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ON SOME MATRIX INEQUALITIES

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Abstract. In this paper, we show the convergence of matrix series and the conditions for their convergence by finding an upper bound for some specific matrix inequalities. Finally, we introduce a new form of arithmetic-geometric matrix series and analyze their convergence.

1. Introduction

Let $\mathbb{M}n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. The singular values $s1(A), \dots, s_n(A)$ of a matrix $A \in \mathbb{M}n(\mathbb{C})$ are the eigenvalues of the matrix $(A^*A)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. A Hermitian matrix $A \in \mathbb{M}n(\mathbb{C})$ is said to be positive semidefinite, written as $A \geq 0$, if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$, and it is called positive definite, written as $A > 0$, if $x^*Ax > 0$ for all $x \in \mathbb{C}^n$ with $x \neq 0$. The Hilbert-Schmidt norm (or Frobenius norm) $\|\cdot\|_2$ is the norm defined on $\mathbb{M}n(\mathbb{C})$ by $\|A\|_2 = \left(\sum j = 1^n s_j^2(A)\right)^{1/2}$, $A \in \mathbb{M}n(\mathbb{C})$. The Hilbert-Schmidt norm is unitarily invariant; that is, $\|UAV\|_2 = \|A\|_2$ for all $A \in \mathbb{M}n(\mathbb{C})$ and all unitary matrices $U, V \in \mathbb{M}n(\mathbb{C})$. Another property of the Hilbert-Schmidt norm is that $\|A\|_2 = \left(\sum i, j = 1^n \left|fj^* Ae_i\right|^2\right)^{1/2}$, where $e_j j = 1^n$ and $fj_{j=1}^n$ are two orthonormal bases of $\mathbb{C}^n$.

The spectral matrix norm, denoted by $\|\cdot\|$, of a matrix $A \in \mathbb{M}n(\mathbb{C})$ is the norm defined by $\|A\| = \sup \|Ax\| : x \in \mathbb{C}^n, \|x\| = 1$, or equivalently, $\|A\| = s1\,(A)$. For further properties of these norms, the reader is referred to [5,6]. A matrix $A \in \mathbb{M}n(\mathbb{C})$ is called a contraction if $\|A\| \leq 1$, or equivalently, if $A^*A \leq In$, where I_n is the identity matrix in $\mathbb{M}n(\mathbb{C})$. An $n \times n$ matrix $A = (aij)$ is called a doubly stochastic matrix if and only if $a_{ij} \geq 0$ for all $i, j = 1, \dots, n$, and the sums of all rows and columns are equal to one; see [2].

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Doubly stochastic matrices are of particular interest in matrix analysis due to their connections with geometric problems, such as the Birkhoff-von Neumann theorem. This theorem states that the set of $n \times n$ doubly stochastic matrices forms a convex polytope whose vertices are precisely the permutation matrices [2]. In our work, doubly stochastic matrices play a crucial role in establishing convergence conditions for matrix series, as their spectral properties (e.g., eigenvalues bounded by 1) ensure the applicability of contraction-based inequalities. For instance, Corollary 2.3 leverages their properties to derive bounds for series involving these matrices.

In this paper, we define matrix series convergence and determine the conditions for their convergence by finding an upper bound for certain specific matrix inequalities.

In Section 2, we introduce a type of harmonic series inequality with its lower and upper bounds and present its matrix form. In Section 3, we construct a matrix form for new types of inequalities for the Hilbert-Schmidt and spectral norms, as the sum approaches infinity. In Section 4, we study arithmetic-geometric inequalities and their convergence for matrices.

2. Inequalities for harmonic series

We consider the harmonic series in the following form: $\sum_{n=1}^{\infty} \frac{1}{n}$, or, in general form, $\sum_{n=1}^{\infty} \frac{1}{an+b}$, where $a \neq 0, b$ are real numbers and $\frac{a}{b}$ is positive. A generalization of the harmonic series is the p-series (or hyperharmonic series), defined as $\sum_{n=1}^{\infty} \frac{1}{n^p}$. In [1], the author shows the following inequality; with respect to the harmonic series, the following lemma holds.

Lemma 2.1. *Let $n > 1$ be positive integer. Then*

$$2\sqrt{n+1} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1. \tag{1}$$

Hence, we can see that

$$\begin{aligned} \sum_{k=1}^{n} \sqrt{k} = & \sum_{k=1}^{n} \frac{k}{\sqrt{k}} \leq \left(\sum_{k=1}^{n} k\right)\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right) \\ \leq & \frac{n(n+1)}{2}\left(2\sqrt{n}-1\right) = n(n+1)(\sqrt{n}-0.5) \end{aligned} \tag{2}$$

We have a more general representation of this through the following theorem.

Theorem 2.2. *Let $A \in \mathbb{M}n(\mathbb{C})$ be a positive definite matrix. Then*

$$\left\|\sum_{k=1}^{n} \sqrt{k} A^k\right\|_2 < \frac{n(n+1)(\sqrt{n}-0.5)(1-\|A\|_2)^{n+1}}{1-\|A\|_2}.$$

Proof. Let us consider that A has singular values $s_1(A) \geq \ldots \geq s_n(A)$ and let U be a unitary matrix such that $A = U \operatorname{diag}(s_1(A), \dots, s_n(A))U^*$. Then

$$\begin{aligned}
\left\| \sum_{k=1}^{n} \sqrt{k} A^k \right\|_2 &= \left\| \sum_{k=1}^{n} \operatorname{diag}(\sqrt{k} s_1^k(A), \dots, \sqrt{k} s_n^k(A)) \right\|_2 \\
&< \sum_{k=1}^{n} \sqrt{k} \left\| \operatorname{diag}(s_1^k(A), \dots, s_n^k(A)) \right\|_2 \\
&\le n(n+1)(\sqrt{n} - 0.5) \sum_{k=1}^{n} \left\| \operatorname{diag}(s_1^k(A), \dots, s_n^k(A)) \right\|_2 \\
&\le n(n+1)(\sqrt{n} - 0.5) \sum_{k=1}^{n} (\| \operatorname{diag}(s_1(A), \dots, s_n(A)) \|_2)^k \\
&= n(n+1)(\sqrt{n} - 0.5) \sum_{k=1}^{n} (\|A\|_2)^k \\
&= \frac{n(n+1)(\sqrt{n} - 0.5)(1 - \|A\|_2)^{n+1}}{1 - \|A\|_2}. \qquad \square
\end{aligned}$$

The following corollary shows the result of the theorem when applied to doubly stochastic matrices.

Corollary 2.3. *Let A be a positive definite doubly stochastic matrix. Then*

$$\left\| \sum_{k=1}^{n} \sqrt{k} A^k \right\|_2 \le n^2(n+1)(\sqrt{n} - 0.5).$$

Proof. Since A is a positive definite doubly stochastic matrix, $0 < \|A\|_2 \le 1$. $\square$

Lemma 2.4. *Let $n > 1$ be a positive integer and x_k, $k = 1, 2, \dots, n$, be positive numbers. Then*

$$\frac{\min(x_k)_{1 \le k \le n}}{n(n+1)(\sqrt{n} - 0.5)} \lessdot \sum_{k=1}^{n} \frac{x_k}{\sqrt{k}} < n(2\sqrt{n} - 1) \max(x_k)_{1 \le k \le n}. \tag{3}$$

Proof. From Lemma 1, we have

$$\sum_{k=1}^{n} \frac{x_k}{\sqrt{k}} \le \left(\sum_{k=1}^{n} x_k \right) \left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \right) < n(2\sqrt{n} - 1) \max(x_k)_{1 \le k \le n}.$$

Then, from inequality (2), for the left-hand side we obtain

$$\sum_{k=1}^{n} \frac{x_k}{\sqrt{k}} \ge \frac{\sum\limits_{k=1}^{n} x_k}{\sum\limits_{k=1}^{n} \sqrt{k}} > \frac{n \min(x_k)_{1 \le k \le n}}{n(n+1)(\sqrt{n} - 0.5)} = \frac{\min(x_k)_{1 \le k \le n}}{(n+1)(\sqrt{n} - 0.5)} \tag{4}$$

This completes the proof. $\square$

Based on Corollary 3, we have the following theorem.

THEOREM 2.5. *Let* $A, X \in \mathbb{M}_n(\mathbb{C})$ *be positive definite matrices and* $\lfloor s_k(A) \rfloor$, $\lceil s_k(A) \rceil \leq n$. *Then*

$$\frac{\lfloor s_k(A) \rfloor s_n(X)}{(\lfloor s_k(A) \rfloor + 1)\left(\sqrt{\lfloor s_k(A) \rfloor} - 0.5\right)} \lessdot \sum_{k=1}^{\lfloor s_k(A) \rfloor} s_k(XA^{-0.5}) < \lfloor s_k(A) \rfloor s_1(X)\left(2\sqrt{\lfloor s_k(A) \rfloor} - 1\right)$$

for $k = 1, 2, \ldots, n$.

Proof. Since, from Lemma 2.4, we have

$$\begin{aligned}
\sum_{k=1}^{\lfloor s_k(A) \rfloor} s_k(XA^{-0.5}) &\leq \left(\sum_{k=1}^{\lfloor s_k(A) \rfloor} s_1(X)\right)\left(\sum_{k=1}^{\lfloor s_k(A) \rfloor} s_k(A^{-0.5})\right) \\
&= \lfloor s_k(A) \rfloor s_1(X)\left(\sum_{k=1}^{\lfloor s_k(A) \rfloor} s_k(A^{-0.5})\right) \\
&\leq \lfloor s_k(A) \rfloor s_1(X)\left(\sum_{k=1}^{\lfloor s_k(A) \rfloor} \frac{1}{\lfloor s_k(A) \rfloor^{0.5}}\right) \\
&< \lfloor s_k(A) \rfloor s_1(X)(2\sqrt{\lfloor s_k(A) \rfloor} - 1),
\end{aligned}$$

then, for the left-hand side of the inequality, we obtain

$$\begin{aligned}
\sum_{k=1}^{\lfloor s_k(A) \rfloor} s_k(XA^{-0.5}) &\geq \left(\sum_{k=1}^{\lfloor s_k(A) \rfloor} s_n(X)\right)\left(\sum_{k=1}^{\lfloor s_k(A) \rfloor} s_k(A^{-0.5})\right) \\
&= \lfloor s_k(A) \rfloor s_n(X)\left(\sum_{k=1}^{\lfloor s_k(A) \rfloor} s_k(A^{-0.5})\right) \\
&\geq \lfloor s_k(A) \rfloor s_n(X)\left(\sum_{k=1}^{\lfloor s_k(A) \rfloor} \frac{1}{\lceil s_k(A) \rceil^{0.5}}\right) \\
&> \frac{\lfloor s_k(A) \rfloor s_n(X)}{(\lfloor s_k(A) \rfloor + 1)\left(\sqrt{\lfloor s_k(A) \rfloor} - 0.5\right)} \text{ by Lemma 2.4.} \qquad \square
\end{aligned}$$

In the next section, we formally define the concept of convergent matrix series and establish their convergence criteria.

The convergence of a matrix series $\sum_{k=0}^{\infty} A_k$ is defined as the convergence of its partial sums $S_N = \sum_{k=0}^{N} A_k$ in a given matrix norm (e.g., spectral or Frobenius), following standard definitions in matrix analysis (see [7]).

3. Matrix series and their convergence

We begin this section by formally defining the convergence of matrix series, following standard results in matrix analysis. A matrix series $\sum_{k=0}^{\infty} A_k$ is said to converge if the sequence of partial sums S_N converges in norm. This is equivalent to $\lim_{N,M\to\infty} |S_N - S_M| = 0$, according to the Cauchy criterion. We now state a key theorem on convergence criteria (see [7]).

Theorem 3.1. *If $A \in \mathbb{M}n(\mathbb{C})$, then the series $\sum k = 0^{\infty} a_k A^k$ converges if there exists a matrix norm $\|\cdot\|$ on $\mathbb{M}n(\mathbb{C})$ such that the numerical series $\sum k = 0^{\infty} |a_k| \left\|A^k\right\|$ converges, or if the partial sums of the series are bounded.*

By Theorems 2.5 and 3.1, we can apply matrix series to obtain a more general convergence formula.

Theorem 3.2. *Let $A \in \mathbb{M}n(\mathbb{C})$ and $p > 1$. Then*

$$\left\|\sum k = 0^{\infty} \frac{1}{k^{n+p}} A^k\right\| \le \frac{1}{p-1} e^{|||A|||},$$

and $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^{n+p}} A^k$ *converges.*

Proof. Since

$$\begin{aligned}\left\|\sum_{k=0}^{\infty} \frac{1}{k^{n+p}} A^k\right\| &= \left\|\sum_{k=0}^{\infty} \frac{k!}{k^{n+p}} \frac{A^k}{k!}\right\| \le \sum_{k=0}^{\infty} \frac{k!}{k^{n+p}} \left\|\sum_{k=0}^{\infty} \frac{A^k}{k!}\right\| \le \sum_{k=0}^{\infty} \frac{1}{k^{p}} \left\|\sum_{k=0}^{\infty} \frac{A^k}{k!}\right\| \\ &\le \frac{1}{p-1} \left\|e^{A}\right\| = \frac{1}{p-1} e^{|||UA|||} = \frac{1}{p-1} e^{|||A|||},\end{aligned}$$

by Theorem 3.1, $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^{n+p}} A^k$ converges. ☐

From [8], we have the following lemma for unitarily invariant norms involving powers of singular values.

Lemma 3.3. *Let $A, B, X \in \mathbb{M}n(\mathbb{C})$, where A, B are positive definite matrices and X is Hermitian. Then*

$$\left|\left|\left|A^{sn(A)} X + X B^{s_n(B)}\right|\right|\right| \ge 2e^{-e^{-1}} |||X||| .$$

This lemma can be generalized in the following theorem.

Theorem 3.4. *Let $A_i, B_i, X \in \mathbb{M}n(\mathbb{C})$, where Ai, B_i, $i = 1, \ldots, m$, are positive definite matrices and X is Hermitian. Then*

$$\left|\left|\left|\sum_{i=0}^{m} A_i^{s_n(A_i)} X^i + X^i B_i^{s_n(B_i)}\right|\right|\right| \ge 2e^{-e^{-1}} \left|\left|\left|\frac{I - X^{m+1}}{I - X}\right|\right|\right| \ge \frac{2e^{-e^{-1}}(1 - s_1^{m+1}(X))}{1 - s_n(X)}.$$

If $X = I$, then

$$\left|\left|\left|\sum_{i=0}^{m} A_i^{s_n(A_i)} + B_i^{s_n(B_i)}\right|\right|\right| \geq 2(m+1)e^{-e^{-1}}.$$

Proof. Since $a^a \geq e^{-e}$ for any positive number and X is Hermitian, then X^i is Hermitian for every $i = 1, \dots, m$, and for any positive definite matrix $A \geq s_n(A)I$. We obtain:

$$\begin{aligned}
\left|\left|\left|\sum_{i=0}^{m} A_i^{s_n(A_i)}X^i + X^iB_i^{s_n(B_i)}\right|\right|\right| &\geq \left|\left|\left|\sum_{i=0}^{m} s_n^{s_n(A_i)}(A_i)X^i + X^i s^{s_n(B_i)}(B_i)\right|\right|\right| \\
&= \left|\left|\left|\sum_{i=0}^{m} (s_n^{s_n(A_i)}(A_i) + s^{s_n(B_i)}(B_i))X^i\right|\right|\right| \\
&\geq \left|\left|\left|\sum_{i=0}^{m} 2e^{-e^{-1}}X^i\right|\right|\right| = 2e^{-e^{-1}}\left|\left|\left|\sum_{i=0}^{m} X^i\right|\right|\right| \\
&= 2e^{-e^{-1}}\left|\left|\left|\frac{I - X^{m+1}}{I - X}\right|\right|\right| \geq \frac{2e^{-e^{-1}}(1 - s_1^{m+1}(X))}{1 - s_n(X)}.
\end{aligned}$$

If $X = I$, then

$$\begin{aligned}
\left|\left|\left|\sum_{i=0}^{m} A_i^{s_n(A_i)} + B_i^{s_n(B_i)}\right|\right|\right| &\geq \left|\left|\left|\sum_{i=0}^{m} (s_n^{s_n(A_i)}(A_i) + s^{s_n(B_i)}(B_i))I\right|\right|\right| \\
&\geq \left|\left|\left|\sum_{i=0}^{m} 2e^{-e^{-1}}I\right|\right|\right| \\
&= \sum_{i=0}^{m} 2e^{-e^{-1}}|||I||| = 2(m+1)e^{-e^{-1}}.
\end{aligned}$$ □

4. Arithmetic-geometric matrix series and their convergence

From the following lemma we have the arithmetic geometric mean inequality.

Lemma 4.1. *Let a_i, for $i = 1, \dots, n$, be nonnegative real number, then*

$$\sqrt[n]{a_1.a_2 \dots .a_n} \leq \frac{\sum\limits_{i=1}^{n} a_i}{n}.$$

Equality holds for $a_i = 0, 1$, $i = 1, \dots, n$.

So, from this lemma we can conclude the following.

Lemma 4.2. *Let a_i, for $i = 1, \dots, n$, be positive real numbers, then*

$$(\sum_{i=1}^{n} a_i)(\sum_{i=1}^{n} a_i^{-1}) \geq n^2.$$

The equality holds for $a_i = 1$, $i = 1, \dots, n$.

Proof. From Lemma 4.1, we obtain:

$$1 = a_1.a_2 \dots a_n.\frac{1}{a_1}.\frac{1}{a_2} \dots \frac{1}{a_n} = \sqrt[n]{a_1.a_2 \dots a_n}\sqrt[n]{\frac{1}{a_1}.\frac{1}{a_2} \dots \frac{1}{a_n}} \leq \frac{\sum\limits_{i=1}^{n} a_i}{n}.\frac{\sum\limits_{i=1}^{n} a_i^{-1}}{n}. \qquad \square$$

The last theorem implies that when $m \longrightarrow \infty$ it diverges, while it tends to partial sum when it is finite.

From [4], we have the following lemma.

Lemma 4.3. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be positive definite matrices. Then*

$$\left\|(A+B)^2\right\|_2 \geq 4\left\|AB\right\|_2. \tag{5}$$

We can see that if we replace B by A^{-1}, then (2) becomes:

$$\left\|(A+A^{-1})^2\right\|_2 \geq 4\sqrt{n}.$$

The Lemma 4.3 can be generalized to the following theorem.

Theorem 4.4. *Let* $A_k \in \mathbb{M}_n(\mathbb{C})$, $k = 1, \dots, m$ *be positive definite matrices, then*

$$\left\|(\sum_{k=1}^{m} A_k)^2\right\|_2 \geq 2^m \left\|\prod_{k=1}^{m} A_k\right\|_2.$$

Proof. By Lemma 4.3, we have for $m = 2$:

$$\sum_{k=1}^{m} \left\|(A_k)^2\right\|_2 \geq \left\|\sum_{k=1}^{m-1} (A_k + A_{k+1})^2\right\|_2.$$

Assume the statement holds when $m = l$. Then

$$\left\|(\sum_{k=1}^{l} A_k)^2\right\|_2 \geq 2^l \left\|\prod_{k=1}^{l} A_k\right\|_2. \tag{6}$$

We need to show that it is true when $m = l + 1$.

$$\begin{aligned}
\left\|(\sum_{k=1}^{l+1} A_k)^2\right\|_2 &= \left\|(\sum_{k=1}^{l} A_k + A_{l+1})^2\right\|_2 \\
&= \left\|(\sum_{k=1}^{l} A_k)^2 + 2A_{l+1}\sum_{k=1}^{l} A_k + (A_{l+1})^2\right\|_2 \\
&\geq \left\|\sum_{k=1}^{l} (A_k)^2\right\|_2^{\frac{1}{2}} + \left\|(A_{l+1})^2\right\|_2^{\frac{1}{2}} \\
&\geq 2\left\|\sum_{k=1}^{l} (A_k)^2\right\|_2 \left\|(A_{l+1})^2\right\|_2 \quad \text{(by Lemma 4.1)}, \\
&\geq 2^l \left\|\prod_{k=1}^{l} A_k\right\|_2 2\left\|A_{l+1}\right\|_2 \quad \text{(by (6))}
\end{aligned}$$

$$\geq 2^{l+1}\left\|\prod_{k=1}^{l} A_k A_{l+1}\right\|_2 = 2^{l+1}\left\|\prod_{k=1}^{l+1} A_k\right\|_2 . \qquad \square$$

From [3], we have another form of inequalities by the following lemma.

Lemma 4.5. *Let* $A, B \in \mathbb{M}n(\mathbb{C})$ *be positive definite contractions. If* $r \leq sn(B)$ *and* $t \leq s_n(A)$, *then*

$$s_j^{-1}(A^r + B^t) + \frac{1}{4}(s_j(A) + s_n(B)) \leq \frac{1}{2}\left(\frac{s_j(A)}{s_n(B)} + \frac{s_n(B)}{s_j(A)}\right),$$

with equality holding for $A = B = I$.

If we have A_i, B_i, $i = 1, \ldots, m$, with $A^0 = I$ this gives a generalization of Lemma 4.5 by the following theorem.

Theorem 4.6. *Let* A , $B \in \mathbb{M}_n(\mathbb{C})$ *be positive definite contractions matrices. If* $r^i \leq s_n(B^i), t^i \leq s_n(A^i)$ *and* $d^i = \max(s_j(A^i), s_n(B^i))$, *then:*

$$\begin{aligned} s_j^{-1}\Big(\sum_{i=0}^{m}(A^{r^i} + B^{t^i})\Big) &+ \frac{1}{4}\Big(\sum_{i=0}^{m}(s_j(A^i) + s_n(B^i))\Big) \\ &\leq \frac{1}{2}\sum_{i=0}^{m}\Big(\frac{s_j(A^i)}{s_n(B^i)} + \frac{s_n(B^i)}{s_j(A^i)}\Big) && (7) \\ &\leq \frac{1}{2}\left(\sum_{i=0}^{m}\left(\frac{d}{r}\right)^i + \sum_{i=1}^{m}\left(\frac{d}{t}\right)^i\right) && (8) \\ &= \frac{1}{2}\left(\frac{\left(r^{m+1} - d^{m+1}\right)}{r^m\,(r-d)} + \frac{t^{m+1} - d^{m+1}}{t^m\,(t-d)}\right) && (9) \end{aligned}$$

and the series in (9) *does not converge because one of its terms tends to infinity as* $m \to \infty$, *given* $d > r$ *and* $d > t$.

Proof. We prove the equation (7) when $i = 1$.

$$s_j^{-1}(A^r + B^t) + \frac{1}{4}(s_j(A) + s_n(B)) \leq \frac{1}{2}\Big(\frac{s_j(A)}{s_n(B)} + \frac{s_n(B)}{s_j(A)}\Big) \quad \text{(by Lemma 4.5)} \qquad (10)$$

Let the statement be true when $m = k$:

$$s_j^{-1}\Big(\sum_{i=1}^{k}(A^{r^i} + B^{t^i})\Big) + \frac{1}{4}\Big(\sum_{i=1}^{k}(s_j(A^i) + s_n(B^i))\Big) \leq \frac{1}{2}\sum_{i=}^{k}\Big(\frac{s_j(A^i)}{s_n(B^i)} + \frac{s_n(B^i)}{s_j(A^i)}\Big). \qquad (11)$$

We want to show it is true for $m = k+1$, then

$$s_j^{-1}\Big(\sum_{i=1}^{k+1}(A^{r^i} + B_i^{t^i})\Big) + \frac{1}{4}\Big(\sum_{i=1}^{k+1}(s_j(A^i) + s_n(B^i))\Big)$$

$$= s_j^{-1}\Big(\sum_{i=1}^{k}(A^{r^i} + B_i^{t^i}) + (A^{r^{k+1}} + B^{t^{k+1}})\Big) + \frac{1}{4}\Big(\sum_{i=1}^{k}(s_j(A^i) + s_n(B^i) + (s_j(A^{k+1}) + s_n(B^{k+1}))\Big)$$

$$\leq \left(s_j^{-1}(\sum_{i=1}^{k}(A^{r^i}+B_i^{t^i})+\frac{1}{4}(\sum_{i=1}^{k}(s_j(A^i)+s_n(B^i)) \right)$$
$$+s_j^{-1}(A^{r^{k+1}}+B^{t^{k+1}})+\frac{1}{4}(s_j(A^{k+1})+s_n(B^{k+1}))$$
$$\leq \frac{1}{2}\sum_{i=}^{k}(\frac{s_j(A^i)}{s_n(B^i)}+\frac{s_n(B^i)}{s_j(A^i)})+\frac{1}{2}(\frac{s_j(A^{k+1})}{s_n(B^{k+1})}+\frac{s_n(B^{k+1})}{s_j(A^{k+1})})$$
$$= \frac{1}{2}\sum_{i=}^{k+1}(\frac{s_j(A^i)}{s_n(B^i)}+\frac{s_n(B^i)}{s_j(A^i)}) \quad \text{(by (10) and (11)).}$$

To prove (8), it is sufficient to replace $s_n(B^i)$ with r^i, $s_n(A^i)$ with t^i and d^i with $\max(s_j(A^i), s_n(B^i))$, then we obtain a geometric series with summation like in (9). ☐

This series always diverges since one of the terms tends to ∞. Where $d > r$ and $t, m \to \infty$.

THEOREM 4.7. *Let* $\{f_{i.}(t) = \|A\|^{\alpha_i t} : i = 1, 2, \ldots, m\}$ *be a set of functions* , $A \in \mathbb{M}_n(\mathbb{C})$ *be positive definite matrix, and* $\sum_{i=1}^{n} \alpha_i = 1$ *where* $\alpha_i \in [0, 1]$. *Then*

$$\sum_{i=1}^{m}\prod_{k=1}^{n} f_i^k(t) = \frac{\|A\|^t - \left(\|A\|^t\right)^{n+1}}{1 - \|A\|^t}$$

Proof.

$$\sum_{i=1}^{m}\prod_{k=1}^{n} f_i^k(t) = 1 + \prod_{k=1}^{n} f_i(t) + \ldots + \prod_{k=1}^{n} f_i^m(t)$$
$$= 1 + \|A\|^{\alpha_1 t}\,\|A\|^{\alpha_2 t}\ldots + \|A\|^{\alpha_n t} + \ldots + \left(\|A\|^{\alpha_1 t}\,\|A\|^{\alpha_2 t}\ldots + \|A\|^{\alpha_n t}\right)^n$$
$$= 1 + \|A\|^t + \|A\|^{2t} + \ldots + \|A\|^{nt} = \frac{\|A\|^t - \left(\|A\|^t\right)^{n+1}}{1 - \|A\|^t}.$$ ☐

This geometric series is convergent when $n \to \infty$, and $\|A\| < 1$, for $\frac{\|A\|^t}{1-\|A\|^t}$, then if A doubly stochastic matrix, the sum converges.

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