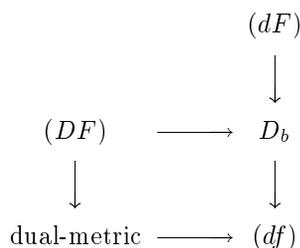


**WEAK TOPOLOGY IN LOCALLY CONVEX SPACES
WITH A FUNDAMENTAL SEQUENCE OF BOUNDED SUBSETS**

Stojan Radenović and Zoran Kadelburg

Abstract. The result of Krassowska and Sliwa [6] about weak topologies in (DF) -spaces is extended to various other classes of locally convex spaces with a fundamental sequence of bounded subsets.

Among many known classes of locally convex linear topological spaces (lcs), the spaces with a fundamental sequence of the family of all bounded subsets or some of its subfamilies (such as precompact or compact subsets), play an important role. The following five classes of spaces



have been studied from various aspects (inheritance properties, three-space-problem, etc). Let us recall the definitions.

An $lcs (E, t)$ is called *countably-quasibarrelled* (resp. *σ -quasibarrelled*; *sequentially quasibarrelled*) if each $\beta(E', E)$ -bounded subset which is a countable union of t -equicontinuous subsets (resp. $\beta(E', E)$ -bounded sequence; $\beta(E', E)$ -convergent sequence) is a t -equicontinuous subset. If, besides that, (E, t) possesses a fundamental sequence of t -bounded subsets, it is said to be of the type (DF) (resp. *dual-metric*; (df)). A barrel T of the space (E, t) is a *p-barrel* (resp. *b-barrel*) if

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its intersection with each t -precompact (resp. t -bounded) absolutely convex subset is a relative t -neighbourhood of the origin. The space (E, t) is p -barrelled (resp. b -barrelled) if each p -barrel (resp. b -barrel) in it is a t -neighbourhood of the origin. We say that the space (E, t) is of the type D_b (or $g(DF)$ by some authors) if it is b -barrelled with a fundamental sequence of t -bounded subsets. The lcs (E, t) is a (dF) -space if it is p -reflexive (i.e. p -barrelled and p -complete [1]) with a fundamental sequence of compact subsets. Remark that the space (E, t) is p -barrelled if and only if each E'_p -precompact subset is t -equicontinuous [2].

It is well known that the main fact that led A. Grothendieck to introduce the class of (DF) -spaces in [3] was that strong duals of Fréchet spaces are of that type. An interesting question may be the following: if (E, t) is a Hausdorff lcs , do the spaces $(E, \sigma(E, E'))$ and $(E', \sigma(E', E))$ with the weak topologies belong to some of the mentioned five classes of spaces? The first answer to such kind of question was given by S. O. Iyahan in [4] when he proved that the space $(c_0, \sigma(c_0, l^1))$ is not of the type (DF) . In [8] it was proved that neither of the spaces $(E, \sigma(E, E'))$ and $(E', \sigma(E', E))$ is of the type (DF) if E is a Banach space of infinite dimension. D. Krassowska and W. Sliwa showed in [6] the most general result: for each dual pair $\langle E, E' \rangle$, the space $(E, \sigma(E, E'))$ (resp. $(E', \sigma(E', E))$) is of the type (DF) if and only if $\dim E < \infty$.

In this short note we show that the result of D. Krassowska and W. Sliwa is true also for the other four classes of spaces from the previous diagram.

THEOREM. *If (E, t) is a Hausdorff lcs , then the space $(E, \sigma(E, E'))$ (resp. $(E', \sigma(E', E))$) is of the type (df) (D_b ; dual-metric; (dF)) if and only if $\dim E < \infty$.*

Proof. Taking into account the diagram, it is enough to consider the case of (df) -spaces. The 'only if' part is nontrivial. So, let $(E, \sigma(E, E'))$ be a (df) -space. Then each $E'_p = \beta(E', E)$ -precompact subset is $\sigma(E, E')$ -equicontinuous, and so $(E, \sigma(E, E'))$ is p -barrelled and also b -barrelled space. As the space $(E, \sigma(E, E'))$ possesses a fundamental sequence of bounded subsets, it is of the type D_b , and so by [7, Cor. 3.2.2] the space $(E'^*, \sigma(E'^*, E'))$, as its completion, is a D_b -space, too. Hence, the lcs $(E'^*, \sigma(E'^*, E'))$ has a fundamental sequence of bounded subsets, wherefrom it follows that the space E' equipped with the finest locally convex topology $\tau(E', E'^*) = \beta(E', E'^*)$ is metrizable. But, the only case when this is possible ([9; Ex. II,7]) is when E' , and so also E , is of finite dimension. ■

REMARK. As for each (df) -space (E, t) , $\beta(E', E)$ -precompact subsets are t -equicontinuous, the proof of the theorem could be derived in the similar way as in [6]. But we proceeded in the other way—using the fact that for each infinite-dimensional vector space E , the lcs $(E^*, \sigma(E^*, E) = \beta(E^*, E))$ has no fundamental sequence of bounded subsets (E^* is the algebraic dual).

COROLLARY. *If (E, t) is a metrizable and barrelled lcs of infinite dimension, then $(E', \sigma(E', E))$ is not a sequentially quasibarrelled space.*

Proof. The space $(E', \sigma(E', E))$ has a fundamental sequence of bounded subsets, but by the Theorem it is not of the type (df) . ■

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Faculty of Sciences, Radoja Domanovića 12, 34000 Kragujevac, Yugoslavia

Faculty of Mathematics, Studentski trg 16, P.O.Box 550, 11000 Beograd, Yugoslavia

e-mail: kadelbur@matf.bg.ac.yu