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# A STUDY ON GENERALIZED RICCI 2-RECURRENT SPACES

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**Abstract.** The object of the present paper is to study some properties of generalized Ricci 2-recurrent spaces. At first it is proved that every 3-dimensional generalized Ricci 2-recurrent space is a generalized 2-recurrent space. In section 3, it is shown that for such a space Ricci-principal invariant is $1/2R$. In section 4 we find a necessary condition for such a space to be a Ricci-recurrent space. Next it is proved that a conformally symmetric Ricci 2-recurrent space is a generalized 2-recurrent space and a conformally symmetric generalized Ricci 2-recurrent space with definite metric and zero scalar curvature can not exist. Lastly an example of a generalized Ricci 2-recurrent space is also constructed.

## 1. Preliminaries

A non flat Riemanian space $V_n$ $(n > 3)$ is called a generalized 2-recurrent space [1] if its curvature tensor satisfies

$$R_{hijk,lm} = \lambda_m R_{hijk,l} + a_{lm} R_{hijk} \tag{1.1}$$

where $a_{lm}$ is non-zero and a comma denotes covariant differentiation with respect to the metric tensor $g_{ij}$. $\lambda_m$ and $a_{lm}$ are called its vector and tensor of recurrence. Such a space has been denoted by $G(2k_n)$. In generalizing this concept we intend to study Riemannian space whose Ricci tensor is non-zero and satisfies a relation of the form

$$R_{ij,lm} = \lambda_m R_{ij,l} + a_{lm} R_{ij} \tag{1.2}$$

where $\lambda_m$ and $a_{lm}$ have the same meaning as before. Such a space shall be called a generalized Ricci 2-recurrent space and will be denoted by $G(2R_n)$. If in particular, $\lambda_m = 0$, then the space reduces to a Ricci 2-recurrent space introduced by Chaki and Roychowdhary [2]. In 1952, Patterson [3] introduced a type of Riemannian space $V_n$ $(n \geqslant 3)$ the Ricci tensor of which satisfies $R_{ij,k} = \lambda_k R_{ij}$ and $R_{ij} \neq 0$ for some non-zero vector $\lambda_k$. He called such a space Ricci-recurrent and denoted an $n$-dimensional space of this kind by $R_n$. Now from (1.1) and (1.2) it is easily seen that every $G(2k_n)$ is a $G(2R_n)$, but the converse is not in general true. Here we prove that every $G(2R_3)$ is a $G(2k_3)$. According to Chaki and Gupta [4], an

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$n$-dimensional ($n > 3$) Riemannian space is called conformally symmetric if its Weyl's conformal curvature tensor

$$C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}(g_{ij}R^h_k - g_{ik}R^h_{j,} + \delta^h_k R_{ij} - \delta^h_{j,} R_{ik}) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_{j,} g_{ik}) \tag{1.3}$$

satisfies

$$C^h_{ijk,l} = 0 \tag{1.4}$$

where $R$ is the scalar curvature.

In the present paper we consider $G(2R_n)$ for $n > 3$.

## 2. 3-dimensional generalized Ricci 2-recurrent space

It is known [5] that for a $V_3$

$$R_{hijk} = g_{hk}\pi_{ij} - g_{hj}\pi_{ik} + g_{ij}\pi_{hk} - g_{ik}\pi_{hj} \tag{2.1}$$

where

$$\pi_{ij} = (R_{ij} - \tfrac{R}{4}g_{ij}). \tag{2.2}$$

Now for a $G(2R_3)$ we have

$$R_{ij,lm} = \lambda_m R_{ij,l} + a_{lm}R_{ij}. \tag{2.3}$$

Transvecting (2.3) with $g^{ij}$ we get

$$R_{,lm} = \lambda_m R_{,l} + a_{lm}R. \tag{2.4}$$

From (2.2) we have by virtue of (2.3) and (2.4)

$$\pi_{ij,lm} - \lambda_m\pi_{ij,l} - a_{lm}\pi_{ij} = R_{ij,lm} - \lambda_m R_{ij,l} - a_{lm}R_{ij} - (R_{,lm} - \lambda_m R_{,l} - a_{lm}R)\frac{g_{ij}}{4} = 0$$

or $\pi_{ij,lm} = \lambda_m\pi_{ij,l} + a_{lm}\pi_{ij}$. Therefore from (2.1) it follows that

$$R_{hijk,lm} = \lambda_m R_{hijk,l} + a_{lm}R_{hijk}.$$

Thus we can state the following theorem:

THEOREM 1. *Every $G(2R_3)$ is a $G(2k_3)$.*

## 3. Tensor of recurrence and Ricci principal invariant in a $G(2R_n)$ with non-zero scalar curvature

We see from (2.4) that if $R$ is constant, then $R = 0$ for $a_{lm} \neq 0$. Again from (2.4)

$$\lambda_m R_{,l} - \lambda_l R_{,m} + (a_{lm} - a_{ml})R = R_{,lm} - R_{,ml} = 0.$$

Hence if $a_{lm}$ is symmetric, then $\lambda_m$, $R_{,l}$ are co-directional.

From Bianchi identity we get

$$R^h_{ijk,h} + R_{ik,j} - R_{ij,k} = 0. \tag{3.1}$$

Covariant differentiation of (3.1) gives $R^h_{ijk,hm} + R_{ik,jm} - R_{ij,km} = 0$. Now by virtue of (1.2)

$$R^h_{ijk,hm} = a_{km}R_{ij} - a_{jm}R_{ik} + \lambda_m(R_{ij,k} - R_{ik,j}). \tag{3.2}$$

Trnsvecting (3.2) with $g^{ij}$ and using the formula $R^r_{i,r} = \frac{1}{2}R_{,i}$ we obtain

$$\frac{1}{2}R_{,km} = a_{km}R - a_{jm}R^j_k + \frac{\lambda_m}{2}R_{,k} \tag{3.3}$$

whence

$$a_{jm}R^j_k = \frac{1}{2}a_{km}R. \tag{3.4}$$

Now by the similar argument as in [2] we get the following theorem:

THEOREM 2. *In a $G(2R_n)$ with non-zero scalar curvature the tensor of recurrence $a_{lm}$ is not symmetric in general and its rank is less than $n$. Also $a_{lm}$ is symmetric if and only if $\lambda_m$ and $R_{,l}$ are co-directional. Further, for such a space, one Ricci principal invariant is $\frac{1}{2}R$.*

## 4. $G(2R_n)$ ($R \neq 0$) of definite metric

In this section we consider a $G(2R_n)$ with non-zero scalar curvature for which

$$R^{ij}R_{ij} = \frac{1}{2}R^2 \tag{4.1}$$

holds. Then from (4.1) it follows $2R^{ij}R_{ij,l} = RR_{,l}$. Differentiating both sides of the previous equation covariantly, we get

$$R^{ij}_{,m}R_{ij,l} + R^{ij}R_{ij,lm} = \frac{1}{2}R_{,l}R_{,m} + \frac{1}{2}RR_{,lm}. \tag{4.2}$$

But

$$\begin{aligned} R^{ij}R_{ij,lm} &= R^{ij}(a_{lm}R_{ij} + \lambda_m R_{ij,l}) = a_{lm}R^{ij}R_{ij} + \lambda_m R^{ij}R_{ij,l} \\ &= \frac{1}{2}a_{lm}R^2 + \frac{1}{2}\lambda_m RR_{,l} = \frac{1}{2}R(a_{lm}R + \lambda_m R_{,l}) = \frac{1}{2}RR_{,lm}. \end{aligned}$$

By virtue of this, (4.2) reduces to $R^{ij}_{,m}R_{ij,l} = \frac{1}{2}R_{,l}R_{,m}$. Put $S_{ijk} = R_{ij,k} - \bar{\lambda}_k R_{ij}$, where $\bar{\lambda}_k = R_{,k}/R$. Then

$$\begin{aligned} S^{ijk}S_{ijk} &= g^{mk}R^{ij}_{,m}R_{ij,k} - \bar{\lambda}_m g^{mk}R^{ij}R_{ij,k} - \lambda_k g^{mk}R^{hl}R_{hl,m} + g^{mk}\bar{\lambda}_m\bar{\lambda}_k R^{ij}R_{ij} \\ &= \frac{1}{2}g^{mk}R_{,m}R_{,k} - g^{mk}\bar{\lambda}_m RR_{,k} + \frac{1}{2}\bar{\lambda}_m\bar{\lambda}_k g^{mk}R^2 = 0. \end{aligned} \tag{4.3}$$

If the space is of definite metric, then (4.3) gives $S_{ijk} = 0$, whence $R_{ij,k} = \bar{\lambda}_k R_{ij}$. We can therefore state the following theorem:

THEOREM 3. *Every $G(2R_n)$ of definite metric whose scalar curvature is different from zero and for which $R^{ij}R_{ij} = \frac{1}{2}R^2$, is an $R_n$.*

## 5. Conformally symmetric $G(2R_n)$

It is well known that for a conformally symmetric Riemannian space, it holds

$$R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)}(R_{,k}g_{ij} - R_{,j}g_{ik}). \tag{5.1}$$

Let us suppose that a $G(2R_n)$ with $R \neq 0$, is conformally symmetric. The conformal curvature tensor can be written in the form

$$C_{hijk} = R_{hijk} - D_{hijk} \tag{5.2}$$

where

$$D_{hijk} = \pi_{hk}g_{ij} - \pi_{hj}G_{ik} + \pi_{ij}g_{hk} - \pi_{ik}g_{hj}, \tag{5.3}$$

$$\pi_{ij} = \left(R_{ij} - \frac{R}{2(n-1)}g_{ij}\right). \tag{5.4}$$

Now

$$R_{hijk,l} - D_{hijk,l} = C_{hijk,l} = 0. \tag{5.5}$$

On account of (5.1) and (5.4),

$$\pi_{ij,k} - \pi_{ik,j} = \frac{1}{n-2}\left[R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)}(R_{,j}g_{ik} - R_{,k}g_{ij})\right] = 0.$$

Hence

$$\pi_{ij,k} = \pi_{ik,j}. \tag{5.6}$$

From (5.4) we have as a consequence of (1.2)

$$\pi_{ij,kl} = a_{kl}\pi_{ij} + \lambda_l \pi_{ij,k}. \tag{5.7}$$

Now (5.5) gives

$$R_{hijk,lm} = D_{hijk,lm}. \tag{5.8}$$

On account of (5.8) and (5.3) we have $R_{hijk,lm} = a_{lm}R_{hijk} + \lambda_m R_{hijk,l}$. Also equations (5.6) and (5.7) give $a_{kl}\pi_{ij} = a_{ij}\pi_{ik}$. Multiplying both sides by $g^{kl}$ we get $\theta\pi_{ij} = a_{jl}g^{kl}\pi_{ik}$ where $\theta = g^{kl}a_{kl}$.

Now considering $a_{ij}$ is symmetric we obtain

$$\theta R_{ij} = \frac{R}{2}\frac{n-2}{n-1}a_{ij} + \frac{R}{2}\frac{\theta}{n-1}g_{ij}. \tag{5.9}$$

Multiplying the above equation by $R^{ij}$ we have

$$\theta R_{ij}R^{ij} = \frac{R}{2}\frac{n-2}{n-1}a_{ij}R^{ij} + \frac{R\theta}{2(n-1)}g_{ij}R^{ij}. \tag{5.10}$$

But multiplying (3.5) by $g^{km}$ we obtain $a_{jm}R^{jm} = \frac{1}{2}R\theta$. Hence (5.10) gives $\theta R_{ij}R^{ij} = \dfrac{nR^2\theta}{4(n-1)}$. Since $R \neq 0$, if $\theta = 0$, (5.10) would give $a_{ij} = 0$. Hence $\theta \neq 0$. Therefore $R_{ij}R^{ij} = \dfrac{nR^2}{4(n-1)}$. Thus we get

Theorem 4. *A conformally symmetric $G(2R_n)$ is a $G(2k_n)$ and when the tensor of recurrence is symmetric then the length of Ricci tensor is* $\dfrac{nR^2}{4(n-1)}$.

Now by covariant differentiation of (5.1) it follows

$$R_{ij,kl} - R_{ik,jl} = \frac{1}{2(n-1)}(R_{,kl}g_{ij} - R_{,jl}g_{ik}). \tag{5.11}$$

By virtue of (2.4) and (1.2) the equation (5.11) reduces to the form

$$a_{kl}\left(R_{ij} - \frac{1}{2(n-1)}Rg_{ij}\right) + \lambda_l\left(R_{ij,k} - \frac{1}{2(n-1)}R_{,k}g_{ij}\right) =$$
$$= a_{jl}\left(R_{ik} - \frac{1}{2(n-1)}Rg_{ik}\right) + \lambda_l\left(R_{ik,j} - \frac{1}{2(n-1)}R_{,j}g_{ik}\right).$$

Hence on account of (5.1) we obtain

$$a_{kl}\left(R_{ij} - \frac{1}{2(n-1)}Rg_{ij}\right) = a_{il}\left(R_{ik} - \frac{1}{2(n-1)}Rg_{ik}\right). \tag{5.12}$$

Transvecting (5.12) with $R^j_p$ and using the relation (3.5),

$$a_{kl}\left(R_{ri}R^r_p - \frac{1}{2(n-1)}RR_{ip}\right) = \frac{1}{2}Ra_{pl}\left(R_{ik} - \frac{1}{2(n-1)}Rg_{ik}\right).$$

But it follows from (5.12) that

$$\frac{1}{2}R\left(R_{ik} - \frac{1}{2(n-1)}Rg_{ik}\right)a_{pl} = \frac{1}{2}R\left(R_{ip} - \frac{1}{2(n-1)}Rg_{ip}\right)a_{kl}.$$

Hence

$$\left(R_{ri}R^r_p - \frac{1}{2(n-1)}RR_{ip}\right)a_{kl} = \frac{1}{2}R\left(R_{ip}\frac{1}{2(n-1)}Rg_{ip}\right)a_{kl}.$$

Therefore

$$R_{ri}R^r_p = \frac{n}{2(n-1)}RR_{ip} - \frac{1}{4(n-1)}R^2 g_{ip}. \tag{5.13}$$

Now if $R = 0$, (5.13) reduces to $R_{ri}R^r_p = 0$ or $R^{ri}R_{ri} = 0$ (by contraction with $g^{ip}$). So, for definite metric $R_{ij} = 0$, which is not possible. Hence we obtain the following theorem:

Theorem 5. *A conformally symmetric generalized Ricci 2-recurrent space with definite metric and zero scalar curvature can not exist.*

## 6. Example of a generalized Ricci 2-recurrent space

For this section let the greek index runs over 2, 3, ..., $n-1$ and the latin index runs over 1, 2, ..., $n$. We define the metric $g$ in $R^n$, $n \geqslant 4$ by the formula [6]

$$ds^2 = Q(dx^1)^2 + K_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n \tag{6.1}$$

where $[K_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constants, and $Q$ is independent of $x^n$.

The only components of Christoffel symbols $R_{hijk}$, $R_{ij}$, not identically zero are those related to

$$\left\{ \begin{matrix} \lambda \\ 11 \end{matrix} \right\} = -\frac{1}{2} K^{\alpha\beta} Q.\beta, \quad \left\{ \begin{matrix} n \\ 11 \end{matrix} \right\} = -\frac{1}{2} Q.1, \quad \left\{ \begin{matrix} n \\ 1\alpha \end{matrix} \right\} = -\frac{1}{2} Q.\alpha,$$
$$R_{1\alpha\beta 1} = \frac{1}{2} Q.\alpha\beta, \quad R_{11} = -\frac{1}{2} K^{\alpha\beta} Q.\alpha\beta \tag{6.2}$$

where $[K^{\alpha\beta}] = [K_{\alpha\beta}]^{-1}$.

Let $Q = K_{\alpha\beta} x^\alpha x^\beta e^{2x^1}$ where

$$[K_{\alpha\beta}] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

So $[K^{\alpha\beta}] = [K_{\alpha\beta}]$. Now

$$K_{\alpha\beta} K^{\alpha\beta} = n - 2, \quad Q.\alpha\beta = 2K_{\alpha\beta} e^{2x^1}, \quad Q.\alpha\beta\nu = 0, \quad K^{\alpha\beta} Q.\alpha\beta = 2(n-2) e^{2x^1}. \tag{6.3}$$

Hence from (6.2) and (6.3) the only non zero components of $R_{ij}$, $R_{ij,l}$, $R_{ij,lm}$ are

$$R_{11} = (n-2) e^{2x^1}, \quad R_{11,1} = 2(n-2) e^{2x^1}, \quad R_{11,11} = 4(n-2) e^{2x^1}.$$

So, $R_{11,11} = R_{11,1} + 2R_{11}$. Hence $R_{ij,lm} = a_{lm} R_{ij} + \lambda_m R_{ij,l}$ where $a_{lm} = \begin{cases} 2, & \text{for } l = m = 1, \\ 0, & \text{otherwise,} \end{cases}$ and $\lambda_m = \begin{cases} 1, & \text{for } m = 1, \\ 0, & \text{otherwise.} \end{cases}$ Hence $V_n$ is a generalized Ricci 2-recurrent space.

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