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ON LAPLACE AUTOREGRESSIVE TIME SERIES MODELS

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Abstract. This paper presents some recent time series models (the so-called NAREX(1) models) for Laplace variables with first order autoregressive structures. They are analogs of standard AR(1) model and of the LAR(1), NLAR(1) and AREX(1) models, introduced by Andel, Lawrance, Lewis, Jevremović and others. Some of their models can be obtained from NAREX models as special cases. The distribution of the innovation sequence (a probability mixture) and the autoregressive structure of NAREX processes are discussed as well.

1. Introduction

It is a well known fact that the standard first-order autoregressive model (AR(1) model) for a stationary sequence of random variables $\{X_t, t \in T = \{0, \pm 1, \pm 2, \dots\}\}$ is defined by difference equation $X_t = \beta X_{t-1} + \varepsilon_t$, $t \in T$, where β is a parameter ($\beta \in (0,1)$) and $\{\varepsilon_t, t \in T\}$ is a sequence of independent and identically distributed (i.i.d.) random variables.

In standard time-series analysis one assumes that the marginal distributions of $\{X_t\}$ are normal. However, a Gaussian distribution will not always be appropriate. In earlier works stationary non-Gaussian time-series models were developed for variables with positive and highly skewed marginal distributions.

There still remain some situations where Gaussian marginals are inappropriate, i.e., where the marginal time-series variable being modeled, although not skewed or inherently positive valued, has a large kurtosis or long-tailed distribution. The position of errors in a large navigation system, timing device errors under periodic excitation and speech-waves have such a distribution that needs to be modeled using Laplace variables.

The Laplace distribution is also known as the double exponential. In general, the density of Laplace distributed variables L has two parameters—a location parameter $-\infty < \mu < \infty$ and a scale parameter $\lambda > 0$. In this case the parameter μ is fixed at zero.

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For $-\infty < x < \infty$ we have $\varphi(x,\lambda) = (1/2\lambda)\exp(-|x|/\lambda)$. The characteristic function of Laplace variable is $k_X(t) = (1+\lambda^2t^2)^{-1}$. Some earlier works on Laplace time series models can be summarized as follows.

Andel (1983) considered an AR(1) model, with a Laplace $L(\lambda)$ marginal distribution, in the following form

$$X_t = \begin{cases} \beta X_{t-1}, & \text{w.p. } \beta^2, \\ \beta X_{t-1} + L_t, & \text{w.p. } 1-\beta^2, \end{cases}$$

where β is a parameter $(0 \leqslant \beta \leqslant 1)$ and L_t, $t \geqslant 0$ are i.i.d. Laplace random variables $L(\lambda)$. This is an LAR(1) model.

Lawrance and Lewis (1981) discussed some new AR models using Laplace variables—NLAR(1)

$$X_t = \begin{cases} \beta X_{t-1} + \varepsilon_t, & \text{w.p. } \rho, \\ \varepsilon_t, & \text{w.p. } 1-\rho. \end{cases}$$

Jevremović (1991) considered some new AR models with a Laplace $L(\lambda)$ marginal distribution—AREX(1) model

$$X_t = \begin{cases} \varepsilon_t, & \text{w.p. } p_0, \\ \alpha X_{t-1}, & \text{w.p. } p_1, \\ \beta X_{t-1} + \varepsilon_t, & \text{w.p. } q_1, \end{cases}$$

where $0 \leqslant p_0, p_1, q_1 \leqslant 1$, $p_0 + p_1 + q_1 = 1$ and $0 < \alpha, \beta < 1$.

In this paper we present a new form of time series models where marginal distributions are in fact Laplace distributions. The forms of these models (the so-called NAREX models) are quite distinct from earlier forms of time-series models using Laplace variables.

2. NAREX(1) models

Let the stationary sequence of random variables $\{X_t, t \in T\}$ be defined by the equation

$$X_t = \begin{cases} \alpha X_{t-1} + \varepsilon_t; & p_0, \\ \beta X_{t-1} + \varepsilon_t; & p_1, \\ \gamma X_{t-1}; & p_2, \end{cases} \tag{2.1}$$

where $0 \leqslant p_0, p_1, p_2 \leqslant 1$, $p_0 + p_1 + p_2 = 1$, $0 < \alpha, \beta, \gamma < 1$ and ε_t are some i.i.d. random variables. Let us also suppose that $\{X_t\}$ and $\varepsilon_t\}$ are "semi-independent", i.e. that X_t and ε_m are independent if $t < m$.

Our first purpose is to obtain the distribution of the i.i.d. sequence $\{\varepsilon_t\}$ which will ensure that the sequence $\{X_n\}$ in (2.1) has a Laplace marginal distribution $\varepsilon(\lambda)$.

Let the characteristic function of the X and ε variables be denoted by

$$k_X(t) = E(e^{itX}), \qquad k_\varepsilon(t) = E(e^{it\varepsilon}). \tag{2.2}$$

We know that the following applies

$$k_X(t) = \frac{1}{1+\lambda^2 t^2} \tag{2.3}$$

if X has an $L(\lambda)$ distribution. If we assume stationary and "semi-independence", then (2.1), (2.2) and (2.3) give

$$k_X(t) = p_0 k_X(\alpha t)\cdot k_\varepsilon(t) + p_1 k_X(\beta t)\cdot k_\varepsilon(t) + p_2 k_X(\gamma t),$$

$$k_\varepsilon(t) = \frac{k_X(t) - p_2 k_X(\gamma t)}{p_0 k_X(\alpha t) + p_1 k_X(\beta t)} = \frac{\dfrac{1}{1+\lambda^2 t^2} - p_2 \dfrac{1}{1+\gamma^2\lambda^2 t^2}}{p_0 \dfrac{1}{1+\alpha^2\lambda^2 t^2} + p_1 \dfrac{1}{1+\beta^2\lambda^2 t^2}}$$

$$= \frac{(1+\alpha^2\lambda^2 t^2)(1+\beta^2\lambda^2 t^2)[p_0 + p_1 + (\gamma^2 - p_2)\lambda^2 t^2]}{(1+\lambda^2 t^2)(1+\gamma^2\lambda^2 t^2)[p_0 + p_1 + (p_0\beta^2 + p_1\alpha^2)\lambda^2 t^2]}. \tag{2.4}$$

Let $\alpha, \beta, \gamma, p_0, p_1$ and p_2 be chosen so that in (2.4) there are no common fractions. In this case we shall have

$$k_\varepsilon(t) = A_0 + \frac{A_1 + B_1 t}{1+\lambda^2 t^2} + \frac{A_2 + B_2 t}{1+\gamma^2\lambda^2 t^2} + \frac{A_3 + B_3 t}{p_0 + p_1 + (p_0\beta^2 + p_1\alpha^2)\lambda^2 t^2}.$$

Further calculation gives

$$A_0 = \frac{\alpha^2\beta^2(\gamma^2 - p_2)}{\gamma^2(p_0\beta^2 + p_1\alpha^2)}, \quad A_1 = \frac{(1-\alpha^2)(1-\beta^2)}{p_0 + p_1 - (p_0\beta^2 + p_1\alpha^2)},$$

$$A_2 = \frac{p_2(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)}{\gamma^2[p_0\beta^2 + p_1\alpha^2 - (p_0 + p_1)\gamma^2]},$$

$$A_3 = (p_0 + p_1)\frac{p_0 p_1(\alpha^2 - \beta^2)^2(p_1\alpha^2 + p_0\beta^2 + p_2 - \gamma^2)}{(p_0\beta^2 + p_1\alpha^2)[p_0 + p_1 - (p_0\beta^2 + p_1\alpha^2)][p_0\beta^2 + p_1\alpha^2 - (p_0 + p_1)\gamma^2]},$$

$$B_1 = 0, \quad B_2 = 0, \quad B_3 = 0.$$

Now we have

$$k_\varepsilon(t) = A_0 + \frac{A_1}{1+\lambda^2 t^2} + \frac{A_2}{1+\gamma^2\lambda^2 t^2} + \frac{A_3}{p_0 + p_1 + (p_0\beta^2 + p_1\alpha^2)\lambda^2 t^2}.$$

It is obvious that A_1, A_2, A_3 and $\frac{A_3}{p_0+p_1}$ are probabilities whose sum equals to 1.

1. If $\alpha < \sqrt{p_2 + \alpha p_1 + \beta p_0} < \beta$ (or $\beta < \sqrt{p_2 + \alpha p_1 + \beta p_0} < \alpha$) it follows $\gamma \in (\sqrt{p_2 + \alpha p_1 + \beta p_0}, \beta)$ (or $\gamma \in (\sqrt{p_2 + \alpha p_1 + \beta p_0}, \alpha)$).

2. If $\sqrt{p_2} < \alpha < \sqrt{\dfrac{\alpha p_1 + \beta p_0}{p_0 + p_1}} < \beta$ (or $\sqrt{p_2} < \beta < \sqrt{\dfrac{\alpha p_1 + \beta p_0}{p_0 + p_1}} < \alpha$) it follows $\gamma \in (\sqrt{p_2}, \alpha)$ (or $\gamma \in (\sqrt{p_2}, \beta)$).

Then ε_t is the following mixture

$$R(\varepsilon_t) = \begin{cases} 0; & A_0, \\ L(\lambda); & A_1, \\ L(\gamma\lambda); & A_2, \\ L\left(\lambda\sqrt{\dfrac{p_0\beta^2 + p_1\alpha^2}{p_0 + p_1}}\right); & \dfrac{A_3}{p_0 + p_1}. \end{cases}$$

Taking the expectation of both sides in (2.1) we have

$$E\{X_t\} = p_0[\alpha E\{X_{t-1}\} + E\{\varepsilon_t\}] + p_1[\beta E\{X_{t-1}\} + E\{\varepsilon_t\}] + p_2\gamma E\{X_{t-1}\}$$

or

$$E(\varepsilon_t) = \frac{1 - (\alpha p_0 + \beta p_1 + \gamma p_2)}{p_0 + p_1} \cdot \frac{1}{\lambda}.$$

It follows that the autocovariance function for NAREX(1) time series model (2.1) is defined by

$$\begin{aligned}\gamma(h) &= E(X_t X_{t-h}) - E(X_t)E(X_{t-h}) = (\alpha p_0 + \beta p_1 + \gamma p_2)\gamma(h-1)\\ &= \cdots = (\alpha p_0 + \beta p_1 + \gamma p_2)^h \gamma(0),\end{aligned}$$

so that autocorrelation function of $\{X_t\}$ is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = (\alpha p_0 + \beta p_1 + \gamma p_2)^h, \quad h > 0.$$

3. Some special cases

a) If $p_0 = p_2 = 0$ the LAR(1) model will apply.

b) If $\alpha = p_2 = 0$ the NLAR(1) model will apply.

c) If $\alpha = 0$, $\gamma = \alpha$, $p_2 = p_1$ and $p_1 = q_1$ the AREX(1) model will apply.

d) FNAREX(1)

Let $p_2 = 0$. Then $X_t = \begin{cases} \alpha X_{t-1} + \varepsilon_t, & \text{w.p. } p_0,\\ \beta X_{t-1} + \varepsilon_t, & \text{w.p. } p_1 = 1 - p_0, \end{cases}$ and

$$k_\varepsilon(t) = \frac{\dfrac{1}{1+\lambda^2 t^2}}{\dfrac{p_1}{1+\alpha^2\lambda^2 t^2} + \dfrac{1-p_1}{1+\beta^2\lambda^2 t^2}} = \frac{(1+\alpha^2\lambda^2 t^2)(1+\beta^2\lambda^2 t^2)}{(1+\lambda^2 t^2)[1+(\alpha^2 + p_1\beta^2 - p_1\alpha^2)\lambda^2 t^2]}.$$

We shall have

$$k_\varepsilon(t) = A_0 + \frac{A_1 + B_1 t}{1+\lambda^2 t^2} + \frac{A_2 + B_2 t}{1+(\alpha^2 + p_1\beta^2 - p_1\alpha^2)\lambda^2 t^2},$$

$$R(\varepsilon_t) = \begin{cases} 0; & A_0,\\ L_1(\lambda); & A_1,\\ L_1(\lambda\sqrt{p_1\beta^2 + \alpha^2 - p_1\alpha^2}); & A_2. \end{cases}$$

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