbf{E^2}$. Moreover, F is only boundedly Lipschitzian on $\boldsymbol{E}^2$ (see [9]).

The numerical results are calculated with $x^0 = (2, 1), x^1 = (1, 2), \alpha_k = 1/2$, $s = 2$, i.e., $L^{k,1} = \{1, 2, 3, 4\}$ and $L^{k,2} = \{5, 6, 7, 8\}$ and $t_k = 1/(k + 1)$. We take $L^{k,1} = L_1^{k,1} \cup L_2^{k,1}$ where $L_1^{k,1} = \{1, 2\}$, $L_2^{k,1} = \{3, 4\}$ and $L^{k,2} = L_1^{k,2} \cup L_2^{k,2}$ with $L_1^{k,2} = \{5, 6\}$, $L_2^{k,2} = \{7, 8\}$.

The numerical result in Table 1 shows the effectiveness of the introduced iterative scheme.

k	x_1^{k+1}	x_2^{k+1}
5	0.0027502427	0.0215071004
10	-0.0000888099	0.0005021674
15	-0.0000047155	0.0000124442
20	-0.0000001741	0.0000003201
25	-0.0000000006	0.0000000008

Table 1: Numerical results calculated by our iterative scheme

5. Conclusion

In this paper, for solving the variational inequality problem, governed by a boundedly Lipschitzian and strongly monotone mapping, over the set of common fixed points of a large finite family of nonexpansive mappings, we suggested a strongly convergent self-adaptive hybrid steepest-descent block-iterative scheme, strong convergence of which is proved without any additional condition on the family of fixed point sets as well as the dimension of setting spaces and condition (t') needed in the recent literature. We also gave applications to some networked systems and the convex optimization over the intersection of a finite family of closed convex subsets with computational experiments for illustration.

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