A NOTE ON $\alpha$-EQUIVALENT TOPOLOGIES

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Abstract. This paper responds to the question of when are two topologies $\alpha$-equivalent by using some recently introduced classes of sets as well as the classes of regular open sets, nowhere dense sets and dense sets.

1. Introduction

In [10] Njastad gave a characterization of $\alpha$-equivalent topologies on a given set by means of semi-open sets. It is natural to ask whether $\alpha$-equivalent topologies can be characterized by means of some other classes of subsets which are shared by $\alpha$-equivalent topologies, that is by means of classes of regular open, preopen, semi-preopen, nowhere dense and dense sets. We answer that question in the affirmative and show that two topologies have the same collection of $\alpha$-sets if and only if they share both the semi-regularisation topology and the $\gamma$-topology.

We first recall some definitions. Let $A$ be a subset of a topological space $(X, T)$. The closure of $A$ and the interior of $A$ with respect to $T$ are denoted by $\text{cl} A$ and $\text{int} A$, respectively.

Definition. A subset $A$ of $(X, T)$ is called
(i) an $\alpha$-set if $A \subset \text{int}(\text{cl}(\text{int} A))$,
(ii) a semi-open set if $A \subset \text{cl}(\text{int} A)$,
(iii) a preopen set if $A \subset \text{int}(\text{cl} A)$,
(iv) a semi-preopen set if $A \subset \text{cl}(\text{int}(\text{cl} A))$.

The first three notions were introduced by Njastad [10], Levine [8] and Mashhour et al. [9], respectively. The fourth concept was introduced by Abd El-Monsef et al. [1] under the name of $\beta$-open set and it was called semi-preopen set in [3]. The classes of these sets in a space $(X, T)$ are denoted by $T_\alpha$, $\text{SO}(T)$, $\text{PO}(T)$ and $\text{SPO}(T)$, respectively. All of these are larger than $T$ and are closed under arbitrary unions. Njastad [10] showed that $T_\alpha$ is a topology on $X$.

For a space $(X, T)$ the family $\{ A \subset X \mid A \cap B \in \text{PO}(T) \text{ whenever } B \in \text{PO}(T) \}$ will be denoted by $T_\gamma$. It was shown in [4] that $T_\gamma$ is a topology on $X$ larger than $T_\alpha$. It will be called the $\gamma$-topology of $T$. 
The classes of regular open sets, nowhere dense sets and dense sets in \((X,T)\) will be denoted by \(\text{RO}(T)\), \(\text{N}(T)\) and \(\text{D}(T)\) respectively. The complement of a regular open set is called regular closed, and the complement of a dense set is called codense.

The following results will be needed in the sequel.

**Proposition 1.** ([10], [2] and [3]) Let \((X,T)\) be a space. Then:

(i) \(T_a = T_{aa}\),
(ii) \(\text{SO}(T) = \text{SO}(T_a)\),
(iii) \(\text{PO}(T) = \text{PO}(T_a)\),
(iv) \(\text{SPO}(T) = \text{SPO}(T_a)\),
(v) \(\text{RO}(T) = \text{RO}(T_a)\),
(vi) \(\text{D}(T) = \text{D}(T_a)\),
(vii) \(\text{N}(T) = \text{N}(T_a)\).

**Proposition 2.** ([3]) A subset of a space \((X,T)\) is semi-preopen if and only if \(\text{cl} A\) is regular closed.

**Proposition 3.** ([3]) In a space \((X,T)\), the intersection of every open and each semi-preopen set is semi-preopen.

**Proposition 4.** ([7]) In a space \((X,T)\), a subset \(A\) is semi-open if and only if it is semi-preopen and \(\text{int(}\text{cl} A\text{)} \subseteq \text{cl(}\text{int} A\text{)}\).

**Proposition 5.** ([7]) If the topologies \(T\) and \(U\) on a set \(X\) have the same \(\gamma\)-topology, then they have the same class of nowhere dense sets.

**Proposition 6.** ([7]) Two topologies on a set \(X\) have the same class of semi-preopen sets if and only if their \(\gamma\)-topologies are the same.

2. \(\alpha\)-equivalent topologies

**Definition.** ([10]) Two topologies on a set \(X\) are called \(\alpha\)-equivalent if their \(\alpha\)-topologies are the same.

In the same paper Njastad obtained the following result on \(\alpha\)-equivalence in terms of semi-open sets.

**Proposition 7.** Two topologies on a set \(X\) are \(\alpha\)-equivalent if and only if they have the same class of semi-open sets.

We consider the same question in relation to the other classes of subsets which are shared by \((X,T)\) and \((X,T_a)\) mentioned in Proposition 1, to obtain analogous characterizations of \(\alpha\)-equivalence. We begin with a consequence of Propositions 7 and 1.

**Proposition 8.** Let \(T\) and \(U\) be two \(\alpha\)-equivalent topologies. Then:

(i) \(\text{PO}(T) = \text{PO}(U)\),
(ii) \(\text{SPO}(T) = \text{SPO}(U)\),
(iii) \(\text{RO}(T) = \text{RO}(U)\),
(iv) \(\text{D}(T) = \text{D}(U)\),
(v) \(\text{N}(T) = \text{N}(U)\).

It should be noted that none of these five conditions is equivalent to the condition \(\text{SO}(T) = \text{SO}(U)\). One can easily find the examples. But it is a natural question whether there are two conditions among these five which together imply \(\text{SO}(T) = \text{SO}(U)\). We prove that any pair of independent conditions does so. From Propositions 5 and 6 it follows immediately that for the statements in Proposition
8, (i) \(\implies\) (ii) \(\implies\) (v) hold in general. The following result shows that (iv) \(\implies\) (v) is also true. The closure and the interior of a set \(A\) in \((X, U)\) will be denoted by \(\text{cl}_U A\) and \(\text{int}_U A\) respectively.

**Proposition 9.** Let \(T\) and \(U\) be topologies on \(X\) having the same class of dense sets. Then their classes of nowhere dense sets coincide.

**Proof.** Suppose that \(A \in N(T) - N(U)\). Then \(G = \text{int}_T (\text{cl}_U A) \neq \emptyset\) because \((X, T)\) and \((X, U)\) have the same class of codense sets. Define \(W = \text{int}_U (G - \text{cl}_A)\). Since \(A\) is \(T\)-nowhere dense, \(G - \text{cl}_A\) is \(T\)-open and non-empty. Hence \(W \subset \text{cl}_U A - A\) and \(W \neq \emptyset\), a contradiction. Therefore \(N(T) \subset N(U)\). The reverse inclusion is shown in an analogous way.

In order to prove our main result we first establish two lemmas.

**Lemma 1.** If \(T\) and \(U\) are topologies on \(X\) sharing the classes of semi-preopen sets and regular open sets, then they have the same class of dense sets.

**Proof.** Assume that \(\text{cl}_A = X\). Then \(A \in \text{SPO}(T) = \text{SPO}(U)\) and so \(\text{cl}_U A\) is \(U\)-regular closed by Proposition 2. Hence \(\text{cl}_U A\) is \(T\)-regular closed and thus \(\text{cl}_U A = X\), i.e. \(A \in D(U)\). The reverse inclusion is obtained analogously.

**Lemma 2.** Let \(T\) and \(U\) be topologies on \(X\) having the same class of dense sets, and let \(A\) be a subset of \(X\). Then \(\text{int}(\text{cl}_A) \subset \text{cl}(\text{int}_A)\) if and only if \(\text{int}_U \text{cl}_U A \subset \text{cl}_U \text{int}_U A\).

**Proof.** Suppose that \(\text{int}(\text{cl}_A) \subset \text{cl}(\text{int}_A)\) and let \(W = \text{int}_U (\text{cl}_U A - \text{int}_U A)\) be non-empty. Then \(G = \text{int} W \neq \emptyset\) because \((X, T)\) and \((X, U)\) share the class of codense sets. Put \(G_1 = G - \text{cl}_A\) and \(G_2 = G \cap \text{int}_A\) and let \(W_1 = \text{int}_U G_1\) and \(W_2 = \text{int}_U G_2\). We observe that \(W_1 \subset \text{cl}_U A - \text{cl}_A\) which implies \(W_1 = \emptyset\) and so \(G_1 = G_1 = \emptyset\). On the other hand, \(W_2 \subset A\) and \(W_2 \cap \text{int}_U A \subset W \cap \text{int}_U A = \emptyset\). Hence \(W_2 = \emptyset\) and thus \(G_2 = G_2 = \emptyset\). Hence both \(G_1\) and \(G_2\) are empty, \(G \subset \text{cl}_A - \text{int}_A\) and so \(G \subset \text{int}(\text{cl}_A) - \text{cl}(\text{int}_A) = \emptyset\), a contradiction. Therefore \(W = \emptyset\), that is \(\text{int}_U \text{cl}_U A \subset \text{cl}_U \text{int}_U A\).

**Theorem 1.** Let \(T\) and \(U\) be topologies on a set \(X\). Then the following are equivalent:

(a) \(T\) and \(U\) are \(\alpha\)-equivalent,
(b) \(\text{RO}(T) = \text{RO}(U)\) and \(\text{PO}(T) = \text{PO}(U)\),
(c) \(\text{RO}(T) = \text{RO}(U)\) and \(\text{SPO}(T) = \text{SPO}(U)\),
(d) \(\text{RO}(T) = \text{RO}(U)\) and \(\text{N}(T) = \text{N}(U)\),
(e) \(\text{RO}(T) = \text{RO}(U)\) and \(\text{D}(T) = \text{D}(U)\),
(f) \(\text{PO}(T) = \text{PO}(U)\) and \(\text{D}(T) = \text{D}(U)\),
(g) \(\text{SPO}(T) = \text{SPO}(U)\) and \(\text{D}(T) = \text{D}(U)\).

**Proof.** (a) \(\implies\) (b), (b) \(\implies\) (c) and (e) \(\implies\) (d) follow from Propositions 8, 5 and 6. Also, (a) \(\implies\) (e) and (a) \(\implies\) (f) follow from Proposition 8, (e) \(\implies\) (d) from Proposition 9 and (f) \(\implies\) (g) from Proposition 6.

(d) \(\implies\) (c): Suppose \(A \in \text{SPO}(T)\) and put \(B = A - \text{cl}_U \text{int}_U \text{cl}_U A\). Then \(B \in \text{N}(U) = \text{N}(T)\) by (d). On the other hand, \(\text{cl}_U \text{int}_U \text{cl}_U A\) is \(U\)-regular closed
and so $T$-regular closed. Hence $B \in \text{SPO}(T)$ by Proposition 3 and thus $B = \emptyset$, that is $A \in \text{SPO}(U)$.

(c) $\implies$ (a): Suppose that $A \in \text{SO}(U)$. According to Lemma 1, $(X,T)$ and $(X, U)$ share the class of codense sets and so $\text{int}_U (A - \text{cl}(\text{int} A)) = \emptyset$, that is $\text{int}_U A \subseteq \text{cl}_U \text{cl}(\text{int} A)$. Therefore $A \subseteq \text{cl}_U \text{int}_U A \subseteq \text{cl}_U \text{cl}(\text{int} A)$. On the other hand, $\text{cl}(\text{int} A)$ is $T$-regular closed and so $U$-regular closed by (c). Hence $A \subseteq \text{cl}(\text{int} A)$, that is $A \in \text{SO}(T)$. The reverse inclusion is obtained analogously.

(g) $\implies$ (a): Suppose that $A \in \text{SO}(T)$. Then $A \in \text{SPO}(T)$ and $\text{int}(\text{cl} A) \subseteq \text{cl}(\text{int} A)$ by Proposition 4. According to Lemma 2 and (g) we have that $A \in \text{SPO}(U)$ and $\text{int}_U \text{cl}_U A \subseteq \text{cl}_U \text{int}_U A$ and thus $A \in \text{SO}(U)$ again by Proposition 4. The reverse inclusion is obtained analogously. ■

For a space $(X, T)$ the topology $T_s$ on $X$ which has as a base the class $\text{RO}(T)$ is called the semiregularisation topology of $(X, T)$. Recall that $T_{ss} = T_s$ and $\text{RO}(T) = \text{RO}(U)$ if and only if $T_s = U_s$ [6]. Theorem 1 and Proposition 6 give the following characterization.

**Theorem 2.** Let $T$ and $U$ be topologies on a set $X$. Then $T_a = U_a$ if and only if $T_\gamma = U_\gamma$ and $T_s = U_s$. ■

**Corollary 1.** Let $(X, T)$ be a space. Then $T_{sa} = T_a$ if and only if $T_{sa} = T_s$. ■

It was shown in [5] that the topologies $T$ and $T_s$ on a set $X$ are $\alpha$-equivalent if and only if they share the class of dense sets. Our Theorem 3 will be a slight improvement of this result. For convenience we first establish two simple lemmas.

**Lemma 3.** ([4]) Let $(X, T)$ be a space. Then:

(i) $\text{cl}_T G = \text{cl} G$ for any $T$-open set $G$,
(ii) $\text{int}_T F = \text{int} F$ for any $T$-closed set $F$. ■

**Lemma 4.** Let $T$ and $U$ be topologies on $X$ such that $U \subseteq T_{\gamma}$. Then $\text{cl}(\text{int} A) \subseteq \text{cl}_U \text{int}_U A$ for any subset $A$.

**Proof.** By the assumption and Lemma 3 we have that $\text{cl}(\text{int} A) = \text{cl}_{T_{\gamma}} \text{int} A \subseteq \text{cl}_U \text{int}_U A$. ■

**Theorem 3.** Let $T$ and $U$ be topologies on a set $X$ satisfying $T \subseteq U_s$ and $U \subseteq T_s$. Then $T$ and $U$ are $\alpha$-equivalent if and only if they have the same class of dense sets.

**Proof.** Assume $D(T) = D(U)$ and let $A \in \text{SO}(T)$. According to Proposition 4 and Lemma 2 it suffices to show that $A \in \text{SPO}(U)$. By Lemma 4 we have $\text{cl}_U A \subseteq \text{cl}_U \text{cl}(\text{int} A) \subseteq \text{cl}_U \text{int}_U A$ and so $\text{cl}_U A = \text{cl}_U \text{int} A$. Since $\text{int} A \in T \subseteq U_s \subseteq \text{PO}(U)$ we have by Proposition 2 that $\text{cl}_U A$ is $U$-regular closed and hence $A \in \text{SPO}(U)$. ■
REFERENCES


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