ON S-CLOSED AND EXTREMALLY DISCONNECTED FUZZY TOPOLOGICAL SPACES

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Abstract. The concepts of a filter-base and $s$-convergence and $\theta$-convergence of a filter-base in a fuzzy setting are defined and investigated. Fuzzy filter-base is used to characterize fuzzy $S$-closed and extremally disconnected spaces. Several other properties of these two types of spaces and comparison between different forms of compactness in fuzzy topology are established.

Introduction

The concept of filters in fuzzy set theory was introduced by Lowen and at the same time by Katsaras who studied in his work [11] fuzzy filters, ultra filters, clusters and the convergence of filters in fuzzy setting. In this paper we have developed the theory of filters a little further and introduced $s$-convergence and $\theta$-convergence of a filter (filter-base).

We offer several characterizations of fuzzy $S$-closed and fuzzy extremally disconnected spaces in terms of fuzzy filter-bases and fuzzy nets. The results are parallel to ones which have been found in general topology. A systematic discussion of these properties in general topology is given in [7], [10], [16] and [17].

In the last section we study the implications between different forms of compactness and comparison with $S$-closedness and extremal disconnectedness in a fuzzy setting.

1. Preliminaries

Throughout the paper by $(X, \tau_X)$, or simply by $X$, we mean a fuzzy topological space (fts, shortly) of Chang [3]. A fuzzy singleton with support $x$ and value $\alpha$ ($0 < \alpha \leq 1$) will be denoted by $x_\alpha$. Two fuzzy sets $\lambda$ and $\mu$ are said to be $q$-coincident if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$ and by $q$ we denote “is not q-coincident”. A fuzzy set $\lambda$ is said to be a $q$-neighbourhood (q-nbd) of $x_\alpha$ if there is a fuzzy open set $\mu$ such that $x_\alpha q \mu$ and $\mu \subseteq \lambda$, where $\lambda \leq \mu$ if $\lambda(x) \leq \mu(x)$.

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for all \( x \in X \). We denote \( N(x_\alpha) \) \((N_q(x_\alpha))\) the neighbourhood (q-nbd) system of \( x_\alpha \) \([12]\). For a fuzzy set \( \lambda \) in an fts \( X \) by \( \text{Cl}\lambda, \text{SCI}\lambda, \text{Int}\lambda \) we denote the closure, semi-closure, interior and support of \( \lambda \), respectively. By \( 0_X \) and \( 1_X \) we mean the constant fuzzy sets taking the values 0 and 1 on \( X \), respectively. For the definition of fuzzy regularly open \((\text{RO}(X))\), regularly closed \((\text{RC}(X))\), semi-open \((\text{SO}(X))\) and semi-closed \((\text{SC}(X))\) sets we refer to Azad \([2]\). A fuzzy point \( x_\alpha \in \text{Cl}\lambda \) \((x_\alpha \in \text{Cl}_0\lambda)\) if \( qU \) implies \( U \cap q \lambda \), for each \( U \in \text{RO}(X) \) \((x_\alpha \cap U \) implies \( \text{Cl}U \cap q \lambda \) for each \( U \in \tau_X \) resp.). A fuzzy set \( \lambda \) is called \( \delta\)-closed \((\theta\)-closed resp.) if \( \lambda = \text{Cl}_0 \lambda \). It is known \([15]\) that for any set \( \lambda \), \( \lambda \leq \text{SCI} \lambda \leq \text{Cl} \lambda \leq \text{Cl}_0 \lambda \leq \text{Cl}_\theta \lambda \).

An fts \((X, \tau_X)\) is said to be semi-regular if the fuzzy regularly open sets of \( X \) form a base for \( \tau_X \) \([2]\). For \((X, \tau_X), X_S\) denotes the semiregularization of \( X \), i.e. \( X_S = (X, \tau_S) \) and \( N_\omega(x_\alpha) \) the nbd-system of \( x_\alpha \) in \( X_S \). An fts \( X \) is called fuzzy almost regular (fuzzy regular, resp.) if every fuzzy regularly open \((\text{resp. fuzzy open})\) set \( \lambda \) of \( X \) can be expressed as a union of fuzzy regularly open \((\text{fuzzy open, resp.})\) sets \( U_\alpha \) such that \( \text{Cl}U_\alpha \leq \lambda \), for all \( \alpha \). Fuzzy regularity implies fuzzy semi-regularity as well as fuzzy almost regularity (see \([1]\) and \([15]\)).

## 2. Nets and filters in fuzzy topology

**Definition 2.1** \([12]\) Let \((D, \geq)\) be a directed set. Let \( X \) be an ordinary set. Let \( \mathcal{J} \) be the collection of all fuzzy points in \( X \). A function \( S : D \to \mathcal{J} \) is called a fuzzy net in \( X \). For \( n \in D \), \( S(n) \) is often denoted by \( x^n_\alpha \), where \( x^n \) is the support and \( \alpha_n \) is the value of the \( n\)-th member of the fuzzy set. Hence the net \( S \) is often denoted by \( \{x^n_\alpha, n \in D\} \) and it is called \( \alpha\)-net if \( \alpha_n \to \alpha \).

**Definition 2.2** \([12]\) A fuzzy net \( \{x^n_\alpha\} \) is said to be \( q\)-coincident with \( \lambda \in I^X \) if for each \( n \in D \), \( x^n_\alpha \) is \( q\)-coincident with \( \lambda \); it is said to be eventually \( q\)-coincident \((\text{or q-final})\) with \( \lambda \) if there is an \( m \in D \) such that if \( n \in D \) and \( n \geq m \), then \( x^n_\alpha \) is \( q\)-coincident with \( \lambda \); it is said to be frequently \( q\)-coincident \((\text{or q-cofinal})\) with \( \lambda \) if for each \( m \in D \), there is an \( n \in D \) such that \( n \geq m \) and \( x^n_\alpha \) is \( q\)-coincident with \( \lambda \).

**Definition 2.3.** \([12]\) A fuzzy net \( \{x^n_\alpha\} \) in an fts \( X \) is said to converge to a fuzzy point \( x_\alpha \) if \( \{x^n_\alpha\} \) is eventually \( q\)-coincident with each q-nbd of \( x_\alpha \).

**Definition 2.4.** \([11]\) Let \( B \) be a nonempty family of fuzzy subsets of \( I^X \). Then \( B \) is called a base for a fuzzy filter on \( X \) (or a fuzzy filter-base) if the following two conditions are satisfied:

1. \( 0_X \notin B \);
2. if \( \lambda_1, \lambda_2 \in B \), then \( \lambda_1 \wedge \lambda_2 \in B \).

If \( B \) has the property

3. \( \lambda \in B \) and \( \lambda \leq \mu \) implies \( \mu \in B \),

then \( B \) is called a fuzzy filter on \( X \).

A maximal, with respect to set inclusion, fuzzy filter on \( X \) is called a fuzzy ultra-filter or a maximal filter (or filter-base). If \( B \) is a base for a fuzzy filter on
X, the collection \( F_{eB} = \{ \mu \in I^X : \exists \lambda \in B \text{ with } \lambda \leq \mu \} \) is the fuzzy filter generated by \( B \). We say that a filter \( F_1 \) is finer than a filter \( F_2 \) if \( F_2 \prec F_1 \), i.e. if for each \( \lambda \in F_2 \), there exists \( \mu \in F_1 \) such that \( \mu \leq \lambda \).

Below are listed some results on fuzzy filter bases (ffb, shortly) which one can prove in a straightforward manner.

**Theorem 2.1.** (1) Let \( F_1, F_2 \) be any ffb on fts \( X \). Then the family \( F_1 \lor F_2 = \{ \lambda_1 \lor \lambda_2 : \lambda_1 \in F_1, \lambda_2 \in F_2 \} \) is an ffb on \( X \).

(2) If \( \lambda_1 \land \lambda_2 \neq 0 \) for each \( \lambda_1 \in F_1 \) and each \( \lambda_2 \in F_2 \), then \( F_1 \land F_2 = \{ \lambda_1 \land \lambda_2 : \lambda_1 \in F_1, \lambda_2 \in F_2 \} \) is an ffb on \( X \).

(3) A nonempty family \( B \subseteq I^X \) is an ffb on \( X \) iff for any finite collection \( \{ \lambda_i \}_i \) from \( B \), \( \bigwedge_{i=1}^n \lambda_i \neq 0 \).

(4) Let \( B \) be an ffb on \( X \) and let \( f : X \to Y \) be a function. Then \( f(B) = \{ f(\lambda) : \lambda \in B \} \) is an ffb on \( Y \). If \( f \) is onto and \( B \) is an ffb on \( Y \), then \( f^{-1}(B) = \{ f^{-1}(\mu) : \mu \in B \} \) is an ffb on \( X \).

**Definition 2.5.** A fuzzy point \( x_\alpha \) is said to be a cluster point of a filter-base \( B \) (i.e. \( F_{eB} \)) if every q-nbd of \( x_\alpha \) is q-coincident with each member of \( B \).

**Proposition 2.1.** A fuzzy point \( x_\alpha \) \((0 < \alpha \leq 1)\) in an fts \( X \) is a cluster point of a filter-base \( B \) iff \( x_\alpha \in \text{Cl}_\lambda \), for each \( \lambda \in B \).

**Proof.** It is straightforward. This proposition is equivalent to the definition 3.2 [6].

**Definition 2.6.** [6] A filter-base \( B \) is said to converge to \( x_\alpha \) (denoted by \( B \prec x_\alpha \)) if every q-nbd of \( x_\alpha \) contains a member of \( B \) and \( x_\alpha \in \text{Cl}_\lambda \), for every \( \lambda \in B \).

**Theorem 2.2.** Let \( F \) be an ffb on fts \( X \) and \( x_\alpha \) a fuzzy point. Then:

(1) \( F \prec x_\alpha \) iff \( N_q(x_\alpha) < F \).

(2) If \( B \) is a base for the q-nbd system of \( x_\alpha \), then \( B \prec x_\alpha \).

(3) If \( x_\alpha \) is a cluster point of an ffb \( F \) on \( X \) and \( U \) is a q-nbd of \( x_\alpha \), then \( G = \{ \lambda \land U : \lambda \in F \} \) is finer than \( F \) and \( G \prec x_\alpha \).

(4) Let \( \lambda \) be a non-empty fuzzy set. If \( F \prec x_\alpha \) and there exists \( \mu \in F \) such that \( \mu \leq \lambda \), then \( x_\alpha \in \text{Cl}_\lambda \).

The proof is straightforward.

**Theorem 2.3.** Let \( x_\alpha \) be a fuzzy point in an fts \( X \). Then

(1) \( x_\alpha \) is a cluster point of an ffb \( F \) iff there exists an ffb \( B \) finer than \( F \) and \( B \prec x_\alpha \).

(2) If \( F \prec x_\alpha \), then \( x_\alpha \) is a cluster point of \( F \).

(3) If \( x_\alpha \) is a cluster point of a fuzzy ultra-filter \( F \), then \( F \prec x_\alpha \).
Proof. (1) $B = N_q(x_a) \vee \{ \lambda \wedge U : \lambda \in F, U \in N_q(x_a) \}$ is an filter finer than $F$ and $B \rightarrow x_a$. Conversely, let $B \rightarrow x_a$ and let $B < F$. Let $\lambda \in F$. Since $B < F$, there exists $\mu \in F$ such that $\mu \leq \lambda$; $x_a \in \text{Cl} \mu$ implies $x_a \in \text{Cl} \lambda$, for each $\lambda \in F$, i.e. $x_a$ is a cluster point of $F$.

(2) It follows from (1).

(3) Since $x_a$ is a cluster point of an ultra-filter $F$, that means that for each $U \in N_q(x_a)$, $U q \lambda$, for each $\lambda \in F$, that implies $U \wedge \lambda \neq 0$ and hence $U \in F$. Therefore $F \rightarrow x_a$.  

**Corollary 2.1.** If $x_a$ is a cluster point of a filter $F_1$ that is finer than $F_2$, then $x_a$ is a cluster point of the filter $F_2$. If $x_a$ is a limit of the filter $F_2$, then $x_a$ is the limit of every filter $F_1$ finer than $F_2$.

**Proposition 2.2.** [11] Let $X, Y$ be fts’s and $x_a$ be a fuzzy point in $X$. If $f$ is a mapping from $X$ to $Y$, continuous at $x_a$, then for every filter-base $B$, $B \rightarrow x_a$ implies $f(B) \rightarrow f(x_a)$.

**Definition 2.7.** We say that a fuzzy point $x_a \in \text{SCl} \lambda$ if for each semi open fuzzy set $U$, $x_a q U$ implies $\text{Cl} U q \lambda$.

**Definition 2.8.** A fuzzy filter (or a filter-base) $F$ is said to:

1. $\delta$-accumulate to $x_a$ if $x_a \in \text{Cl} \lambda$, for each $\lambda \in F$ [6].
2. $s$-accumulate to $x_a$ if $x_a \in \text{SCl} \lambda$, for each $\lambda \in F$.
3. $\theta$-accumulate to $x_a$ if $x_a \in \text{Cl} \lambda$, for each $\lambda \in F$.

**Definition 2.9.** Let $(X, \tau_X)$ be an fts and let $N^q_\lambda(x_a)$ be the q-nbd filter in $X$, i.e. the filter generated by regular open fuzzy sets; $S(x_a)$ be the filter generated by the family $SO(x_a) = \{ \text{Cl} \lambda : x_a q \lambda \in SO(\tau_X) \}$, and $X(x_a)$ be the filter generated by $C(x_a) = \{ \text{Cl} \lambda : x_a q \lambda \in \tau_X \}$. We say that a filter (or a filter-base) $F$

1. $\delta$-converges to $x_a$ if $N^q_\lambda(x_a) < F$;
2. $s$-converges to $x_a$ if $S(x_a) < F$;
3. $\theta$-converges to $x_a$ if $C(x_a) < F$.

**Proposition 2.3.** A filter (or a filter-base) $F$ in an fts $X$

1. $\delta$-converges to $x_a$ iff every fuzzy regular open q-nbd of $x_a$ contains some member of $F$ and $x_a \in \text{Cl} \lambda$ for each $\lambda \in F$;
2. $s$-converges to $x_a$ iff for each fuzzy semi open q-nbd $\mu$ of $x_a$ there is a $\lambda \in F$ such that $\lambda \leq \text{Cl} \mu$ and $x_a \in \text{SCl} \lambda$, for each $\lambda \in F$;
3. $\theta$-converges to $x_a$ iff for every open q-nbd $\mu$ of $x_a$ there is $\lambda \in F$ such that $\lambda \leq \text{Cl} \mu$ and $x_a \in \text{Cl} \lambda$ for each $\lambda \in F$.

**Proof.** Easy.  

**Theorem 2.4.** A filter-base $B$ (or a filter) on $X$ $s$-accumulates ($\theta$-accumulates, resp.) iff there exists a filter $F$ finer than $B$ which $s$-converges ($\theta$-converges, resp.).
3. S-closed and extremally disconnected fuzzy topological spaces

Definition 3.1. [3,5] A family \{\lambda_\alpha : \alpha \in I\} of fuzzy open subsets of a fuzzy topological space \((X, \tau_X)\) is called a cover if \(\bigcup\{\lambda_\alpha : \alpha \in I\} = X\). A fuzzy topological space is called compact (shortly FC) if every open cover has a finite subcover. An fts \(X\) is said to be fuzzy nearly (almost) compact (shortly FNC (FAC)) if every open cover contains a finite subfamily \(\{\lambda_\alpha : i = 1, \ldots, n\}\) such that \(X = \bigcup_{i=1}^{n} \text{Int} \lambda_\alpha_i\) \((X = \bigcup_{i=1}^{n} \text{Cl} \lambda_\alpha_i)\).

Definition 3.2. [15] An fts \(X\) is fuzzy \(\delta\)-compact (denoted F\(\delta\)C) if every open cover contains a finite \(\delta\)-open subcover.

Definition 3.3. [4] An fts \(X\) is S-closed if every fuzzy semi-open cover \(\{\lambda_\alpha : \alpha \in I\}\) contains a finite subfamily \(\{\lambda_\alpha_i\}_{i=1}^n\) such that \(X = \bigcup_{i=1}^{n} \text{Cl} \lambda_\alpha_i\).

Definition 3.4. [9] An fts \(X\) is extremally disconnected (denoted FED) if the closure of every fuzzy open set is open.

Using all these definitions it is easy to see that the following implications hold:

\[ F\delta\text{C} \implies FC \implies FNC \implies FAC. \]

Theorem 3.1. A semi-regular fts \(X\) is F\(\delta\)C iff it is FC.

Proof. The proof follows easily from the fact that \(\tau_S = \tau_X\). \(\blacksquare\)

Using results from [5] and [15] and theorem 3.1, we have:

Theorem 3.2. a) If fts \(X\) is semi-regular, then F\(\delta\)C \(\sim\) FC \(\sim\) FNC;

d) if \(X\) is fuzzy regular, then F\(\delta\)C \(\sim\) FC \(\sim\) FNC \(\sim\) FAC (\(\sim\) means “is equivalent”).

Theorem 3.3. Let an fts \((X, \tau_X)\) be S-closed. Then for every family of fuzzy open sets \(\{\lambda_\alpha : \alpha \in I\}\) of \(X\) with finite intersection property (FIP, shortly) it holds \(\bigcap_\alpha \text{Cl} \lambda_\alpha = 0\).

Proof. Let \(\{\lambda_\alpha\}_{\alpha \in I}\) be a family of fuzzy open sets with FIP. If \(\bigcap_\alpha \text{Cl} \lambda_\alpha = 0\), then \(\bigcup_\alpha (1 - \text{Cl} \lambda_\alpha) = 1_X\) and \(\{1 - \text{Cl} \lambda_\alpha\}_\alpha\) is a semi-open cover of \(X\). Hence, there exists a finite subcollection \(\{\lambda_\alpha_i\}_{i=1}^n\) such that \(\bigcup_{i=1}^{n} (1 - \text{Cl} \lambda_\alpha_i) = 1_X\). Therefore \(\bigcap_{i=1}^{n} \lambda_\alpha_i \subseteq \bigcap_{i=1}^{n} (1 - \text{Cl} (1 - \text{Cl} \lambda_\alpha_i)) = 0\), a contradiction. Hence \(\bigcap_\alpha \text{Cl} \lambda_\alpha \neq 0\). \(\blacksquare\)

Theorem 3.4. If an fts \((X, \tau_X)\) is extremally disconnected and for any family of fuzzy open sets \(\{\lambda_\alpha\}_\alpha\) of \(X\) with FIP it holds \(\bigcap_\alpha \text{Cl} \lambda_\alpha = 0\), then \(X\) is S-closed.
Proof. Let $\{\lambda_\alpha\}_{\alpha \in I}$ be a semi-open cover of $X$ and suppose that $X$ is not $S$-closed. Hence for each finite family $\{\lambda_\alpha\}_{\alpha \in I}, \bigvee_{i=1}^n \mathrm{Cl}\lambda_{\alpha_i} < 1. Therefore \bigwedge_{i=1}^n (1 - \mathrm{Cl}\lambda_{\alpha_i}) \neq 0, i.e. \{1 - \mathrm{Cl}\lambda_{\alpha_i}\}_{\alpha} is a family of fuzzy open sets with FIP. Hence $\bigwedge_{i=1}^n \mathrm{Cl}(1 - \mathrm{Cl}\lambda_{\alpha_i}) \neq 0.$ Since $X$ is FED and since $\mathrm{Cl}\lambda_{\alpha} = \mathrm{Cl} \mathrm{Int} \lambda_{\alpha},$ for each $\lambda_{\alpha} \in SO(X)$ it follows that $\mathrm{Cl}\lambda_{\alpha}$ is fuzzy open for each $\alpha \in I,$ i.e. $1 - \mathrm{Cl}\lambda_{\alpha}$ is fuzzy closed. Thus

$$\bigvee_{\alpha} \lambda_{\alpha} \leq \bigvee_{\alpha} \mathrm{Cl}\lambda_{\alpha} = \bigvee_{\alpha} (1 - (1 - \mathrm{Cl}\lambda_{\alpha})) = \bigvee_{\alpha} (1 - \mathrm{Cl}(1 - \mathrm{Cl}\lambda_{\alpha})) < 1_X,$$

a contradiction (since $\{\lambda_\alpha\}_{\alpha}$ is a cover of $X$). \qed

**Proposition 3.1.** An fts $X$ is $S$-closed iff any cover by regular closed sets has a finite subcover.

**Proof.** It is trivial, since the closure of a semi-open set is regular closed, therefore semi-open. Even more, it is easy to see that $RC(X) = \{\mathrm{Cl}\lambda_{\alpha} : \lambda_{\alpha} \in SO(X)\}.$ \qed

**Corollary 3.1.** $S$-closedness is a fuzzy semiregular property, i.e. $X$ is $S$-closed iff $X_S$ is $S$-closed.

**Proof.** The proof is straightforward since an fts and its semiregularization have the same fuzzy regular closed sets. \qed

**Theorem 3.5.** For an fts $X$ the following are equivalent:

1. $X$ is $S$-closed;
2. each filter-base in $X$ $s$-accumulates;
3. every maximal filter-base on $X$ $s$-converges.

**Proof.** (2) $\iff$ (3). See theorem 2.1.

(1) $\iff$ (3). Let $X$ be $S$-closed and let $F$ be a maximal filter-base on $X$ which doesn't $s$-converge. Therefore, it doesn't $s$-accumulate to any point. This implies that for each point $x_{\alpha}$ there exists a semi-open set $\mu_{\alpha} \in S(x_{\alpha})$ and an element $\lambda_{\alpha} \in F$ such that $x_{\alpha} \not\in \mu_{\alpha}$ and $\mathrm{Cl}\mu_{\alpha} \nsubseteq \lambda_{\alpha}.$ Without loss of generality, we may assume that $C = \{\mu_{\alpha}\}_{x_{\alpha} \in X}$ is a semi-open cover of $X,$ because if $x_{\alpha} \not\in \mu_{\alpha},$ then $x_{\alpha} = x_{\alpha'} \in \mu_{\alpha},$ for each $\alpha \in (0,1).$ Since $X$ is $S$-closed, there exists a finite subcollection $\{\mu_{\alpha_i}\}_{i=1}^n$ such that $X = \bigvee_{i=1}^n \mathrm{Cl}\mu_{\alpha_i}. Since F is a maximal filter-base there exists $\lambda \neq 0, \lambda \in F$ such that $\lambda \leq \bigwedge_{i=1}^n \lambda_{\alpha_i}$ and $\lambda_{\alpha} \not\in \mathrm{Cl}\mu_{\alpha_i},$ for each $i.$ This implies that $\lambda \not\in \mathrm{Cl}\mu_{\alpha_i},$ for each $i = 1, \ldots, n,$ i.e. $\lambda \not\in \bigvee_{i=1}^n \mathrm{Cl}\mu_{\alpha_i} = X,$ a contradiction, since $\lambda \neq 0.$

Conversely, suppose $X$ is not $S$-closed and let $\{\lambda_{\alpha}\}_{\alpha \in I}$ be a semi-open cover of $X$ such that $\bigvee_{i=1}^n \mathrm{Cl}\lambda_{\alpha_i} < X$ for every finite family $\{\lambda_{\alpha_i} : i = 1, \ldots, n\}.$ Then there exists an $x_{t}$ such that $\mathrm{Cl}\lambda_{\alpha_i}(x_{t}) < t,$ for each $1 \leq i \leq n.$ Hence $\mu_{\alpha_i}(x_t) = \{1 - \mathrm{Cl}\lambda_{\alpha_i}(x_t) > 1 - t,$ i.e. $\mu_{\alpha_i} \not\in SO(X).$ Therefore $\bigvee_{i=1}^n \mu_{\alpha_i} \neq 0,$ for each finite intersection. Hence $\{\mu_{\alpha_i}\}$ forms a filter-base $B$ which $s$-accumulates to $x_{t}.$ Then there exists a maximal filter-base $F$ which $s$-converges to $x_{t},$ and this
implies that \( \bigvee_a \text{Cl} \lambda_a \neq X \). Therefore \( \bigvee_a \lambda_a \neq X \), a contradiction, since \( \{\lambda_a\}_a \) is a cover of \( X \). ■

It is known that the product of S-closed spaces is not necessarily S-closed, even in general topology. In fuzzy topology for fuzzy spaces \( X \) and \( Y \), where \( X \) is product related to \( Y \) (cf. [2, Definition 3.7; Theorem 3.10; Theorem 4.6]) we have the following result:

**Theorem 3.6.** Let \((X, \tau_X)\) and \((Y, \tau_Y)\) be S-closed fts's such that \( X \) is product related to \( Y \). Then fuzzy topological product \( X \times Y \) is S-closed.

**Proof.** Let \( \{\lambda_a\}_a \) and \( \{\mu_\beta\}_\beta \) be semi-open covers for \( X \), respectively \( Y \). Then \( \{\lambda_a \times \mu_\beta\}_{a,\beta} \) is fuzzy semi-open cover of \( X \times Y \). For finite \( n \in \mathbb{N} \) we have \( \bigvee_{i=1}^n \text{Cl} \lambda_{a_i} \times \bigvee_{i=1}^n \text{Cl} \mu_{\beta_i} = \bigvee_{i=1}^n \text{Cl}(\lambda_{a_i} \times \mu_{\beta_i}) \), i.e. \( X \times Y = \bigvee_{i=1}^n \text{Cl}(\lambda_{a_i} \times \mu_{\beta_i}) \). Thus \( X \times Y \) is S-closed. ■

**Definition 3.5.** [14] A function \( f: X \to Y \) is said to be fuzzy semi-continuous (resp. irresolute) of \( f^{-1}(\mu) \in SO(X) \), for any fuzzy open (resp. semi-open) set \( \mu \) of \( Y \).

**Theorem 3.7.** [4] If \( f: X \to Y \) is an irresolute surjection from an S-closed fts \( X \) to an fts \( Y \), then \( Y \) is S-closed.

Since fuzzy semi-continuous and almost open (i.e. \( f^{-1}(\text{Cl} \lambda) < \text{Cl} f^{-1}(\lambda) \), for each \( \lambda \in \tau_Y \)) mapping implies that \( f \) is irresolute [6] we have the following corollary:

**Corollary 3.2.** An image of an S-closed space under an almost open fuzzy continuous surjection is S-closed.

**Definition 3.6.** A property of fts is called fuzzy semi-topological if it is preserved by fuzzy semi-homeomorphism (i.e. bijection and such that the images of semi-open sets are semi-open and the inverses of semi-open sets are semi-open).

**Corollary 3.3.** To be an S-closed fts is a fuzzy semi-topological property.

**Theorem 3.8.** For an fts \((X, \tau_X)\) the following are equivalent:

1. \( X \) is FED.
2. If a filter base on \( X \) \( \delta \)-converges, then it \( s \)-converges.
3. A filter-base \( X \) \( s \)-converges iff it \( \theta \)-converges.
4. If a filter-base on \( X \) converges with respect to the topology \( \tau_X \), then it \( s \)-converges.

**Proof.** We shall first prove the following lemma.

**Lemma 3.1.** If an fts \( X \) is FED, then \( \text{SCL} \lambda = \text{Cl}_\delta \lambda \), for each \( \lambda \in SO(X) \).

**Proof of Lemma.** \( \text{SCL} \lambda \subseteq \text{Cl} \lambda \subseteq \text{Cl}_\delta \lambda \subseteq \text{Cl}_\delta \lambda \), for each \( \lambda \in I^X \). We shall prove that \( \text{Cl}_\delta \lambda \leq \text{SCL} \lambda \), for \( \lambda \in SO(X) \). Let \( x_\alpha \in \text{Cl}_\delta \lambda \), then there exists \( U \in SO(X) \) such that \( x_\alpha \in U \) and \( U \notin \lambda \), which implies that \( U \leq \lambda^c \), where
\[ \lambda^c = 1 - \lambda. \] Hence \( \text{Cl} U \subseteq \text{Cl}(\lambda^c). \) Since \( X \) is FED, then \( \text{Cl} U \) is open. Therefore \( \text{Cl} U \subseteq \text{Int} \text{Cl}(\lambda^c) \subseteq \text{SCl}(\lambda^c) = \lambda^c. \) Hence \( \text{Cl} U \not\subseteq \text{Cl}(\lambda). \) \]

**Corollary 3.4.** If \( X \) is FED, then \( \text{SCl} \lambda = \text{Cl} \lambda = \text{Cl}_\theta \lambda = \text{Cl}_\delta \lambda, \) for each \( \lambda \in SO(X). \)

Using the results above, the proof of theorem 3.8 is now straightforward. \( \Box \)

**Remark 3.1.** From the above corollary and Lemma 7 and Theorem 12 in [9] we have the following: \( \text{SCl} \lambda = \text{Cl} \lambda = \text{Cl}_\theta \lambda = \text{Cl}_\delta \lambda, \) for every fuzzy set \( \lambda \in SO(X) \cup PO(X), \) where \( PO(X) \) is the set of all fuzzy pre-open sets in \( X. \) Also it is easy to prove that definition of re-convergence [9] is equivalent to our definition of \( \text{s-} \) convergence.

**Lemma 3.2.** Let \( X \) be fuzzy almost regular and \( S \)-closed. Then \( X \) is FED and fuzzy nearly compact (FNC).

**Proof.** Suppose that \( X \) is not FED. Then there exists a fuzzy regular open set \( \lambda \) such that \( \text{Cl} \lambda(x) > \lambda(x) \) for some \( x \in X \) and \( \text{Cl} \lambda \not\subseteq \{x\}. \) Let \( x_\alpha \in \text{Cl} \lambda \) and \( x_\alpha \not\subseteq \lambda. \) For every open q-nbd \( U \) of \( x_\alpha, \) we have \( U \not\subseteq \lambda \not\subseteq 0. \) Therefore \( F = \{U \wedge \lambda : U \in N_q(x_\alpha)\} \) forms a filter-base in \( \text{Cl} \lambda. \) Since \( \text{Cl} \lambda \) is \( S \)-closed relative to \( X, \) then \( F \rightarrow^S y_\beta, \) i.e. \( F \) \( s \)-converges to some point \( y_\beta \in \text{Cl} \lambda. \) If \( y_\beta \not\subseteq \lambda, \) then \( y_\beta \wedge \lambda = 1 - \lambda \in RC(X), \) hence \( 1 - \lambda \) is semi-open and \( 1 - \lambda \in S(y_\beta). \) Therefore, there exists \( \mu \in F \) such that \( \mu \leq 1 - \lambda, \) i.e. \( \mu \not\subseteq \lambda \). Since \( F \rightarrow y_\beta \) in the usual sense and \( y_\beta \in \text{Cl} u \leq 1 - \lambda, \) therefore every member of \( F \) is \( q \)-coincident with \( 1 - \lambda, \) what is impossible. Hence \( y_\beta \subseteq \lambda. \) Almost regularity and \( \lambda \in RO(X) \) imply that \( \lambda \in S(y_\beta), \) i.e. there exists \( V \in RO(X) \) such that \( y_\beta \wedge q V \subseteq \text{Cl} V \subseteq \lambda. \) Since \( x_\alpha \not\subseteq \lambda, \) then \( x_\alpha \wedge \text{Cl} V^c = 1 - \text{Cl} V \) is an open q-nbd of \( x_\alpha. \) Since \( N_q(x_\alpha) \rightarrow x_\alpha, \) there exists a q-nbd \( U \) of \( x_\alpha, \) such that \( (U \wedge \lambda) \subseteq 1 - \text{Cl} V, \) i.e. \( (U \wedge \lambda) \not\subseteq \text{Cl} V. \) But that contradicts the fact that \( F \rightarrow^S y_\beta. \) Therefore it must be \( x_\alpha = y_\beta, \) i.e. \( \text{Cl} \lambda = \lambda. \) Hence \( X \) is FED.

To prove nearly compactness, let \( \mathcal{U} \) be a maximal filter-base. \( S \)-closedness implies that \( \mathcal{U} \) \( s \)-converges. Theorem 3.8.(3) implies that \( \mathcal{U} \) \( \theta \)-converges. Almost regularity implies that \( \mathcal{U} \) \( \delta \)-converges. Since \( X \) is fuzzy nearly compact (see Theorem 3.9 [13] and Theorem 2.3) iff every maximal filter-base \( \delta \)-converges, then this completes the proof. \( \Box \)

**Corollary 3.5.** Let \( X \) be an almost regular fts. Then \( X \) is \( S \)-closed iff \( X \) is \( \text{FNC} \) and \( \text{FED} \) iff \( X_S \) is regular, compact and \( \text{FED}. \)

**Theorem 3.9.** Let fts \( (X, \tau_X) \) be a fuzzy regular space. Then the following are equivalent:

1. \( X \) is compact and \( \text{FED}; \)
2. \( X \) is \( S \)-closed;
3. \( X \) is \( \text{FAC} \) and \( \text{FED}. \)

**Proof.** (1) \( \implies \) (2). See Theorem 3.4.
(2) \(\implies\) (3). That S-closed implies FAC, follows directly from the definitions. Using lemma 3.2 we have the proof of (2) \(\implies\) (3).

(3) \(\implies\) (1). The proof is straightforward. 

**Corollary 3.6.** (cf. [4, Theorem 3.7; Corollary 3.8]) Let \(X\) be fuzzy regular and extremally disconnected. Then the following is valid:

\[
FC \sim F8C \sim S\text{-closed} \sim FAC \sim FNC.
\]

**REFERENCES**


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