

## SOME $d$ -TRANSFORM FUNCTORS BASED ON AN IDEAL

A. Makhsoos, N. Zamani and M. S. Sayedsadeghi

**Abstract.** Let  $R$  be a commutative Noetherian ring,  $\mathfrak{b}$  be an ideal of  $R$ ,  $M$  an  $R$ -module and let  $d$  be a non-negative integer. We introduce a general  $d$ -transform functor  $T_{d,\mathfrak{b}}(M, -)$  and its right derived functors  $T_{d,\mathfrak{b}}^i(M, -)$ ,  $i \in \mathbb{N}_0$ , on the category of  $R$ -modules and study their various properties. The connection of these functors with some kind of generalized local cohomology functors  $H_{d,\mathfrak{b}}^i(M, -)$  is discussed. When both  $M$  and  $N$  are finitely generated, some finiteness results on  $T_{d,\mathfrak{b}}^i(M, N)$  and  $H_{d,\mathfrak{b}}^i(M, N)$  are concluded. Then, we study how the depth and dimension of certain subsets of  $\text{Spec}(R)$  affect the behavior and vanishing of these modules.

### 1. Introduction

Throughout this paper,  $R$  denotes a commutative Noetherian ring with a non-zero identity, and  $M$  and  $N$  are two  $R$ -modules. Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be two ideals of  $R$ , and let  $d$  be a non-negative integer. The basic theory of the  $\mathfrak{a}$ -transform functor  $D_{\mathfrak{a}}(-) := \varinjlim_{n \in \mathbb{N}} \text{Hom}_R(\mathfrak{a}^n, -)$  and its connections with the local cohomology functor have appeared and been studied in [6] (see also [5, 10]). For an  $R$ -module  $X$ , the  $R$ -module  $D_{\mathfrak{a}}(X)$  is an important algebraic tool in studying ordinary local cohomology modules  $H_{\mathfrak{a}}^i(M)$ .

In [15], the authors defined a new functor  $T_d(-)$  and its right derived functors and studied its connection with the functor  $U_d(-)$  and with the cohomology functors  $H_d^i(-)$ ,  $i \in \mathbb{N}_0$ , defined in [3]. In this paper, we study to what extent these results are valid for generalized local cohomology. To be precise, let  $\mathcal{I}(R)$  be the set of all ideals of  $R$ , and put  $\Sigma_d = \{\mathfrak{a} \in \mathcal{I}(R) : \dim(R/\mathfrak{a}) \leq d\}$ .

We define the set  $\Gamma_{d,\mathfrak{b}}(X)$  consisting of all elements  $x \in X$  such that  $\mathfrak{a}x \subseteq \mathfrak{b}x$  for some  $\mathfrak{a} \in \Sigma_d$ . Then  $\Gamma_{d,\mathfrak{b}}(-)$  is an  $R$ -linear left exact functor on the category of  $R$ -modules  $\mathcal{C}(R)$ , and we consider its  $i$ th ( $i \geq 0$ ) right derived functor, denoted by  $H_{d,\mathfrak{b}}^i(-)$ .

---

2020 Mathematics Subject Classification: 13C05, 13C12, 16D70

Keywords and phrases: Local cohomology; generalized local cohomology; projective dimension;  $d$ -transform; injective modules.

Now, for two modules  $M$  and  $N$ , we put  $H_{d,b}^i(M, N) := H^i(\Gamma_{d,b}(\text{Hom}_R(M, \mathbf{E}^N)))$ , where  $i \geq 0$  and  $\mathbf{E}^N$  is any injective resolution of  $N$ .

To give a more concrete description of the aim of the paper, we consider three other sets  $\tilde{W}(d, \mathfrak{b}) = \{I \in \mathcal{I}(R) : \exists \mathfrak{a} \in \Sigma_d, \mathfrak{a} \subseteq I + \mathfrak{b}\}$ ,  $W(d, \mathfrak{b}) := \{\mathfrak{p} \in \text{Spec}(R) \mid \exists \mathfrak{a} \in \Sigma_d, \mathfrak{a} \subseteq \mathfrak{p} + \mathfrak{b}\}$  and  $S_k^*(M, N) := \{\mathfrak{p} \in \text{Supp}(M) \mid \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) \leq k\}$ ,  $k \in \mathbb{N}_0$  (cf. [12, 14]). Then, with the reverse inclusion,  $\tilde{W}(d, \mathfrak{b})$  (and also  $\Sigma_d$ ) is a system of ideals in  $R$  in the sense of [6, p. 21]. Now we define the  $d$ -transform functor  $T_{d,b}(M, -)$  supported in  $\mathfrak{b}$  by  $T_{d,b}(M, N) = \varinjlim_{\mathfrak{a} \in \tilde{W}(d,b)} \text{Hom}(\mathfrak{a}M, N)$ , for all  $N \in \mathcal{C}(R)$ .

Evidently,  $T_{d,b}(M, -)$  is  $R$ -linear and left exact, and its right derived functors  $T_{d,b}^i(M, -)$ ,  $i \in \mathbb{N}_0$ , form a strongly connected sequence in the sense of [6, Definition 1.3.1]. It is convenient to mention that, using standard arguments of homological algebra [11, Theorem 6.68] (Axioms for covariant  $\text{Ext}$ ), we have  $H_{d,b}^i(M, N) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d,b)} \text{Ext}_R^i(M/\mathfrak{a}M, N)$ . From this fact and using [11, Lemma 5.30], it is easily seen that for each  $i \in \mathbb{N}_0$  we have  $\Gamma_{d,b}(H_{d,b}^i(M, N)) = H_{d,b}^i(M, N)$ .

The aim of this paper is to study the structure of the modules  $T_{d,b}^i(M, N)$  and their connections with  $H_{d,b}^i(M, N)$ ,  $i \in \mathbb{N}_0$ . In Section 2, we present some basic auxiliary facts concerning the modules  $\Gamma_{d,b}(M)$ ,  $\Gamma_{d,b}(M, N)$ ,  $H_{d,b}^i(M, N)$ , and  $T_{d,b}^i(M, N)$ . It is shown, for example, that when  $M$  is finitely generated, or when  $\Gamma_{d,b}(N) = N$ , these modules exhibit interesting properties and coincide with previously known structures. In Section 3, we assume that  $M$  is finitely generated and obtain further isomorphism results, which are collected in Theorem 3.1. It is concluded that whenever both  $M$  and  $N$  are finitely generated, the set of associated primes of  $T_{d,b}^i(M, N)$  is finite. In Section 4, we investigate how the dimensions of the sets  $W(d, \mathfrak{b})$  and  $S^*d, \mathfrak{b}(M, N)$  affect the vanishing of the modules  $H^i d, \mathfrak{b}(M, N)$  and  $T_{d,b}^i(M, N)$ . For any unexplained notation, we refer the reader to [6, 12].

## 2. Preliminaries

In this section we provide some elementary facts concerning the modules  $\Gamma_{d,b}(M)$ ,  $\Gamma_{d,b}(M, N)$ ,  $H_{d,b}^i(M, N)$  and  $T_{d,b}^i(M, N)$ . It is seen that like  $\Gamma_{\mathfrak{a}}(-)$  and  $\Gamma_d(-)$ , the functor  $\Gamma_{d,b}(-)$  keeps the injective property. We collect the properties in the following lemma. For an  $R$ -module  $X$ , the projective dimension of  $X$  will be denoted by  $\text{pd}(X)$ .

**PROPOSITION 2.1.** *Let  $M$  and  $N$  be two  $R$ -modules. The following statements hold.*

- (i) *If  $M$  is an injective  $R$ -module, then  $\Gamma_{d,b}(M)$  is also an injective  $R$ -module.*
- (ii) *If  $M$  is finitely generated, then  $H_{d,b}^i(M, N) \cong H^i(\text{Hom}_R(M, \Gamma_{d,b}(\mathbf{E}^N)))$ .*
- (iii) *If  $M$  is finitely generated and  $\Gamma_{d,b}(N) = N$ , then  $H_{d,b}^i(M, N) \cong \text{Ext}_R^i(M, N)$  for all  $i \geq 0$ . In particular  $H_{d,b}^i(M, \Gamma_{d,b}(N)) \cong \text{Ext}_R^i(M, \Gamma_{d,b}(N))$ , for  $i \geq 0$ .*
- (iv) *If  $\text{pd}(M) = p$ , then  $H_{d,b}^i(M, N) \cong H_{d,b}^i(M, N/\Gamma_{d,b}(N))$ , for all  $i > p$ .*
- (v) *For each  $i \in \mathbb{N}_0$ ,  $H_{d,b}^i(M, N) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d,b)} H_{\mathfrak{a}}^i(M, N)$ .*

(vi) For each  $i \in \mathbb{N}_0$ ,  $H_{d,b}^i(M, N) \cong \varinjlim_{\mathfrak{a} \in \Sigma_d} H_{\mathfrak{a},b}^i(M, N)$ , where  $H_{\mathfrak{a},b}^i(M, N)$  is the  $i^{\text{th}}$  generalized local cohomology relative to  $(\mathfrak{a}, \mathfrak{b})$ , studied in [9, 13].

(vii)  $T_{d,b}(M, N) \cong \varinjlim_{\mathfrak{a} \in \Sigma_d} D_{\mathfrak{a},b}(M, N)$ , where  $D_{\mathfrak{a},b}(M, N) = \varinjlim_{\mathfrak{c} \in \tilde{W}(\mathfrak{a}, \mathfrak{b})} D_{\mathfrak{c}}(M, N)$ .

(viii) There exists a long exact sequence of  $R$ -modules

$$0 \rightarrow \Gamma_{d,b}(M, N) \rightarrow \text{Hom}_R(M, N) \rightarrow T_{d,b}(M, N) \rightarrow H_{d,b}^1(M, N) \rightarrow \text{Ext}_R^1(M, N) \\ \rightarrow T_{d,b}^1(M, N) \rightarrow H_{d,b}^2(M, N) \rightarrow \cdots$$

(ix) If  $M$  is projective or  $N$  is injective, then we have the sequence

$$0 \rightarrow \Gamma_{d,b}(M, N) \rightarrow \text{Hom}_R(M, N) \rightarrow T_{d,b}(M, N) \rightarrow H_{d,b}^1(M, N) \rightarrow 0.$$

(x) If  $\text{pd}(M) = p$ , then  $T_{d,b}^i(M, N) \cong H_{d,b}^{i+1}(M, N)$ , for all  $i > p$ .

(xi) For each  $i \in \mathbb{N}_0$ ,  $T_{d,b}^i(M, -) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d,b)} \text{Ext}_R^i(\mathfrak{a}M, -)$ .

*Proof.* (i) Note that by a similar argument as in [12, Theorem 3.2], we have  $\Gamma_{d,b}(M) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d,b)} \Gamma_{\mathfrak{a}}(M)$ . As the direct limit of injective modules is injective (note that  $R$  is Noetherian), this gives the claim.

To prove (ii), by our definition we have

$$H_{d,b}^0(M, N) = \Gamma_{d,b}(\text{Hom}_R(M, N)) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d,b)} \Gamma_{\mathfrak{a}}(\text{Hom}_R(M, N)) \\ \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d,b)} \text{Hom}(M, \Gamma_{\mathfrak{a}}(N)) \cong \text{Hom}(M, \varinjlim_{\mathfrak{a} \in \tilde{W}(d,b)} \Gamma_{\mathfrak{a}}(N)) \cong \text{Hom}(M, \Gamma_{d,b}(N)),$$

which is the  $0^{\text{th}}$ -term of the connected sequence  $H^i(\text{Hom}_R(M, \Gamma_{d,b}(\mathbf{E}^-)))$ ,  $i \in \mathbb{N}_0$ , evaluated in  $N$  (to see why the second isomorphism holds true, see for example [9, Proposition 2.2]) with  $\mathfrak{b} = 0$ , and for the third isomorphism see [1, Proposition 7.7]. Now, as both connected sequences  $H_{d,b}^i(M, -)$  and  $H^i(\text{Hom}_R(M, \Gamma_{d,b}(\mathbf{E}^-)))$  are zero in injective modules, by the standard argument of homology theory (see [11, Theorem 6.68]) the result follows.

(iii) By a similar argument as in [14, Lemma 2.1(3)], there exists an injective resolution of  $N$ , say  $\mathbf{E}^N$ , such that  $\Gamma_{d,b}(\mathbf{E}^N) = \mathbf{E}^N$ . By part (ii), we see that

$$H_{d,b}^i(M, N) \cong H^i(\text{Hom}_R(M, \Gamma_{d,b}(\mathbf{E}^N))) \cong H^i(\text{Hom}_R(M, \mathbf{E}^N)) \cong \text{Ext}_R^i(M, N).$$

In particular, as  $\Gamma_{d,b}(\Gamma_{d,b}(N)) = \Gamma_{d,b}(N)$ , the desired isomorphism is clear.

(iv) From the short exact sequence  $0 \rightarrow \Gamma_{d,b}(N) \rightarrow N \rightarrow N/\Gamma_{d,b}(N) \rightarrow 0$ , we obtain the exact sequence

$$H_{d,b}^i(M, \Gamma_{d,b}(N)) \rightarrow H_{d,b}^i(M, N) \rightarrow H_{d,b}^i(M, N/\Gamma_{d,b}(N)) \rightarrow H_{d,b}^{i+1}(M, \Gamma_{d,b}(N)),$$

for all  $i \geq 0$ . Now, since  $\Gamma_{d,b}(\Gamma_{d,b}(N)) = \Gamma_{d,b}(N)$  and  $\text{Ext}_R^i(M, \Gamma_{d,b}(N)) = 0$ , for all  $i > p$ , we get the result by part (iii).

(v) This follows from

$$\Gamma_{d,b}(M, N) = \Gamma_{d,b}(\text{Hom}_R(M, N)) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d,J)} \Gamma_{\mathfrak{a}}(\text{Hom}_R(M, N)) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d,J)} \Gamma_{\mathfrak{a}}(M, N),$$

and the standard homology arguments.

(vi) First, for an  $R$ -module  $X$ , we show that  $\Gamma_{d,b}(X) \cong \varinjlim_{\mathfrak{a} \in \Sigma_d} \Gamma_{\mathfrak{a},b}(X)$ . As  $\Sigma_d$  is a system of ideals, it is enough to show that  $\Gamma_{d,b}(X) = \cup_{\mathfrak{a} \in \Sigma_d} \Gamma_{\mathfrak{a},b}(X)$ . To do this, let  $x \in \Gamma_{d,b}(X)$ . Then there exists  $\mathfrak{a} \in \Sigma_d$  such that  $\mathfrak{a}x \subseteq \mathfrak{b}x$ . Hence  $x \in \Gamma_{\mathfrak{a},b}(X)$  and so  $x \in \cup_{\mathfrak{a} \in \Sigma_d} \Gamma_{\mathfrak{a},b}(X)$ . For the opposite inclusion, we just note that for each  $\mathfrak{a} \in \Sigma_d$  and each  $i \in \mathbb{N}_0$ ,  $\mathfrak{a}^i \in \Sigma_d$ . Now, since both  $H_{d,b}^i(M, -)$  and  $\varinjlim_{\mathfrak{a} \in \Sigma_d} H_{\mathfrak{a},b}^i(M, -)$ ,  $i \in \mathbb{N}_0$ , are strongly connected the result follows again by [11, Theorem 6.68].

(vii) Using the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_{d,b}(M, N) & \longrightarrow & \text{Hom}_R(M, N) & \longrightarrow & T_{d,b}(M, N) & \longrightarrow & H_{d,b}^1(M, N) & \longrightarrow & \text{Ext}_R^1(M, N) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \twoheadrightarrow & \varinjlim_{\mathfrak{a} \in \Sigma_d} \Gamma_{\mathfrak{a},b}(M, N) & \twoheadrightarrow & \text{Hom}_R(M, N) & \twoheadrightarrow & \varinjlim_{\mathfrak{a} \in \Sigma_d} D_{\mathfrak{a},b}(M, N) & \twoheadrightarrow & \varinjlim_{\mathfrak{a} \in \Sigma_d} H_{\mathfrak{a},b}^1(M, N) & \twoheadrightarrow & \text{Ext}_R^1(M, N) \end{array}$$

with exact rows, this follows by (vi) and the Five Lemma.

(viii) Let  $\mathfrak{a}$  and  $\mathfrak{c}$  be two ideals in  $\tilde{W}(d, \mathfrak{b})$  such that  $\mathfrak{a} \subseteq \mathfrak{c}$ . From the commutative diagram of  $R$ -modules and  $R$ -homomorphisms with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{a}M & \longrightarrow & M & \longrightarrow & M/\mathfrak{a}M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{c}M & \longrightarrow & M & \longrightarrow & M/\mathfrak{c}M \longrightarrow 0, \end{array}$$

we obtain the commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow \text{Hom}_R(M/\mathfrak{c}M, N) & \rightarrow & \text{Hom}_R(M, N) & \rightarrow & \text{Hom}_R(\mathfrak{c}M, N) & \rightarrow & \text{Ext}_R^1(M/\mathfrak{c}M, N) \rightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow \text{Hom}_R(M/\mathfrak{a}M, N) & \rightarrow & \text{Hom}_R(M, N) & \rightarrow & \text{Hom}_R(\mathfrak{a}M, N) & \rightarrow & \text{Ext}_R^1(M/\mathfrak{a}M, N) \rightarrow \cdots \end{array}$$

with exact rows. Now, passing to the direct limit over  $\tilde{W}(d, \mathfrak{b})$  and using standard homology arguments gives the desired sequence.

(ix) Since in either case  $\text{Ext}_R^1(M, N) = 0$ , this follows by part (viii).

(x) Apply part (viii) and use the fact that  $\text{Ext}_R^i(M, N) = 0$ , for all  $i > p$ .

(xi) Let  $T(-) = T_{d,b}(M, -)$  and  $T^i(-) = \varinjlim_{\mathfrak{a} \in \tilde{W}(d, \mathfrak{b})} \text{Ext}_R^i(\mathfrak{a}M, -)$ . Then,  $T^i(-)_{i \in \mathbb{N}_0}$  is a negative strongly connected sequence of covariant functors and  $T^0(-)$  is naturally equivalent to  $T(-)$ . Since  $T^i(E) = 0$ , for all  $i \geq 1$  and all injective  $R$ -module  $E$ , the result follows by [11, Theorem 6.68].  $\square$

### 3. Finiteness results

In this section we generally assume that  $M$  is finitely generated and study basic properties of the functor  $T_{d,b}(M, -)$  and its right derived functors  $T_{d,b}^i(M, -)$ . The end of this section contains a study of the sets  $\text{Ass}(T_{d,b}(M, N))$  and  $\text{Ass}(T_{d,b}^i(M, N))$ .

THEOREM 3.1. *Let  $M$  be a finitely generated  $R$ -module and  $N$  be an  $R$ -module. The following statements hold.*

- (i) *If  $\Gamma_{d,b}(N) = N$ , then  $T_{d,b}^i(M, N) = 0$  for all  $i \geq 0$ .*
- (ii)  *$T_{d,b}^i(M, N) \cong T_{d,b}^i(M, N/\Gamma_{d,b}(N)) \cong T_{d,b}^i(M, T_{d,b}(N))$  for all  $i \geq 0$ .*
- (iii)  *$T_{d,b}(T_{d,b}(M, N)) \cong T_{d,b}(M, N)$ .*
- (iv)  *$T_{d,b}(\text{Hom}_R(M, N)) \cong \text{Hom}_R(M, T_{d,b}(N))$ .*
- (v)  *$T_{d,a}(T_{d,b}(M, N)) = T_{d,b}(T_{d,a}(M, N))$ .*
- (vi)  *$\Gamma_{d,b}(T_{d,b}(M, N)) = H_{d,b}^1(T_{d,b}(M, N)) = 0$ .*

*Proof.* (i) By Proposition 2.1 (viii) we have the long exact sequence

$$0 \rightarrow \Gamma_{d,b}(M, N) \rightarrow \text{Hom}_R(M, N) \rightarrow T_{d,b}(M, N) \rightarrow H_{d,b}^1(M, N) \rightarrow \text{Ext}_R^1(M, N).$$

Thus by Proposition 2.1 (iii) we conclude that  $T_{d,b}(M, N) = 0$ . The proof will be complete if we show  $T_{d,b}^i(M, N) = 0$  for all  $i \geq 1$ . Since  $\Gamma_{d,b}(N) = N$ , there is an injective resolution  $\mathbf{E}^N$  of  $N$  such that  $\Gamma_{d,b}(E^i) = E^i$  for all  $i \geq 0$ . So as we did for  $N$ , we will have  $T_{d,b}(M, E^i) = 0$  for all  $i \geq 0$ . Therefore  $T_{d,b}^i(M, N) = 0$  for all  $i \geq 0$ .

(ii) From the short exact sequence  $0 \rightarrow \Gamma_{d,b}(N) \rightarrow N \rightarrow N/\Gamma_{d,b}(N) \rightarrow 0$ , we obtain the long exact sequence

$$0 \rightarrow T_{d,b}(M, \Gamma_{d,b}(N)) \rightarrow T_{d,b}(M, N) \rightarrow T_{d,b}(M, N/\Gamma_{d,b}(N)) \rightarrow \dots \rightarrow T_{d,b}^i(M, N) \rightarrow T_{d,b}^i(M, N/\Gamma_{d,b}(N)) \rightarrow T_{d,b}^{i+1}(M, T_{d,b}(N)) \rightarrow \dots$$

Now, using part (i), we have  $T_{d,b}^i(M, \Gamma_{d,b}(N)) = 0$ , for all  $i \geq 0$ , and thus the first isomorphism holds true.

For the second isomorphism, using Proposition 2.1 (ix) with  $M = R$ , the short exact sequence  $0 \rightarrow N/\Gamma_{d,b}(N) \rightarrow T_{d,b}(N) \rightarrow H_{d,b}^1(N) \rightarrow 0$ , induces the exact sequence

$$0 \rightarrow T_{d,b}(M, N/\Gamma_{d,b}(N)) \rightarrow T_{d,b}(M, T_{d,b}(N)) \rightarrow T_{d,b}(M, H_{d,b}^1(N)) \rightarrow \dots \rightarrow T_{d,b}^i(M, N/\Gamma_{d,b}(N)) \rightarrow T_{d,b}^i(M, T_{d,b}(N)) \rightarrow T_{d,b}^i(M, H_{d,b}^1(N)) \rightarrow \dots$$

Since  $T_{d,b}^i(M, H_{d,b}^1(N)) = 0$ , for all  $i \geq 0$ , we are done.

(iii) We have

$$\begin{aligned} T_{d,b}(T_{d,b}(M, N)) &= \varinjlim_{\mathbf{a} \in \bar{W}(d,b)} \text{Hom}_R(\mathbf{a}, T_{d,b}(M, N)) \cong \varinjlim_{\mathbf{a} \in \bar{W}(d,b)} \text{Hom}_R(\mathbf{a}, \varinjlim_{\mathbf{c} \in \bar{W}(d,b)} \text{Hom}_R(\mathbf{c}M, N)) \\ &\cong \varinjlim_{\mathbf{a} \in \bar{W}(d,b)} ( \varinjlim_{\mathbf{c} \in \bar{W}(d,b)} \text{Hom}_R(\mathbf{a}, \text{Hom}_R(\mathbf{c}M, N)) ) \cong \varinjlim_{\mathbf{a} \in \bar{W}(d,b)} ( \varinjlim_{\mathbf{c} \in \bar{W}(d,b)} \text{Hom}_R(\mathbf{c}, \text{Hom}_R(\mathbf{a}M, N)) ) \\ &\cong \varinjlim_{\mathbf{a} \in \bar{W}(d,b)} ( \varinjlim_{\mathbf{c} \in \bar{W}(d,b)} \text{Hom}_R(\mathbf{c} \otimes_R \mathbf{a}M, N) ) \cong \varinjlim_{\mathbf{a} \in \bar{W}(d,b)} ( \varinjlim_{\mathbf{c} \in \bar{W}(d,b)} \text{Hom}_R(\mathbf{a} \otimes_R \mathbf{c}M, N) ) \\ &\cong \varinjlim_{\mathbf{a} \in \bar{W}(d,b)} ( \varinjlim_{\mathbf{c} \in \bar{W}(d,b)} \text{Hom}_R(\mathbf{a}, \text{Hom}_R(\mathbf{c}M, N)) ) \cong \varinjlim_{\mathbf{a} \in \bar{W}(d,b)} \text{Hom}_R(\mathbf{a}, \varinjlim_{\mathbf{c} \in \bar{W}(d,b)} \text{Hom}_R(\mathbf{c}M, N)) \\ &\cong \varinjlim_{\mathbf{a} \in \bar{W}(d,b)} \text{Hom}_R(\mathbf{a}, T_{d,b}(M, N)) \cong T_{d,b}(M, T_{d,b}(N)) \cong T_{d,b}(M, N), \end{aligned}$$

where in the second and seventh isomorphisms we use again [1, Proposition 7.7].

(iv) We have

$$\begin{aligned} T_{d,\mathfrak{b}}(\text{Hom}_R(M, N)) &= \varinjlim_{\mathfrak{a} \in \tilde{W}(d, \mathfrak{b})} \text{Hom}_R(\mathfrak{a}, \text{Hom}_R(M, N)) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d, \mathfrak{b})} \text{Hom}_R(\mathfrak{a} \otimes_R M, N) \\ &\cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d, \mathfrak{b})} \text{Hom}_R(M, \text{Hom}_R(\mathfrak{a}, N)) \cong \text{Hom}_R(M, \varinjlim_{\mathfrak{a} \in \tilde{W}(d, \mathfrak{b})} \text{Hom}_R(\mathfrak{a}, N)) \cong \text{Hom}_R(M, T_{d,\mathfrak{b}}(N)), \end{aligned}$$

which is the desired isomorphism.

(v) We have

$$\begin{aligned} T_{d,\mathfrak{a}}(T_{d,\mathfrak{b}}(M, N)) &= \varinjlim_{\mathfrak{c} \in \tilde{W}(d, \mathfrak{a})} \text{Hom}_R(\mathfrak{c}, T_{d,\mathfrak{b}}(M, N)) \\ &\cong \varinjlim_{\mathfrak{c} \in \tilde{W}(d, \mathfrak{a})} \text{Hom}_R(\mathfrak{c}, \varinjlim_{\mathfrak{d} \in \tilde{W}(d, \mathfrak{b})} \text{Hom}_R(\mathfrak{d}M, N)) \cong \varinjlim_{\mathfrak{c} \in \tilde{W}(d, \mathfrak{a})} ( \varinjlim_{\mathfrak{d} \in \tilde{W}(d, \mathfrak{b})} \text{Hom}_R(\mathfrak{c}, \text{Hom}_R(\mathfrak{d}M, N)) ) \\ &\cong \varinjlim_{\mathfrak{c} \in \tilde{W}(d, \mathfrak{a})} ( \varinjlim_{\mathfrak{d} \in \tilde{W}(d, \mathfrak{b})} \text{Hom}_R(\mathfrak{c} \otimes_R \mathfrak{d}M, N) ) \cong \varinjlim_{\mathfrak{d} \in \tilde{W}(d, \mathfrak{b})} ( \varinjlim_{\mathfrak{c} \in \tilde{W}(d, \mathfrak{a})} \text{Hom}_R(\mathfrak{d} \otimes_R \mathfrak{c}M, N) ) \\ &\cong \varinjlim_{\mathfrak{d} \in \tilde{W}(d, \mathfrak{b})} ( \varinjlim_{\mathfrak{c} \in \tilde{W}(d, \mathfrak{a})} \text{Hom}_R(\mathfrak{d}, \text{Hom}_R(\mathfrak{c}M, N)) ) \cong \varinjlim_{\mathfrak{d} \in \tilde{W}(d, \mathfrak{b})} \text{Hom}_R(\mathfrak{d}, \varinjlim_{\mathfrak{c} \in \tilde{W}(d, \mathfrak{a})} \text{Hom}_R(\mathfrak{c}M, N)) \\ &\cong \varinjlim_{\mathfrak{d} \in \tilde{W}(d, \mathfrak{b})} \text{Hom}_R(\mathfrak{d}, T_{d,\mathfrak{a}}(M, N)) \cong T_{d,\mathfrak{b}}(T_{d,\mathfrak{a}}(M, N)), \end{aligned}$$

and the claim is proved.

(vi) Replacing  $M$  with  $R$  and  $N$  with  $T_{d,\mathfrak{b}}(M, N)$  in Proposition 2.1 (ix), we have the exact sequence of

$$0 \rightarrow \Gamma_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M, N)) \rightarrow T_{d,\mathfrak{b}}(M, N) \rightarrow T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M, N)) \rightarrow H_{d,\mathfrak{b}}^1(T_{d,\mathfrak{b}}(M, N)) \rightarrow 0.$$

Now since by part (iii),  $T_{d,\mathfrak{b}}(M, N) \cong T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M, N))$ , we obtain the result.  $\square$

In general it is a subtle problem to calculate the modules  $T_{d,\mathfrak{b}}^i(M, N)$  for arbitrary  $R$ ,  $M$  and  $N$  (see for example [6, 12.4.7 Example]). In the following example we calculate these modules for the case  $R = M = N = \mathbb{Z}[x]$ , the ring of polynomials over the integers.

EXAMPLE 3.2. Let  $R = \mathbb{Z}[x] = M = N$ . We calculate  $T_{d,\mathfrak{b}}^i(R, R)$  for  $d = 1, 2$  and for each ideal  $\mathfrak{b}$  of  $\mathbb{Z}[x]$ . Note that  $R$  is Noetherian ring of Krull dimension 2. Thus, for each  $d \geq 2$ , we have  $\Sigma_d = \{\mathfrak{a} \in \mathcal{I}(R) \mid \dim(R/\mathfrak{a}) \leq d\} = \mathcal{I}(R)$  and so  $\Gamma_{d,\mathfrak{b}}(R) = \{f \in R \mid \mathfrak{a}f \subseteq \mathfrak{b}f \text{ for some } \mathfrak{a} \in \Sigma_d\} = R$ . Thus, by Theorem 3.1 (i),  $T_{d,\mathfrak{b}}^i(R, R) = 0$ , for each ideal  $\mathfrak{b}$  and each  $i \geq 0$ . It follows, by Proposition 2.1 (viii), that  $H_{d,\mathfrak{b}}^i(R, R) = 0$  for each for each ideal  $\mathfrak{b}$  and each  $i \geq 1$ . For  $d = 1$ ,  $\Sigma_1 = \{\mathfrak{a} \in \mathcal{I}(R) \mid \dim(R/\mathfrak{a}) \leq 1\} = \mathcal{I}(R) \setminus \{0\}$ . Now, we consider two cases  $\mathfrak{b} = 0$  and  $\mathfrak{b} \neq 0$ .

The case  $\mathfrak{b} = 0$ : It is easily seen that  $\tilde{W}(1, (0)) = \mathcal{I}(R) \setminus \{(0)\}$  and we compute  $T_{1,(0)}(R, R)$ . By definition,  $T_{1,(0)}(R, R) = \varinjlim_{0 \neq \mathfrak{a} \in \mathcal{I}(R)} \text{Hom}_R(\mathfrak{a}, R)$ .

We now show that  $\text{Hom}_R(\mathfrak{a}, R) \cong \mathfrak{a}^{-1}$  for all  $\mathfrak{a} \neq (0)$ , where  $\mathfrak{a}^{-1} = \{u \in \mathbb{Q}(x) \mid u\mathfrak{a} \subseteq R\}$ , where  $\mathbb{Q}(x) = \mathbb{Z}[x]_{\{0\}}$  is the fractions field of  $\mathbb{Z}[x]$ . To do this, fix  $a(x) \in \mathfrak{a}$  and consider the map  $\phi : \text{Hom}_R(\mathfrak{a}, R) \rightarrow \mathfrak{a}^{-1}$  defined by

$$\phi(f) = \frac{f(a(x))}{a(x)} \quad \text{for each } f \in \text{Hom}_R(\mathfrak{a}, R).$$

Then, for an arbitrary element  $b(x) \in \mathfrak{a}$ , we have

$$b(x) \frac{f(a(x))}{a(x)} = \frac{a(x)}{a(x)} f(b(x)) = f(b(x)) \in R,$$

and

$$a(x)f(b(x)) = f(a(x)b(x)) = b(x)f(a(x))$$

so that  $\frac{f(a(x))}{a(x)} = \frac{f(a(x))}{a(x)}$ . These, together, give that  $\phi(f) \in \mathfrak{a}^{-1}$  and that  $\phi$  is well-defined. To see that  $\phi$  is a monomorphism, let  $\phi(f) = 0$  for some  $f \in \text{Hom}_R(\mathfrak{a}, R)$ . Then  $f(a(x)) = 0$ , and thus  $f(b(x)) = b \cdot \phi(f) = 0$  for all  $b \in \mathfrak{a}$ , by what we observed above. Hence,  $f = 0$ . For surjectivity of  $\phi$ , let  $u(x) \in \mathfrak{a}^{-1}$ . Define  $f : \mathfrak{a} \rightarrow R$  by  $f(b(x)) = u(x)b(x)$ . Then,  $\phi(f) = \frac{f(a(x))}{a(x)} = \frac{u(x)a(x)}{a(x)} = u$  and we conclude that  $\phi$  is an isomorphism. Therefore,

$$T_{1,(0)}(R, R) \cong \varinjlim_{0 \neq \mathfrak{a} \in \mathcal{I}(R)} \mathfrak{a}^{-1} = \bigcup_{0 \neq \mathfrak{a} \in \mathcal{I}(R)} \mathfrak{a}^{-1} = \mathbb{Q}(x),$$

where the first equality comes from [2, Exercise 17], and the second one is an easy observation. In this case, as  $\Gamma_{1,(0)}(R, R) = \Gamma_{1,(0)}(\text{Hom}_R(R, R)) = \Gamma_{1,(0)}(R) = 0$ , we obtain again by Proposition 2.1 (viii) that  $H_{1,(0)}^1(R, R) = \mathbb{Q}(x)/\mathbb{Z}[x]$ .

The case  $\mathfrak{b} \neq 0$ : Since  $\mathfrak{b} \in \Sigma_1$  we easily conclude that  $\Gamma_{1,\mathfrak{b}}(R) = R$ . Thus by Theorem 3.1 (i),  $T_{1,\mathfrak{b}}^i(R, R) = 0$  for all  $i \geq 0$ .

**THEOREM 3.3.** *Let  $M$  be finitely generated and let  $f : N \rightarrow N'$  be a homomorphism of  $R$ -modules such that  $\Gamma_{d,\mathfrak{b}}(\text{Ker} f) = \text{Ker} f$  and  $\Gamma_{d,\mathfrak{b}}(\text{Coker} f) = \text{Coker} f$ . Then  $T_{d,\mathfrak{b}}(M, N) \cong T_{d,\mathfrak{b}}(M, N')$ .*

*Proof.* From the short exact sequences

$$0 \rightarrow \text{Ker} f \rightarrow N \rightarrow \text{Im} f \rightarrow 0, \quad 0 \rightarrow \text{Im} f \rightarrow N' \rightarrow \text{Coker} f \rightarrow 0$$

we obtain the exact sequence

$$\begin{aligned} 0 \rightarrow T_{d,\mathfrak{b}}(M, \text{Ker} f) \rightarrow T_{d,\mathfrak{b}}(M, N) \rightarrow T_{d,\mathfrak{b}}(M, \text{Im} f) \rightarrow T_{d,\mathfrak{b}}^1(M, \text{Ker} f) \\ 0 \rightarrow T_{d,\mathfrak{b}}(M, \text{Im} f) \rightarrow T_{d,\mathfrak{b}}(M, N') \rightarrow T_{d,\mathfrak{b}}(M, \text{Coker} f) \rightarrow T_{d,\mathfrak{b}}^1(M, \text{Im} f). \end{aligned}$$

Using Theorem 3.1 (i),  $T_{d,\mathfrak{b}}(M, \text{Ker} f) = T_{d,\mathfrak{b}}(M, \text{Coker} f) = 0$ , and so  $T_{d,\mathfrak{b}}(M, N) \cong T_{d,\mathfrak{b}}(M, \text{Im} f)$  and  $T_{d,\mathfrak{b}}(M, \text{Im} f) \cong T_{d,\mathfrak{b}}(M, N')$ . Hence  $T_{d,\mathfrak{b}}(M, N) \cong T_{d,\mathfrak{b}}(M, N')$ .  $\square$

**THEOREM 3.4.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  be an  $R$ -module. Then for each  $i \in \mathbb{N}_0$   $T_{d,\mathfrak{b}}^i(M, N) \cong T_{d,\mathfrak{b}}^i(\text{Hom}_R(M, N))$ .*

*Proof.* For the case  $i = 0$ , consider the exact sequence

$$0 \rightarrow \Gamma_{d,\mathfrak{b}}(M, N) \xrightarrow{\alpha} \text{Hom}_R(M, N) \xrightarrow{f} T_{d,\mathfrak{b}}(M, N) \xrightarrow{\beta} H_{d,\mathfrak{b}}^1(M, N).$$

We have  $\text{Ker} f = \text{Im} \alpha$  and  $\text{Coker} f = \text{Im} \beta$ . Since  $\Gamma_{d,\mathfrak{b}}(\text{Im} \alpha) = \text{Im} \alpha$  and  $\Gamma_{d,\mathfrak{b}}(\text{Im} \beta) = \text{Im} \beta$ , then using Theorem 3.3,  $T_{d,\mathfrak{b}}(R, \text{Hom}_R(M, N)) \cong T_{d,\mathfrak{b}}(R, T_{d,\mathfrak{b}}(M, N))$ . Thus  $T_{d,\mathfrak{b}}(\text{Hom}_R(M, N)) \cong T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M, N))$ .

Now, by Theorem 3.1 (iv) we have  $T_{d,\mathfrak{b}}(M, N) \cong T_{d,\mathfrak{b}}(\text{Hom}_R(M, N))$ , meaning that  $T_{d,\mathfrak{b}}^0(M, N) \cong T_{d,\mathfrak{b}}^0(\text{Hom}_R(M, N))$ . Now, we use the standard argument of homology and Proposition 2.1 (xi) to deduce the result.  $\square$

THEOREM 3.5. *Let  $M$  and  $N$  be two finitely generated  $R$ -modules. Then*

$$\text{Ass}(T_{d,b}(M, N)) = \text{Supp}(M) \cap \text{Ass}(N/\Gamma_{d,b}(N)).$$

*Proof.* Since  $N$  is finitely generated  $R$ -module, by a modification of [15, Theorem 1], we have  $\text{Ass}(T_{d,b}(N)) = \text{Ass}(N/\Gamma_{d,b}(N))$ .

Now, by Theorem 3.4 and Theorem 3.1 (iv), we have

$$\begin{aligned} \text{Ass}(T_{d,b}(M, N)) &= \text{Ass}(T_{d,b}(\text{Hom}_R(M, N))) = \text{Ass}(\text{Hom}_R(M, T_{d,b}(N))) \\ &= \text{Supp}(M) \cap \text{Ass}(T_{d,b}(N)) = \text{Supp}(M) \cap \text{Ass}(N/\Gamma_{d,b}(N)). \quad \square \end{aligned}$$

In case that  $M$  is finitely generated  $R$ -module, in Theorem 3.1 (iii), we showed that  $T_{d,b}(T_{d,b}(M, N)) \cong T_{d,b}(M, N)$ . Also, in Theorem 3.4, we proved  $T_{d,b}(T_{d,b}(M, N)) \cong T_{d,b}(\text{Hom}_R(M, N))$ . In the next theorem we drop the finiteness assumption on  $M$  and conclude these two isomorphisms with some other assumptions on  $M$ .

THEOREM 3.6. *Let  $M$  and  $N$  be two  $R$ -modules. The followings hold:*

(i) *If  $\text{Ext}_R^1(M, N) = 0$ , then  $T_{d,b}(T_{d,b}(M, N)) \cong T_{d,b}(\text{Hom}_R(M, N))$  and*

$$H_{d,b}^i(T_{d,b}(M, N)) \cong H_{d,b}^i(\text{Hom}_R(M, N)),$$

*for each  $i \geq 1$ .*

(ii) *If  $M$  is a flat  $R$ -module, then*

$$T_{d,b}(T_{d,b}(M, N)) \cong T_{d,b}(M, N),$$

*and*

$$\Gamma_{d,b}(T_{d,b}(M, N)) \cong 0 \cong H_{d,b}^1(T_{d,b}(M, N)).$$

*Proof.* (i) As  $\text{Ext}_R^1(M, N) = 0$ , by Proposition 2.1 (viii) we have the exact sequence

$$0 \rightarrow \frac{\text{Hom}_R(M, N)}{\Gamma_{d,b}(M, N)} \rightarrow T_{d,b}(M, N) \rightarrow H_{d,b}^1(M, N) \rightarrow 0$$

from which the long sequence

$$\begin{aligned} 0 \rightarrow T_{d,b} \left( \frac{\text{Hom}_R(M, N)}{\Gamma_{d,b}(M, N)} \right) &\rightarrow T_{d,b}(T_{d,b}(M, N)) \rightarrow T_{d,b}(H_{d,b}^1(M, N)) \\ &\rightarrow T_{d,b}^1 \left( \frac{\text{Hom}_R(M, N)}{\Gamma_{d,b}(M, N)} \right) \rightarrow \dots \end{aligned}$$

is obtained. But, we have  $\Gamma_{d,b}(H_{d,b}^1(M, N)) = H_{d,b}^1(M, N)$ , which in turn by Theorem 3.1 (i), gives  $T_{d,b}(H_{d,b}^1(M, N)) = 0$ . So

$$T_{d,b} \left( \frac{\text{Hom}_R(M, N)}{\Gamma_{d,b}(M, N)} \right) \cong T_{d,b}(T_{d,b}(M, N)). \quad (1)$$

Also, since  $\Gamma_{d,b}(M, N) \cong \Gamma_{d,b}(\text{Hom}_R(M, N))$  and  $T_{d,b}(\Gamma_{d,b}(M, N)) = 0$ , then from the short exact sequence

$$0 \rightarrow \Gamma_{d,b}(M, N) \rightarrow \text{Hom}_R(M, N) \rightarrow \frac{\text{Hom}_R(M, N)}{\Gamma_{d,b}(M, N)} \rightarrow 0,$$

we have

$$T_{d,b} \left( \frac{\text{Hom}_R(M, N)}{\Gamma_{d,b}(M, N)} \right) \cong T_{d,b}(\text{Hom}_R(M, N)). \quad (2)$$

Comparing (1) and (2), we conclude that  $T_{d,b}(\text{Hom}_R(M, N)) \cong T_{d,b}(T_{d,b}(M, N))$ .

(ii) Since  $M$  is a flat  $R$ -module, by [8, Theorem 7.7], for each ideal  $\mathfrak{a}$ ,  $\mathfrak{a}M \cong \mathfrak{a} \otimes_R M$ . Thus, we have

$$\begin{aligned} T_{d,b}(M, N) &= \varinjlim_{\mathfrak{a} \in \tilde{W}(d,b)} \text{Hom}_R(\mathfrak{a}M, N) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d,b)} \text{Hom}_R(\mathfrak{a} \otimes_R M, N) \\ &\cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d,b)} \text{Hom}_R(\mathfrak{a}, \text{Hom}_R(M, N)) \cong T_{d,b}(\text{Hom}_R(M, N)) \cong T_{d,b}(T_{d,b}(M, N)). \end{aligned}$$

To prove the second part of (ii), one simply use the exact sequence

$$0 \rightarrow \Gamma_{d,b}(T_{d,b}(M, N)) \rightarrow T_{d,b}(M, N) \rightarrow T_{d,b}(T_{d,b}(M, N)) \rightarrow H_{d,b}^1(T_{d,b}(M, N)) \rightarrow 0.$$

□

The next result gives a necessary condition for the functor  $T_{d,b}(M, -)$  to be exact. We were not able to find a better sufficient condition for the exactness of  $T_{d,b}(M, -)$ .

**THEOREM 3.7.** *Let  $M$  be an  $R$ -module. If  $T_{d,b}(M, -)$  is an exact functor, then  $H_{d,b}^i(M, N) \cong \text{Ext}_R^i(M, N)$ , for all  $i \geq 2$  and all  $R$ -module  $N$ . In addition, if  $H_{d,b}^i(M, N) \cong \text{Ext}_R^i(M, N)$  for each  $i \geq 1$  and each finitely generated  $R$ -module  $N$ , then  $T_{d,b}(M, -)$  is exact.*

*Proof.* By the exactness of  $T_{d,b}(M, -)$  we have  $T_{d,b}^i(M, -) = 0$ , for all  $i \geq 1$ . Thus by Proposition 2.1 (xi),  $\varinjlim_{\mathfrak{a} \in \tilde{W}(d,b)} \text{Ext}_R^i(\mathfrak{a}M, -) = 0$ , for all  $i \geq 1$ . Now, for two elements  $\mathfrak{a}$  and  $\mathfrak{b}$  of  $\tilde{W}(d, \mathfrak{b})$  with  $\mathfrak{a} \subseteq \mathfrak{b}$ , we have the commutative diagram

$$\begin{array}{ccccccc} \text{Ext}_R^i(\mathfrak{b}M, N) & \rightarrow & \text{Ext}_R^{i+1}(M/\mathfrak{b}M, N) & \rightarrow & \text{Ext}_R^{i+1}(M, N) & \rightarrow & \text{Ext}_R^{i+1}(\mathfrak{b}M, N) \rightarrow \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Ext}_R^i(\mathfrak{a}M, N) & \rightarrow & \text{Ext}_R^{i+1}(M/\mathfrak{a}M, N) & \rightarrow & \text{Ext}_R^{i+1}(M, N) & \rightarrow & \text{Ext}_R^{i+1}(\mathfrak{a}M, N) \rightarrow \cdots \end{array}$$

with exact rows for each  $i \geq 1$ . Then passing to direct limit over  $\tilde{W}(d, \mathfrak{b})$  we obtain the exact sequence

$$T_{d,b}^i(M, N) \longrightarrow H_{d,b}^{i+1}(M, N) \longrightarrow \text{Ext}_R^{i+1}(M, N) \longrightarrow T_{d,b}^{i+1}(M, N) \quad (3)$$

As both end sides are zero for all  $i \geq 1$ , we get  $H_{d,b}^i(M, N) \cong \text{Ext}_R^i(M, N)$ , for all  $i \geq 2$ .

In addition, as each  $R$ -module can be written as a direct limit of its finitely generated submodules over a suitable directed set, we deduce that  $H_{d,b}^i(M, N) \cong \text{Ext}_R^i(M, N)$  for each  $i \geq 1$ . Now, from the exact sequence (3) we get the exactness of  $T_{d,b}(M, -)$ . □

The following theorem and its proof is a counterpart of [10, Theorem 3.5].

**THEOREM 3.8.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  be an  $R$ -module. Then, for each  $t \geq 0$  we have*

$$\text{Ass}(T_{d,b}^t(M, N)) \subseteq \bigcup_{i=1}^t \text{Ass}(E_{t+2}^{i,t-i}) \cup \text{Ass}(\text{Hom}_R(M, T_{d,b}^t(N))),$$

$$\text{and} \quad \text{Supp}(T_{d,\mathfrak{b}}^t(M, N)) \subseteq \bigcup_{i=0}^t \text{Supp}(\text{Ext}_R^i(M, T_{d,\mathfrak{b}}^{t-i}(N))).$$

*Proof.* We note that by [11, Theorem 10.47] there exists a Grothendieck spectral sequence  $E_2^{p,q} = \text{Ext}_R^p(M, T_{d,\mathfrak{b}}^q(N)) \rightarrow T_{d,\mathfrak{b}}^{p+q}(M, N)$ , and modifying the arguments of the proof of [10, Theorem 3.5] gives the results.  $\square$

**COROLLARY 3.9.** *Let  $M$  be a finitely generated  $R$ -module,  $N$  an  $R$ -module and  $t$  a non-negative integer. If  $\text{Supp}(T_{d,\mathfrak{b}}^i(N))$  is a finite set for all  $i \leq t$ , then  $\text{Supp}(T_{d,\mathfrak{b}}^t(M, N))$  is also a finite set.*

*Proof.* We note that by Theorem 3.8

$$\text{Supp}(T_{d,\mathfrak{b}}^t(M, N)) \subseteq \bigcup_{i=0}^t \text{Supp}(\text{Ext}_R^i(M, T_{d,\mathfrak{b}}^{t-i}(N))) \subseteq \bigcup_{i=0}^t \text{Supp}(T_{d,\mathfrak{b}}^{t-i}(N)),$$

and the result follows.  $\square$

#### 4. Using the sets $W(d, \mathfrak{b})$ and $S_k^*(M, N)$

In this section, we study the connection of the vanishing of the modules  $T_{d,\mathfrak{b}}^i(M, N)$ ,  $i \in \mathbb{N}_0$ , with the dimension of the set

$$W(d, \mathfrak{b}) := \{\mathfrak{p} \in \text{Spec}(R) \mid \exists \mathfrak{a} \in \Sigma_d, \mathfrak{a} \subseteq \mathfrak{p} + \mathfrak{b}\},$$

and the dimension of the sets

$$S_k^*(M, N) := \{\mathfrak{p} \in \text{Supp}(M) \mid \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) \leq k\},$$

$k \in \mathbb{N}_0$  (see [4, 14]).

For simplicity we put  $w = \dim(W(d, \mathfrak{b})) := \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in W(d, \mathfrak{b})\}$ , and  $s_k^*(M, N) = \dim(S_k^*(M, N)) := \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in S_k^*(M, N)\}$ .

The other aim of this section is to give a low vanishing theorem for the modules  $H_{d,\mathfrak{b}}^i(M, N)$  in case that both  $M$  and  $N$  are finitely generated. Recall from [8, Theorem 16.7] that for an  $R$ -module  $M$  and an ideal  $\mathfrak{a}$  of  $R$  with  $\mathfrak{a}M \neq M$ , the length of a maximal  $M$ -sequence contained in  $\mathfrak{a}$ , denoted by  $\text{depth}(\mathfrak{a}, M)$ , is a well determined least integer  $n$  such that  $\text{Ext}_R^n(R/\mathfrak{a}, M) \neq 0$ . It is known that for each ideal  $\text{depth}(\mathfrak{a}, M) = \min\{\text{depth}(\mathfrak{p}, M) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$ . For a local ring  $(R, \mathfrak{m})$  and a nonzero finitely generated  $R$ -module  $M$ ,  $\text{depth}(\mathfrak{m}, M)$  is simply denoted by  $\text{depth}(M)$ . Note that, as usual (see [8, p. 131]), by  $\text{grade}_R(M)$ , we mean  $\inf\{i \in \mathbb{N}_0 \mid \text{Ext}_R^i(M, R) \neq 0\}$ .

**THEOREM 4.1.** *Let  $M$  and  $N$  be two finitely generated  $R$ -modules,  $n \geq 2$  and  $k$  be two non-negative integers such that  $s_{n+k}^*(M, N) \leq k$  for all  $k < w$ . Then  $T_{d,\mathfrak{b}}^i(M, N) \cong \text{Ext}_R^i(M, N)$  for all  $i < n - 1$ .*

*Proof.* Using the exact sequence of Proposition 2.1 (viii), it suffices to prove that  $H_{d,\mathfrak{b}}^i(M, N) = 0$  for all  $i < n$ . Let  $\mathfrak{a} \in \tilde{W}(d, \mathfrak{b})$  and  $\mathfrak{p} \in V(\mathfrak{a} + (0 :_R M))$ . Then

$\mathfrak{p} \in W(d, \mathfrak{b})$ . Put  $k := \dim(R/\mathfrak{p}) - 1$ . Then  $k < w$  and so  $\mathfrak{p} \notin S_{n+k}^*(M, N)$ . Hence, we have

$$\begin{aligned} \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) &> k + n \Rightarrow \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) + k + 1 > k + n \\ &\Rightarrow \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) + 1 > n \Rightarrow \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \geq n. \end{aligned}$$

Then  $\text{depth}(\mathfrak{a} + (0 :_R M), N) \geq n$  and so by [8, Theorem 16.6],  $\text{Ext}_R^i(M/\mathfrak{a}M, N) = 0$  for all  $i < n$ . But, as mentioned in introduction,  $H_{d,\mathfrak{b}}^i(M, N)$  is the direct limit of  $\text{Ext}_R^i(M/\mathfrak{a}M, N)$  over the set  $\tilde{W}(d, \mathfrak{b})$ . Thus, it follows that  $H_{d,\mathfrak{b}}^i(M, N) = 0$  for all  $i < n$ .  $\square$

**COROLLARY 4.2.** *Let  $M$  and  $N$  be two finitely generated  $R$ -modules. The following statements hold.*

- (i) *If  $\text{pd}(N) < \text{grade}_R(M) < w$ ,  $s_{n+k}^*(M, N) \leq k$ , then  $T_{d,\mathfrak{b}}^i(M, N) = 0$  for all  $i < \text{grade}_R(M) - \text{pd}(N) - 1$ .*
- (ii) *If  $(R, \mathfrak{m})$  is local,  $\text{depth}(M) < w$  and  $s_{n+k}^*(M, N) \leq \text{depth}(M)$ , then  $T_{d,\mathfrak{b}}^i(M, N) = 0$  for all  $i < \text{depth}(M) - \dim(N) - 1$ .*

*Proof.* (i) By [8, Theorem 16.9], we have  $\text{Ext}_R^i(M, N) = 0$  for all  $i < \text{grade}_R(M) - \text{pd}(N)$  and the result follows by Theorem 4.1.

(ii) Using [8, Theorem 17.1] and Theorem 4.1, we see that  $T_{d,\mathfrak{b}}^i(M, N) = 0$  for all  $i < \text{depth}(M) - \dim(N) - 1$ .  $\square$

**COROLLARY 4.3.** *Let  $M$  and  $N$  be two finitely generated  $R$ -modules, and let  $n$  and  $k$  be two non-negative integers such that  $s_{n+k}^*(M, N) \leq k$  for all  $k < w$ .*

- (i) *If  $\Gamma_{d,\mathfrak{b}}(N) = N$ , then  $\text{Ext}_R^i(M, N) = 0$  for all  $i < n - 1$ .*
- (ii) *If  $\text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \neq \text{depth}_{R_{\mathfrak{p}}}(\Gamma_{d,\mathfrak{b}}(N)_{\mathfrak{p}})$  for some  $\mathfrak{p} \in \text{Supp}(M)$ , then  $\text{Ext}_R^i(M, N) \cong \text{Ext}_R^i(M, N/\Gamma_{d,\mathfrak{b}}(N))$  for all  $i < n - 1$ .*

*Proof.* (i) As  $\Gamma_{d,\mathfrak{b}}(N) = N$ , by Theorem 3.1 (i) we have  $T_{d,\mathfrak{b}}^i(M, N) = 0$  for all  $i \geq 0$ . On the other hand, by Theorem 4.1,  $\text{Ext}(M, N) \cong T_{d,\mathfrak{b}}^i(M, N)$  for all  $i < n - 1$ . These together gives the result.

(ii) Assume that  $\mathfrak{p} \in \text{Supp}(M)$ . From the short exact sequence

$$0 \rightarrow \Gamma_{d,\mathfrak{b}}(N) \rightarrow N \rightarrow N/\Gamma_{d,\mathfrak{b}}(N) \rightarrow 0,$$

and [7, Proposition 1.29] we have

$$\begin{aligned} \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) &\geq \min\{\text{depth}_{R_{\mathfrak{p}}}(\Gamma_{d,\mathfrak{b}}(N)_{\mathfrak{p}}), \text{depth}_{R_{\mathfrak{p}}}((N/\Gamma_{d,\mathfrak{b}}(N))_{\mathfrak{p}})\}, \\ \text{depth}_{R_{\mathfrak{p}}}(\Gamma_{d,\mathfrak{b}}(N)_{\mathfrak{p}}) &\geq \min\{\text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}), \text{depth}_{R_{\mathfrak{p}}}((N/\Gamma_{d,\mathfrak{b}}(N))_{\mathfrak{p}}) + 1\}, \\ \text{depth}_{R_{\mathfrak{p}}}((N/\Gamma_{d,\mathfrak{b}}(N))_{\mathfrak{p}}) &\geq \min\{\text{depth}_{R_{\mathfrak{p}}}(\Gamma_{d,\mathfrak{b}}(N)_{\mathfrak{p}}) - 1, \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})\}. \end{aligned}$$

Now, according to the our assumption in the statement of the corollary and these inequalities, we obtain  $\text{depth}_{R_{\mathfrak{p}}}((N/\Gamma_{d,\mathfrak{b}}(N))_{\mathfrak{p}}) \leq \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$ . Hence, we have

$$\begin{aligned} \text{depth}_{R_{\mathfrak{p}}}((N/\Gamma_{d,\mathfrak{b}}(N))_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) &\leq \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) \\ \Rightarrow s_{n+k}^*(M, N/\Gamma_{d,\mathfrak{b}}(N)) &\subseteq s_{n+k}^*(M, N) \Rightarrow s_{n+k}^*(M, N/\Gamma_{d,\mathfrak{b}}(N)) \leq s_{n+k}^*(M, N) < k. \end{aligned} \tag{4}$$

Since  $N/\Gamma_{d,b}(N)$  is finitely generated  $R$ -module, then in view of (4) and using Theorem 4.1, we get that  $T_{d,b}^i(M, N) \cong \text{Ext}_R^i(M, N/\Gamma_{d,b}(N))$  and  $T_{d,b}^i(M, N) \cong \text{Ext}_R^i(M, N)$  for all  $i < n - 1$ . Comparing these and using Theorem 3.1 (ii), the proof of the claim is complete.  $\square$

## REFERENCES

- [1] A. Altman, S. Kleiman, *A term of commutative algebra*, Worldwide Center of Mathematics, 2021.
- [2] M. F. Atiyah, I.G. Macdonald. *Introduction to commutative algebra*, University of Oxford, 1969.
- [3] C. Banica, M. Soia, *Singular sets of a module on local cohomology*, Boll. Un. Mat. Ital. B, **16** (1976), 923–934.
- [4] M. H. Bijan-Zadeh, *On the singular sets of a modules*, Commun. Algebra, **21** (1993), 4629–4639.
- [5] M. P. Brodmann, *Finiteness of ideal transforms*, J. Algebra, **63**(1) (1980), 162–185.
- [6] M. P. Brodmann, R. Y. Sharp, *Local Cohomology- An Algebraic Introduction with Geometric Applications*, Cambridge University Press, 1998.
- [7] W. Bruns, J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, revised ed., 1998.
- [8] H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.
- [9] T. T. Nam, N. Minh Tri, *Some properties of generalized local cohomology modules with respect to a pair of ideals*, Int. J. Algebra Comput., **24**(7) (2014), 1043–1054.
- [10] T. T. Nam, N. Minh Tri, *Generalized ideal transforms*, Stud. Sci. Math. Hung., **51**(1) (2014), 67–82.
- [11] J. Rotman, *Introduction to homological algebra*, 2nd edn, Springer, 2009.
- [12] R. Takahashi, Y. Yoshino, T. Yoshizawa, *Local cohomology based on a nonclosed support defined by a pair of ideals*, J. Pure Appl. Algebra, **213**(4) (2009), 582–600.
- [13] N. Zamani, *Generalized local cohomology relative to  $(I, J)$* , Southeast Asian Bull. Math., **35** (2011), 1045–1050.
- [14] N. Zamani, M. H. Bijan-Zadeh, M. S. Sayedsadeghi, *Cohomology with supports of dimension  $\leq d$* , J. Algebra Appl., **15**(3) (2016), Article ID 1650042, 10 pp.
- [15] N. Zamani, M. S. Sayedsadeghi, M. H. Bijan-Zadeh,  *$d$ -Transform Functor and Some Finiteness and Isomorphism Results*, Vietnam. J. Math., **41** (2013), 179–186.

(received 26.01.2025; in revised form 17.07.2025; available online 03.02.2026)

Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran

*E-mail:* ahadmakhsoos@uma.ac.ir

ORCID iD: <https://orcid.org/00009-0004-0620-0334>

Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran

*E-mail:* naserzaka@yahoo.com

ORCID iD: <https://orcid.org/0000-0002-6353-2288>

Department of Mathematics, Payame Noor University (Pnu), P.O. Box 19395-4697, Tehran, Iran

*E-mail:* m.sayedsadeghi@pnu.ac.ir

ORCID iD: <https://orcid.org/0000-0002-0865-037X>