МАТЕМАТІČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
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ON A CLASS OF DEGENERATED NONLOCAL $p(x)$-BIHARMNIC PROBLEM WITH $q(x)$-HARDY POTENTIAL

My Driss Morchid Alaoui

Abstract. This work deals with the study of a class of nonlocal Navier boundary value problems involving the degenerate $p(x)$-biharmonic operator with a potential term $q(x)$-Hardy

$$\begin{cases} \Delta(\omega(|\Delta u|^{p(x)})|\Delta u|^{p(x)-2}\Delta u) - \lambda \frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}} = \mu\vartheta(x)|u|^{q(x)-2}u\left(\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\right)^r & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases}$$

In this new setting, our objectif is to extend the results obtained in the paper [M. Laghzal, A. El Khalil, M. D. Morchid Alaoui, A. Touzani, *Eigencurves of the* $p(\cdot)$*-biharmonic operator with a Hardy-type term potential*, Moroccan J. Pure Appl. Anal., **6(2)** (2020), 198–209] for the nonhomogeneous case $p(x) \neq q(x)$, where ϑ is a weight function. The main results are established by using the variational method and min-max arguments based on Ljusternik-Schnirelmann theory on C^1 manifoleds [A. Szulkin, *Schnirelmann theory on* C^1*-manifolds*, Ann. Inst. Henri Poincaré C, Anal. Non Linéaire, **5(2)** (1988), 119–139]. A direct characterization of the principal curve (first one) is provided.

1. Introduction

In a regular bounded domain $\Omega \subset \mathbb{R}^N, N \geq 3$, with smooth boundary $\partial\Omega$, we consider the nonlinear singular boundary eigenvalue problem

$$\begin{cases} \Delta(\omega(|\Delta u|^{p(x)})|\Delta u|^{p(x)-2}\Delta u) - \lambda \frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}} = \mu\vartheta(x)|u|^{q(x)-2}u\left(\int\limits_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\right)^r & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

$\Delta(|\Delta u|^{p(x)-2}\Delta u$ is the $p(x)$-biharmonic, $\Delta(\omega(|\Delta u|^{p(x)})|\Delta u|^{p(x)-2}\Delta u)$ is a degenerate one, the functions $p(\cdot)$, $q(\cdot)$ are supposed to be continuous on $\overline{\Omega}$, $\delta(x) = \text{dist}(x, \partial\Omega)$

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denotes the distance from x to the boundary $\partial\Omega$, $\vartheta \in L^{m(x)}$ is a nonnegative function with $m \in C_+(\Omega)$, $\lambda, r > 0$ are real parameters and the real μ is a spectral parameter, playing the role of an eigenvalue. The following mapping

$$\theta : \mathbb{R}^N \to \mathbb{R}, \quad \zeta \mapsto \theta(\zeta) = \Lambda(|\zeta|^{p(x)}),$$

is strictly convex, where Λ is the primitive of a real function $\omega : \mathbb{R}^+ \to \mathbb{R}$ that is $\Lambda(t) = \int_0^t \omega(s)\, ds$. The term "nonlocal" refers to the presence of the integral: $\int_\Omega \frac{\vartheta(x)}{q(x)} |u|^{q(x)} dx$, which appears in the equation in (1). This integral represents a nonlocal interaction, since the value of the equation at a point x depends on the average behavior of the solution u over the whole domain Ω. Such terms naturally arise in models of population dynamics, where the growth at a specific location is influenced by the total population; they also appear in various physical and engineering contexts, such as beam vibration problems, image processing, and models involving spatial heterogeneity. These applications typically require sophisticated mathematical tools for analysis due to the nonlocal and singular nature of the problem.

Nonlinear singular eigenvalues boundary problems form a class of important problems in the theory and applications of partial differential equations. The study of this type of problems is motivated by recent advances in mathematical modeling of non-Newtonian fluids and elastic mechanics and nonlinear porous medium, and image processing, in this context we cite as examples the papers [16, 17, 22].

The problem (1) is inspired by recent studies on $p(x)$-biharmonic nonlinear boundary problems related to Hardy-type inequalities. Relevant references include the work [5, 6, 8, 13], where further bibliographic details can be found.

In the particular case where $p(x) \equiv q(x) \equiv p$ El Khalil. et al [7], established the existence of an increasing sequence of positive eigencurves for the following Dirichlet problem:

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda w(x) \frac{|u|^{p-2}u}{\delta(x)^{2p}} + \mu \frac{|u|^{p-2}u}{\delta(x)^{2p}} \text{ in } \Omega, \\ u \in W_0^{2,p}(\Omega), \end{cases}$$

where w is an indefinite weight in $L^\infty(\Omega)$, and $\operatorname{mes}(\{x \in \Omega : W(x) \neq 0\}) \neq 0$.

For the variable exponent case $p(x) = q(x)$ the same authors [12] proved an analogous result for the problem:

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) - \lambda \frac{|u|^{p(x)-2}u}{\delta(x)^{2p(x)}} = \mu |u|^{p(x)-2}u \text{ for } x \in \Omega, \\ u = 0, \quad \text{for } x \in \partial\Omega. \end{cases}$$

In the present work, we study problem (1) in the nonhomogeneous case $p(x) \neq q(x)$, which involves a Hardy-type singular term and a nonlocal term with a weighted integral. Our analysis is conducted under the following set of assumptions:

(H1) $1 < \min_{\overline{\Omega}} q \leq \max_{\overline{\Omega}} q < \min_{\overline{\Omega}} p \leq \max_{\overline{\Omega}} p < \frac{N}{2}$, and $\max_{\overline{\Omega}} q < p_2^*$, where

$$p_2^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)}, & p(x) < \frac{N}{2}, \\ +\infty, & p(x) \geq \frac{N}{2}. \end{cases}$$

(H2) $\forall t \geq 0 : \quad 0 < L \leq \omega(t) \leq K$, where K and L are positive constants.

(H3) $\vartheta \in L^{m(x)}(\Omega)$ such that $m(x) > \frac{N}{2}$, and there is a measurable set $\Omega_0 \subset \Omega$ verifying $\vartheta(x) > 0$ for each $x \in \overline{\Omega}_0$.

(H4) $0 \le \lambda < R$, where R is a positive constant in the Hardy inequality associated with the degenerate $p(x)$-biharmonic operator, which will be stated later in Lemma 3.1.

In this new setting, our objective is to prove the existence of at least one non-decreasing sequence of positive eigencurves $(\mu_k(\lambda))_{k\ge 1}$. We also provide a direct characterization of the principal eigencurve (i.e., the first one) using a variational technique based on the Ljusternik-Schnirelmann theory on C^1-manifolds [18] involving a mini-max argument over sets of genus greater than k.

The structure of the paper is as follows: In Section 2, we introduce some basic properties of the generalized Lebesgue-Sobolev spaces along with several useful lemmas. In Section 3, we present an improved Hardy-type inequality. Finally, in Section 4, we establish the existence of the sequence of eigencurves and provide a characterization of the principal (first) eigencurve.

2. Preliminaries and useful results

We begin by stating some basic properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$. For a comprehensive treatment of these spaces, we refer the reader to the monograph [3] and the references therein. The generalized Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega);= \Big\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}dx < \infty\Big\},$$

endowed with the Luxemburg norm

$$|u|_{p(\cdot)} = \inf\left\{\gamma > 0 : \int_\Omega \left|\frac{u}{\gamma}\right|^{p(x)} dx \le 1\right\}.$$

For each fixed exponent function $p(\cdot)$, the modular functional $\rho_{p(\cdot)}$ defined by

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)}dx.$$

has an important role in manipulating the generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$.

Proposition 2.1 ([20]). *Under hypothesis (H1), the space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate dual space is $L^{p'(\cdot)}(\Omega)$ where $p'(\cdot)$ is the conjugate function of $p(\cdot)$, i.e.,*

$$p'(x) = \frac{p(x)}{p(x)-1} \quad \textit{for all } x \in \Omega.$$

For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ we have

$$\Big|\int_\Omega u(x)v(x)\,dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(\cdot)}|v|_{p'(\cdot)} \le 2|u|_{p(\cdot)}|v|_{p'(\cdot)}.$$

$$\Big|\int_\Omega u(x)v(x)w(x)\,dx\Big| \leq \Big(\frac{1}{p_1^-}+\frac{1}{p_2^-}+\frac{1}{p_3^-}\Big)|u|_{p_1(\cdot)}|v|_{p_2(\cdot)}|w|_{p_3(\cdot)}$$
$$\leq 3|u|_{p_1(\cdot)}|v|_{p_2(\cdot)}|w|_{p_3(\cdot)} \tag{2}$$

Proposition 2.2 ([11, Theorem 1.3]). *Let $u_n,\ u \in L^{p(\cdot)}$, we have*
(a) $|u|_{p(\cdot)} = a \Leftrightarrow \rho_{p(\cdot)}(\frac{u}{a}) = 1$ *for* $u \neq 0$ *and* $a > 0$.
(b) $|u|_{p(\cdot)} < (=;>1) \Leftrightarrow \rho_{p(\cdot)}(u) < (=;>1)$.
(c) $|u_n| \to 0(resp \to +\infty) \Leftrightarrow \rho_{p(\cdot)}(u_n) \to 0, (resp \to +\infty)$.
(d) the following statements are equivalent to one another:
  (i) $\lim_{n\to+\infty}|u_n - u|_{p(\cdot)} = 0$,
  (ii) $\lim_{n\to+\infty}\rho_{p(\cdot)}(u_n - u) = 0$,
  (iii) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to+\infty}J(u_n) = \rho_{p(\cdot)}(u)$.

The Sobolev space with variable exponent $W^{m,p(\cdot)}(\Omega)$ is defined as
$$W^{m,p(\cdot)}(\Omega) = \Big\{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega),\ \ |\alpha| \leq m\Big\},$$
where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}}u$ (the derivative is in the sense of distributions) with a multi-index $\alpha = (\alpha_1,\ldots,\alpha_N)$, $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{m,p(\cdot)}(\Omega)$ equipped with the norm $\|u\|_{m,p(x)} = \sum_{|\alpha|\leq m}|D^\alpha u|_{p(x)}$ is a Banach, separable and reflexive space. For more details, we refer the reader to [9, 11, 15, 19]. We denote by $W_0^{m,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$. Note that weak solutions of problem (1) are considered in the generalized Sobolev space $X = W^{2,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)$, endowed with the norm $\|u\|_X = \inf\Big\{\gamma > 0 : \int_\Omega\Big|\frac{\Delta u}{\gamma}\Big|^{p(x)}dx \leq 1\Big\}$.

Remark 2.3. For all $u \in X$ we have
$$\|u\|_X = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}, \quad \forall u \in X,$$
where $\quad \|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad$ and $\quad \|u\|_{2,p(x)} = \sum_{|\alpha|=2}|D^\alpha u|_{p(x)}$.

Then according to [20] the norm $\|.\|_{2,p(\cdot)}$ is equivalent to the norm $|\Delta.|_{p(\cdot)}$ in the space X for the Lipschitz boundary and the exponent $p(\cdot)$ in the class that keeps the maximal function operator bounded (i.e., $\frac{1}{p(\cdot)}$ is globally log-Hölder continuous). Consequently, the norms $\|.\|_{2,p(\cdot)}$, $\|u\|_X$ and $|\Delta.|_{p(\cdot)}$ are equivalent.

Note that X equipped with the norm $\|\cdot\|_X$, is also a separable, reflexive, and Banach space.

We also recall the following proposition which will be required later.

Proposition 2.4 ([4]). *Suppose that p et q are measurable functions verifying $p \in L^\infty(\Omega)$ and for all $x \in \Omega$ and $1 < p(x)q(x) \leq \infty$. Then we have for all $u \in L^{q(x)}(\Omega), u \neq 0$*
$$|u|_{p(x)} \leq 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \leq \Big||u|^{p(x)}\Big|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)},$$

$$|u|_{p(x)} \geq 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}.$$

THEOREM 2.5 ([1]). *Assume that $p, q \in C_+(\Omega)$. If $q(x) < p_2^*(x)$, then there exists a compact and continuous embedding $X \hookrightarrow L^{q(x)}(\Omega)$.*

Let $m'(x)$ be the conjugate exponent of $m(x)$ and consider $\eta(x) := \frac{m(x)q(x)}{m(x)-q(x)}$. By (H1) one get for all $x \in \overline{\Omega}$, $\eta(x) < p_2^*(x)$ and $m'(x)q(x) < p_2^*(x)$. Hence, according to Theorem 2.5, the embeddings $X \hookrightarrow L^{m'(x)q(x)}(\Omega)$ and $X \hookrightarrow L^{\eta(x)}(\Omega)$ are compact and continuous.

According to Proposition 2.2, the following assertions holds.

PROPOSITION 2.6 ([10]). *Let u_n, $u \in L^{p(\cdot)}$, and $J(u) = \int_\Omega |\Delta u|^{p(x)}dx$, we have*

(i) $\|u\|_X < (=; > 1) \Leftrightarrow J(u) < (=; > 1)$.

(ii) $\begin{cases} \|u\|_X \leq 1 \Rightarrow \|u\|_X^{p^+} \leq J(u) \leq \|u\|_X^{p^-}, \\ \|u\|_X \geq 1 \Rightarrow \|u\|_X^{p^-} \leq J(u) \leq \|u\|_X^{p^+}. \end{cases}$

(iii) $\|u_n\|_X \to 0$ *(resp.* $\to +\infty) \Leftrightarrow J(u_n) \to 0$, *(resp.* $\to +\infty)$.

3. Improved $(p(x), q(x))$-Hardy inequality

First recall that for the classical case $p(\cdot) = q(\cdot) = p$ constant, Davis and Hinz [2] proved that for any $p \in (1, \frac{N}{2})$,

$$\int_\Omega |\Delta u|^p dx \geq \left(\frac{N(p-1)(N-2p)}{p^2}\right)^p \int_\Omega \frac{|u|^p}{\delta(x)^{2p}}dx,$$

whenever $u \in C_c^\infty(\Omega)$. Also, this inequality was proved in [14], for all $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $1 < p < \frac{N}{2}$. For variable exponent case , this inequality was obtained in [5, Lemma 3.1] under the hypothesis (H1), as follows

$$\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx \geq C \int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}dx. \tag{3}$$

where $C = \min\left(\frac{q^-}{p^+}C_{q^-}; \frac{q^-}{p^+}C_{p^+}\right)$, with $C_{q^-} = \left(\frac{N(q^- -1)(N-2q^-)}{(q^-)^2}\right)^{q^-}$ and $C_{p^+} = \left(\frac{N(p^+ -1)(N-2p^+)}{(p^+)^2}\right)^{p^+}$.

Now, we present a new Hardy inequality related to degenerate $p(x)$-biharmonic operator.

LEMMA 3.1. *Assume that (H1) and (H2) holds. Then there exists a positive constant R, such that the $(p(x), q(x))$-Hardy inequality*

$$\int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}dx \leq \frac{1}{R}\int_\Omega \frac{1}{p(x)}\Lambda(|\Delta u|^{p(x)})dx + C^{q^+}\|u\|_X^{q^+},$$

holds for all $u \in X$.

Proof. Define the sets

$$U = \Omega_1 \cap \{x \in \Omega : |u(x)| \leq \delta(x)^2\}, \qquad E = \Omega_2 \cap \{x \in \Omega : |u(x)| \geq \delta(x)^2\},$$
$$V = \Omega_1 \cap \{x \in \Omega : |u(x)| > \delta(x)^2\}, \qquad F = \Omega_2 \cap \{x \in \Omega : |u(x)| < \delta(x)^2\},$$

where $\Omega_1 = \left\{ x \in \Omega : \Lambda(|\Delta u|^{p(x)}) \geq 1 \right\}$ and $\Omega_2 = \left\{ x \in \Omega : \Lambda(|\Delta u|^{p(x)}) < 1 \right\}$. Then $\Omega = U \cup V \cup E \cup F$.

Case 1: $x \in \Omega_1$. By (H2) Λ is bounded, then

$$\int_{\Omega_1} \frac{p^+}{p(x)} \Lambda(|\Delta u|^{p(x)}) dx \geq L \int_{\Omega_1} |\Delta u|^{q^- dx}. \tag{4}$$

In view of (3), there is a positive constant $C_{q^-} = C(N, q^-)$ such that

$$\int_{\Omega_1} |\Delta u|^{q^-} dx \geq C_{q^-} \int_{\Omega_1} \frac{|u|^{q^-}}{\delta(x)^{2q^-}} dx, \tag{5}$$

- If $|u(x)| \leq \delta(x)^2$, we have $\left(\frac{|u|}{\delta(x)^2}\right)^{q(x)} \leq \left(\frac{|u|}{\delta(x)^2}\right)^{q^-}$. Therefore,

$$C_{q^-} \int_{\Omega_1 \cap \{x \in \Omega : |u(x)| \leq \delta(x)^2\}} \left(\frac{|u(x)|}{\delta(x)^2}\right)^{q(x)} dx \leq C_{q^-} \int_{\Omega_1 \cap \{x \in \Omega : |u(x)| \leq \delta(x)^2\}} \left(\frac{|u(x)|}{\delta(x)^2}\right)^{q^-} dx.$$

Then, by (5) we obtain

$$\int_U |\Delta u|^{q^-} dx \geq q^- C_{q^-} \int_U \frac{1}{q^-} \frac{|u|^{q(x)}}{\delta(x)^{2q(x)}} dx.$$

Hence, in view of (4), we conclude that

$$\int_U \frac{p^+}{p(x)} \Lambda(|\Delta u|^{p(x)}) dx \geq q^- L C_{q^-} \int_U \frac{1}{q(x)} \frac{|u|^{q(x)}}{\delta(x)^{2q(x)}} dx.$$

Then

$$\int_U \frac{1}{p(x)} \Lambda(|\Delta u|^{p(x)}) dx \geq R_{q^-} \int_U \frac{1}{q(x)} \frac{|u|^{q(x)}}{\delta(x)^{2q(x)}} dx,$$

where $R_{q^-} = \frac{q^-}{p^+} L C_{q^-}$.

- If $|u(x)| > \delta(x)^2$. We have

$$\int_{\Omega_1} \frac{p^+}{p(x)} \Lambda(|\Delta u|^{p(x)}) dx \geq L \int_{\Omega_1} |\Delta u|^{q^+} dx. \tag{6}$$

By (3), there is a positive constant $C_{q^+} = C(N, q^+)$ such that

$$\int_{\Omega_1} |\Delta u|^{q^+} dx \geq C_{q^+} \int_{\Omega_1} \frac{|u|^{q^+}}{\delta(x)^{2q^+}} dx, \tag{7}$$

Since
$$\left(\frac{|u(x)|}{\delta(x)^2}\right)^{q(x)} \leq \left(\frac{|u(x)|}{\delta(x)^2}\right)^{q^+}.$$

Therefore

$$C_{q^+}\int_{\Omega_1\cap\{x\in\Omega:|u(x)|>\delta(x)^2\}}\frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}dx \le C_{q^+}\int_{\Omega_1\cap\{x\in\Omega:|u(x)|>\delta(x)^2\}}\left(\frac{|u(x)|}{\delta(x)^2}\right)^{q^+}dx.$$

Then, by (7), we obtain

$$\int_V |\Delta u(x)|^{q^+}dx \ge q^- C_{q^+}\int_V \frac{1}{q^-}\frac{|u(x)|^{q^-}}{\delta(x)^{2q(x)}}dxdx.$$

Hence by (6), we conclude that

$$\int_V \frac{p^+}{p(x)}\Lambda(|\Delta u|^{p(x)})dx \ge q^- LC_{q^+}\int_V \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}dx,$$

Then

$$\int_V \frac{p^+}{p(x)}\Lambda(|\Delta u|^{p(x)})dx \ge R_{q^+}\int_V \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}dx,$$

where $R_{q^+} = \frac{q^-}{p^+}LC_{q^+}$.

Case 2: $x \in \Omega_2$. We have

$$\int_{\Omega_2} \frac{p^+}{p(x)}\Lambda(|\Delta u|^{p(x)})dx \ge L\int_{\Omega_2} |\Delta u|^{p^+}dx. \tag{8}$$

By (3), there is a positive constant $C_{p+} = C(N, p^+)$ such that

$$\int_{\Omega_2} |\Delta u|^{p^+}dx \ge C_{p^+}\int_{\Omega_2} \frac{|u|^{p^+}}{\delta(x)^{2p^+}}dx, \tag{9}$$

- If $|u(x)| \ge \delta(x)^2$, we have

$$\left(\frac{|u|}{\delta(x)^2}\right)^{q(x)} \le \left(\frac{|u|}{\delta(x)^2}\right)^{p^+}.$$

Therefore

$$\int_{\Omega_2\cap\{x\in\Omega:|u(x)|\ge\delta(x)^2\}}\left(\frac{|u(x)|}{\delta(x)^2}\right)^{q(x)}dx \le \int_{\Omega_2\cap\{x\in\Omega:|u(x)|\ge\delta(x)^2\}}\left(\frac{|u(x)|}{\delta(x)^2}\right)^{p^+}dx.$$

Then, by (9), we obtain

$$\int_E |\Delta u|^{p^+}dx \ge q^- C_{q^-}\int_E \frac{1}{q^-}\left(\frac{|u|}{\delta(x)^2}\right)^{p^+}dx.$$

Hence, in view of (8), we conclude that

$$\int_E \frac{p^+}{p(x)}\Lambda(|\Delta u|^{p(x)})dx \ge q^- LC_{q^-}\int_E \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}dx,$$

Then

$$\int_E \frac{p^+}{p(x)}\Lambda(|\Delta u|^{p(x)})dx \geq R_{p^+}\int_E \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}dx,$$

where $R_{p^+} = \frac{q^-}{p^+}LC_{p^+}$.

- If $|u(x)| < \delta(x)^2$. Since $q^- < p_2^*$, then by Theorem 2.5 the space X is continuously embedded in $L^{q^+}(\Omega)$, that is, there exists a constant C such that such that

$$\int_\Omega |u(x)|^{q^-}\,dx \leq C^{q^-}\|u\|_X^{q^-},\ \forall u \in X. \tag{10}$$

According to the fact that $|u(x)| < \delta(x)^2$, and for $\delta(x)^2$ large enough, such that $\delta(x)^2 > 1$, we obtain for every $u \in X$

$$\left(\frac{|u(x)|}{\delta(x)^2}\right)^{q(x)} \leq \left(\frac{|u(x)|}{\delta(x)^2}\right)^{q^-} \leq |u(x)|^{q^-},\ \forall x \in \overline{\Omega_2 \cap \{x \in \Omega : |u(x)| < \delta(x)^2\}}.$$

Hence

$$\int_F \frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}dx \leq \int_F |u(x)|^{q^-}\,dx \tag{11}$$

Combining (10) and (11), we obtain

$$\int_\Omega \frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}dx \leq C^{q^-}\|u\|_X^{q^-},\ \forall u \in X.$$

Therefore, by taking $R = \max\{R_{q^-}, R_{q^+}, R_{p^+}\}$, we have the following Hardy type inequality

$$\int_\Omega \frac{1}{q(x)}\frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}dx \leq \frac{1}{R}\int_\Omega \frac{1}{p(x)}\Lambda(|\Delta u|^{p(x)})dx + C^{q^-}\|u\|_X^{q^-},\ \forall u \in X. \qquad \square$$

Definition 3.2. We say that a function $u \in X$ is a weak solution of (P_λ), if for all $v \in X$,

$$\int_\Omega \omega(|\Delta u|^{p(x)})|\Delta u|^{p(x)-2}\Delta u\Delta v\,dx - \lambda\int_\Omega \frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}}v\,dx$$
$$= \mu\int_\Omega |u|^{q(x)-2}uv\,dx\left(\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}\,dx\right)^r. \tag{12}$$

If u is not identically zero, we say that u is an eigenfunction of the degenerate $p(x)$-biharmonic operator associated to the pair of eigenvalues (λ, μ).

Definition 3.3. We define the principal eigencurve of the degenerate $p(x)$-Biharmonic operator, the graph of the function $\lambda \to \mu_1(\lambda) \in \mathbb{R}$, defined by

$$\mu_1(\lambda) = \inf\left\{\frac{\int_\Omega \frac{1}{p(x)}\Lambda(|\Delta u|^{p(x)})\,dx - \lambda\int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}}{\frac{1}{r+1}\left[\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\right]^{r+1}};\ u \in X\setminus\{0\}\right\}.$$

Consider the following functionals defined on X:

$$\Phi_\lambda(u) = \int_\Omega \frac{1}{p(x)}\Lambda(|\Delta u|^{p(x)})dx - \lambda \int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}dx = \Phi - \lambda\varphi(u)$$

$$\Psi(u) = \frac{1}{r+1}\left[\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\right]^{r+1}.$$

Clearly, that Φ, φ and Ψ are even and of class C^1 on X. Moreover Φ', φ' and Ψ : $X \to X^*$ are defined by

$$\langle \Phi'(u), v\rangle := \int_\Omega \omega(|\Delta u|^{p(x)})|\Delta u|^{p(x)\text{-}2}\Delta u \Delta v\, dx,$$

$$\langle \varphi'(u), v\rangle := \int_\Omega m(x)\frac{|u|^{p(x)-2}}{\delta(x)^{2p(x)}}uv\, dx,$$

$$\langle \Psi'(u), v\rangle := \left(\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\right)^r \int_\Omega \vartheta(x)|u|^{q(x)\text{-}2}uv\, dx,$$

We set, $\mathcal{H} = \left\{u \in X;\ \Psi(u) = 1\right\}$.

Lemma 3.4. *$\mathcal{H}$ is a closed C^1-manifold on X.*

Proof. Since $\mathcal{H} = \Psi_\lambda^{-1}\{1\}$, it follows that $\mathcal{M}$ is closed. Therefore, we only need to prove that Ψ' is onto for all $u \in \mathcal{H}$.

Let $u, v \in \mathcal{H}$, for any $t > 0$, the mapping

$$h(t) = \Big\langle \Psi(tu), v\Big\rangle = \frac{1}{r+1}\left[\int_\Omega t^{q(x)}\frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\right]^{r+1},$$

is differentiable and we have

$$\begin{aligned} h'(t) = \Big\langle \Psi'(tu), u\Big\rangle &= \left(\int_\Omega t^{q(x)}\frac{\vartheta(x)}{q(x)}|u|^{q(x)}\, dx\right)^r \int_\Omega t^{q(x)-1}\vartheta(x)|u|^{q(x)}dx \\ &\geq \min\left\{t^{q^-(r+1)-1}, t^{q^+(r+1)-1}\right\}\left[\int_\Omega \vartheta(x)|u|^{q(x)}\, dx\right]^{r+1} q^- \\ &\geq \min\left\{t^{q^-(r+1)-1}, t^{q^+(r+1)-1}\right\}(r+1)q^- > 0. \end{aligned}$$

Hence, h is strictly increasing, further $h(0) = 0$ and $\lim_{t\to\infty} h(t) = \infty$ since for t large one has $h(t) \geq t^{q^-}(r+1)$. Thus for every $\zeta \in \mathbb{R}$, there exists a unique $t_0 > 0$ such that $h(t_0) = \zeta$, so that $\Psi(t_0 u) = \zeta$ and we also have

$$\Big\langle \Psi'(t_0 u), \frac{t_0}{(r+1)q(x)}u\Big\rangle = \left(\int_\Omega \frac{\vartheta(x)}{q(x)}|t_0 u|^{q(x)}dx\right)^r \int_\Omega \frac{1}{r+1}\frac{\vartheta(x)}{q(x)}|t_0 u|^{q(x)}dx = \Psi(t_0 u) = \zeta.$$

This means that $\Psi'(u)$ is onto for all $u \in \mathcal{H}$. Therefore, Ψ is a submersion. Hence, $\mathcal{H}$ is a C^1-manifold . □

Remark 3.5. We write Φ'_λ as $\Phi'_\lambda(\cdot) = \Phi'(\cdot) - \lambda\varphi'((\cdot))$. Then, problem (1) can be equivalently written as

$$\Phi_\lambda(u) = \mu\Psi(u), \quad u \in \mathcal{H}, \tag{13}$$

and the paire (μ, u) solves (13) if and only if u is a critical point of Φ_λ with respect to $\mathcal{H}$.

The operator $\Theta := \Phi' : X \to X^*$ defined as

$$\langle \Theta(u), v\rangle = \int_\Omega \omega(|\Delta u|^{p(x)})|\Delta u|^{p(x)\text{-}2}\Delta u \Delta v\, dx \quad \text{for any } u, v \in X,$$

satisfies the assertions of the following lemma, which is the key to establish our main results in the next section.

LEMMA 3.6. *The following statements hold:*
(i) Θ *is continuous, bounded and strictly monotone.*

(ii) Θ *is of* $(S+)$ *type.*

(iii) Θ *is a homeomorphism.*

Proof. (i) Since Θ is the Fréchet derivative of Φ, it follows that Θ is continuous and bounded so that we deduce that for all $u, v \in X$ such that $u \neq v$, $\langle \Theta(u)-\Theta(v), u-v\rangle > 0$. This means that Θ is strictly monotone.

(ii) First we recall the following well-known inequalities, which hold for any three real ξ_1, ξ_2 and p

$$\Big(\xi_1|\xi_1|^{p-2} - \xi_2|\xi_2|^{p-2}\Big)(\xi_1 - \xi_2) \geq c(p) \begin{cases} |\xi_1 - \xi_2|^p & \text{if } p \geq 2, \\ \frac{|\xi_1-\xi_2|^2}{(|\xi_1|+|\xi_2|)^{2-p}} & \text{if } 1 < p < 2, \end{cases} \tag{14}$$

where $c(p) = 2^{2-p}$ when $p \geq 2$ and $c(p) = p-1$ when $1 < p < 2$. Let $(u_n)_n$ be a sequence of X such that $u_n \rightharpoonup u$ weakly in X and $\limsup_{n\to+\infty}\langle \Theta(u_n), u_n - u\rangle \leq 0$. From (14) we have $\langle \Theta(u_n) - \Theta(u), u_n - u\rangle \geq 0$, and since $u_n \rightharpoonup u$ weakly in X, it follows that $\limsup_{n\to+\infty}\langle \Theta(u_n) - \Theta(u), u_n - u\rangle = 0$. On the other hand, we have

$$\begin{aligned}\langle \Theta(u_n) - \Theta(u), u_n - u\rangle &= \int_\Omega \omega(|\Delta u_n|^{p(x)})|\Delta u_n|^{p(x)\text{-}2}\Delta u_n\Big(\Delta u_n - \Delta u\Big)dx \\ &\quad - \int_\Omega \omega(|\Delta u|^{p(x)})|\Delta u|^{p(x)\text{-}2}\Delta u\Big(\Delta u_n - \Delta u\Big)dx,\end{aligned}$$

and by hypothesese (H2), we obtain

$$\begin{aligned}\langle \Theta(u_n)-\Theta(u), u_n-u\rangle &\geq L\int_\Omega |\Delta u_n|^{p(x)\text{-}2}\Big(\Delta u_n-\Delta u\Big)dx - K\int_\Omega |\Delta u|^{p(x)\text{-}2}\Big(\Delta u_n-\Delta u\Big)dx \\ &\geq \max(L, K)\int_\Omega\Big(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u\Big)\Big(\Delta u_n-\Delta u\Big)dx \\ &\geq K\int_\Omega\Big(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u\Big)\Big(\Delta u_n-\Delta u\Big)dx.\end{aligned}$$

Thus again from (14) we have

$$\int_{\{x\in\Omega:p(x)\geq 2\}} |\Delta u_n - \Delta u|^{p(x)}dx \leq 2^{(p^- -2)}\int_\Omega F(u_n, u)dx,$$

$$\int_{\{x\in\Omega:1<p(x)<2\}}|\Delta u_n-\Delta u|^{p(x)}dx \le (p^+-1)\int_\Omega (F(u_n,u))^{\frac{p(x)}{2}}(G(u_n,u))^{(2-p(x))\frac{p(x)}{2}}dx,$$

where

$$\begin{cases} F(u_n,u)=(|\Delta u_n|^{p(x)-2}\Delta u_n-|\Delta u|^{p(x)-2}\Delta u)(\Delta u_n-\Delta u),\\ G(u_n,u)=(|\Delta u_n|+|\Delta u|)^{2-p(x)}.\end{cases}$$

Since $\int_\Omega F(u_n,u)\,dx=\langle\Theta(u_n)-\Theta(u),u_n-u\rangle$, we can consider $0\le\int_\Omega F(u_n,u)\,dx<1$. We then distinguish two cases.

First, if $\int_\Omega G(u_n,u)\,dx=0$, then $F(u_n,u)=0$, since $F(u_n,u)\ge 0$ a.e. in Ω.

Second, if $0<\int_\Omega F(u_n,u)\,dx<1$, then

$$t^{p(x)}:=\left(\int_{\{x\in\Omega:1<p(x)<2\}}F(u_n,u)\,dx\right)^{-1},$$

is positive and by applying Young's inequality we deduce that

$$\begin{aligned}&\int_{\{x\in\Omega:1<p(x)<2\}}\left[t(F(u_n,u))^{\frac{p(x)}{2}}\right](G(u_n,u))^{(2-p(x))\frac{p(x)}{2}}\,dx\\ &\quad\le\int_{\{x\in\Omega:1<p(x)<2\}}\left(F(u_n,u)(t)^{\frac{2}{p(x)}}+(G(u_n,u))^{p(x)}\right)dx.\end{aligned}$$

Now, by the fact that $\frac{2}{p(x)}<2$, we have

$$\begin{aligned}&\int_{\{x\in\Omega:1<p(x)<2\}}\left(F(u_n,u)(t)^{\frac{2}{p(x)}}+(G(u_n,u))^{p(x)}\right)dx\\ &\le\int_{\{x\in\Omega:1<p(x)<2\}}\left(F(u_n,u)t^2+(G(u_n,u))^{p(x)}\right)dx\\ &\le 1+\int_{\{x\in\Omega:1<p(x)<2\}}(G(u_n,u))^{p(x)}dx.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{\{x\in\Omega:1<p(x)<2\}}|\Delta u_n-\Delta u|^{p(x)}dx\\ &\le\left(\int_{\{x\in\Omega:1<p(x)<2\}}F(u_n,u)dx\right)^{\frac12}\left(1+\int_\Omega(G(u_n,u))^{p(x)}dx\right).\end{aligned}$$

Since $\int_\Omega(G(u_n,u))^{p(x)}\,dx$ is bounded, we have $\int_{\{x\in\Omega:1<p(x)<2\}}|\Delta u_n-\Delta u|^{p(x)}dx\to 0$ as $n\to\infty$.

(iii) Note that the strict monotonicity of Θ implies that Θ is into an operator.

Moreover, Θ is a coercive operator. Indeed, from ((ii)) and since $p^--1>0$, for each $u\in X$ such that $\|u\|_X\ge 1$, we have

$$\frac{\langle\Theta(u),u\rangle}{\|u\|_X}\ge L\frac{J(u)}{\|u\|_X}\ge L\|u\|_X^{p^--1}\to\infty\quad\text{as}\quad\|u\|_X\to\infty.$$

Finally, thanks to the Minty–Browder Theorem [21], the operator T is surjective and admits an inverse mapping.

To complete the proof of (iii), it suffices then to show the continuity of Θ^{-1}. Indeed, let $(g_n)_n$ be a sequence of X^* such that $g_n \to g$ in X^*. Let u_n and u in X such that $\Theta^{-1}(g_n) = u_n$ and $\Theta^{-1}(g) = u$. By the coercivity of Θ, we deduce that the sequence $(u_n)_n$ is bounded in the reflexive space X. For a subsequence, if necessary, we have $u_n \rightharpoonup \widehat{u}$ in X for a some $\widehat{u}$. Then

$$\lim_{n\to+\infty} \langle \Theta(u_n) - \Theta(u), u_n - \widehat{u}\rangle = \lim_{n\to+\infty} \langle g_n - g, u_n - \widehat{u}\rangle = 0.$$

It follows by the assertion (ii) and the continuity of Θ that $u_n \to \widehat{u}$ in X and $\Theta(u_n) \to \Theta(\widehat{u}) = \Theta(u)$ in X^*. Further, since Θ is an into operator, we conclude that $u \equiv \widehat{u}$. □

LEMMA 3.7 ([6, Lemma 3.6-(i)]). *The functional φ' sequentially weakly-strongly continuous, namely, $u_n \rightharpoonup u$ in $X \Rightarrow \varphi'(u_n) \to \varphi'(u)$ in X^*.*

LEMMA 3.8. *For any $\lambda \in \mathbb{R}$, we have*
(a) Ψ' is sequentially weakly-strongly continuous.

(b) Φ_λ is bounded froum below on $\mathcal{H}$.

Proof. (a) Let $u_n \rightharpoonup u$ (weakly) in X we prove that $\Psi'(u_n) \to \Psi'(u)$ in X^*. By Hölder's inequality (2), we have for any $v \in X$,

$$\begin{aligned}
&\Big|\langle \Psi'(u_n) - \langle \Psi'(u_n), v\rangle\Big| \\
&= \Bigg|\bigg(\int_\Omega \frac{\vartheta(x)}{q(x)}|u_n|^{q(x)}dx\bigg)^r \int_\Omega \vartheta(x)|u_n|^{q(x)\text{-}2}u_n v\,dx \\
&\quad - \bigg(\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\bigg)^r \int_\Omega \vartheta(x)|u|^{q(x)\text{-}2}uv\,dx\Bigg| \\
&\le \Bigg|\bigg(\int_\Omega \frac{\vartheta(x)}{q(x)}|u_n|^{q(x)}dx\bigg)^r \int_\Omega \vartheta(x)\Big(|u_n|^{q(x)\text{-}2}u_n - |u|^{q(x)\text{-}2}u\Big)v\,dx\Bigg| \\
&\quad + \Bigg|\bigg[\bigg(\int_\Omega \frac{\vartheta(x)}{q(x)}|u_n|^{q(x)}dx\bigg)^r - \bigg(\int_\Omega \frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\bigg)^r\bigg] \int_\Omega \vartheta(x)|u|^{q(x)\text{-}2}uv\,dx\Bigg|, \\
&\le \frac{1}{(q^-)^r}\Bigg[|\vartheta|_{m(x)}\Big||u_n|^{q(x)}\Big|_{m'(x)} \times 3|\vartheta|_{m(x)}\Big|\big(|u_n|^{q(x)\text{-}2}u_n - |u|^{q(x)\text{-}2}u\big)\Big|_{\frac{q(x)}{q(x\text{-}1}}|v|_{\eta(x)} \\
&\quad + \bigg|\bigg(\int_\Omega \vartheta(x)|u_n|^{q(x)}dx\bigg)^r - \bigg(\int_\Omega \vartheta(x)|u|^{q(x)}dx\bigg)^r\bigg| \times 3|\vartheta|_{m(x)}\Big||u|^{q(x)\text{-}1}\Big|_{\frac{q(x)}{q(x\text{-}1}}|v|_{\eta(x)}\Bigg], \\
&\le \frac{3}{(q^-)^r}\Bigg[|\vartheta|^2_{m(x)}|u_n|^{q^i}_{m'(x)q(x)}\Big|\big(|u_n|^{q(x)\text{-}2}u_n - |u|^{q(x)\text{-}2}u\big)\Big|_{\frac{q(x)}{q(x\text{-}1}}|v|_{\eta(x)} \\
&\quad + \bigg|\bigg(\int_\Omega \vartheta(x)|u_n|^{q(x)}dx\bigg)^r - \bigg(\int_\Omega \vartheta(x)|u|^{q(x)}dx\bigg)^r\bigg| \times |\vartheta|_{m(x)}|u|^{q^j\text{-}1}_{q(x)}|v|_{\eta(x)}\Bigg],
\end{aligned}$$

where $i = +$ if $|u_n|_{m'(x)q(x)} > 1, i = -$ if $|u_n|_{m'(x)q(x)} < 1$ and $j = +$ if $|u|_{q(x)} > 1$, $j = -$ if $|u|_{q(x)} < 1$.

Using the fact that X is continuously embedded in $L^{m'(x)\beta(x)}(\Omega)$, $L^{q(x)}(\Omega)$ and $L^{\eta(x)}(\Omega)$ respectivly, then there is $c_1, c_2, c_3 > 0$ satisfying

$$|u_n|_{m'(x)q(x)} \leq c_1 \|u_n\|_X, \quad |u|_{q(x)} \leq c_2 \|u\|_X \quad \text{and} \quad |v|_{\eta(x)} \leq c_3 \|v\|_X.$$

Thus

$$\Big|\langle \Psi'(u_n) - \langle \Psi'(u_n), v\rangle\Big| \leq \frac{3c_3}{(q^-)^r}\Bigg[c_1^{q^i}|\vartheta|^2_{m(x)}\|u_n\|_X^{q^i}\Big|\big(|u_n|^{q(x)\text{-}2}u_n - |u|^{q(x)\text{-}2}u\big)\Big|_{\frac{q(x)}{q(x\text{-}1}}\|v\|_X$$

$$+\left|\Big(\int_\Omega \vartheta(x)|u_n|^{q(x)}dx\Big)^r - \Big(\int_\Omega \vartheta(x)|u|^{q(x)}dx\Big)^r\right| \times c_2^{q^j\text{-}1}|\vartheta|_{m(x)}\|u\|_X^{q^j\text{-}1}\|v\|_X\Bigg].$$

Since the embedding $X \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact, u_n converges strongly to u in $L^{q(\cdot)}(\Omega)$. Consequently, there exists a positive function $g \in L^{q(\cdot)}(\Omega)$ such that $| u |\leq g$ a.e. in Ω. Since $g \in L^{q(\cdot)-1}(\Omega)$, it follows from the Dominated Convergence Theorem that $|u_n|^{q(x)\text{-}2}u_n \to |u|^{q(x)\text{-}2}u$ in $L^{\frac{q(\cdot)}{q(\cdot)-1}}(\Omega)$. On the other hand, $\vartheta(x)|u_n|^{q(x)} \to \vartheta(x)|u|^{q(x)}$ in $L^1(\Omega)$. Then,

$$\Big(\int_\Omega \vartheta(x)|u_n|^{q(x)}\,dx\Big)^r \to \Big(\int_\Omega \vartheta(x)|u|^{q(x)}dx\Big)^r.$$

Consequently $\Psi'(u_n) \to \Psi'(u)$ in X^*. This achieves the proof of (a).

(b) Let $u \in \mathcal{H}$. We have

$$\Phi_\lambda(u) = \int_\Omega \frac{1}{p(x)}\Lambda(|\Delta u|^{p(x)})\,dx - \lambda\int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\,dx,$$

then by using the hypothesis (H2) and $(p(x), q(x))$-Hardy inequality in Lemma 3.1 and

$$\begin{aligned}\Phi_\lambda(u) &\geq \Big(1 - \frac{\lambda}{R}\Big)\int_\Omega \frac{1}{p(x)}\Lambda(|\Delta u|^{p(x)})\,dx - C^{q^-}\|u\|_X^{q^-}\\ &\geq \frac{L}{p^+}\Big(1 - \frac{\lambda}{R}\Big)J(u) - C^{q^-}\|u\|_X^{q^-},\end{aligned} \tag{15}$$

then for all $||u||_X$ large enough. it follows from Remak 2.6 that

$$\Phi_\lambda \geq \frac{L}{p^+}\Big(1 - \frac{\lambda}{R}\Big)||u||_X^{p^-} - C^{q^-}\|u\|_X^{q^-}. \tag{16}$$

By hypothesis (H4) and since $p^- > q^- > 1$, Φ_λ is coercive and hence bounded below. □

LEMMA 3.9. *The functional Φ_λ satisfies the Palais–Smale condition on $\mathcal{H}$, i.e., for $\{u_n\} \subset \mathcal{H}$, if $\{\Phi_\lambda(u_n)\}_n$ is bounded and*

$$X_n = \Phi'_\lambda(u_n) - Y_n\Psi'_{q(\cdot)}(u_n) \to 0 \quad as \quad n \to +\infty, \tag{17}$$

where $Y_n = \langle \Phi'_\lambda(u_n), u_n\rangle / \langle \Psi'_{q(\cdot)}(u_n), u_n\rangle$, implying that $\{u_n\}_{n\geq 1}$ has a convergent subsequence in X.

Proof. From (16) we have that $\{\Phi_\lambda(u_n)\}_n$ being coercive, then $\{u_n\}_n$ is bounded, hence $u_n \rightharpoonup u \in X$ (weakly) and due the fact that the embedding X in $L^{q(x)}(\Omega)$ is compact, $u_n \to u$ (strongly) . On the other hand, we deduce from the inequality (15), that $J(u_n)$ is bounded in $\mathbb{R}$. Thus, without loss of generality, we can assume that $J(\Delta u_n) \to \ell$. For the remainder, we distinguish two cases:

- If $\ell = 0$, then u_n converges strongly to 0 in X.
- If $\ell \neq 0$, let us prove that $\limsup_{n\to\infty} \big\langle \Theta(u_n), u_n - u\big\rangle \leq 0$.

By hypothesis (H3), we have $\big\langle \Theta(u_n), u_n - u\big\rangle \leq K\rho_{p(\cdot)}(\Delta u_n) - \big\langle \Theta(u_n), u\big\rangle$. Applying X_n of (17) to u, we deduce that

$$Z_n = \big\langle \Theta(u_n), u\big\rangle - \lambda\big\langle \varphi'(u_n), u\big\rangle - Y_n\big\langle \Psi'_{q(\cdot)}(u_n), u\big\rangle \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\big\langle \Theta(u_n), u_n - u\big\rangle \leq K\rho_{p(\cdot)}(\Delta u_n) - \lambda\big\langle \varphi'(u_n), u\big\rangle - Z_n - \frac{\big\langle \Phi'_\lambda(u_n), u_n\big\rangle}{\big\langle \Psi'_{p(\cdot)}(u_n), u_n\big\rangle}\big\langle \Psi'_{p(\cdot)}(u_n), u\big\rangle.$$

That is,

$$\begin{aligned}\big\langle \Theta(u_n), u_n - u\big\rangle \leq{}& \frac{K\rho_{p(\cdot)}(\Delta u_n)}{\big\langle \Psi'_{p(\cdot)}(u_n), u_n\big\rangle}\Big(\big\langle \Psi'_{p(\cdot)}(u_n), u_n\big\rangle - \big\langle \Psi'_{p(\cdot)}(u_n), u\big\rangle\Big)\\ &- Z_n - \lambda\big\langle \varphi'(u_n), u_n\big\rangle + \lambda\frac{\big\langle \varphi(u_n), u_n\big\rangle}{\big\langle \Psi'_{p(\cdot)}(u_n), u_n\big\rangle} \cdot \big\langle \Psi'_{p(\cdot)}(u_n), u_n\big\rangle.\end{aligned}$$

On the other hand, from Lemma 3.7, and Lemma 3.8 (a), φ' and $\Psi_{q(\cdot)}$ are completely continuous. Thus $\varphi'(u_n) \to \varphi'(u)$, $\langle \varphi'(u_n), u_n\rangle \to \langle \varphi'(u), u\rangle$ and $\langle \varphi'(u_n), u\rangle \to \langle \varphi'(u), u\rangle$, $\Psi'_{q(\cdot)}(u_n) \to \Psi'_{q(\cdot)}(u)$, and $\langle \Psi'_{q(\cdot)}(u_n), u_n\rangle \to \langle \Psi'_{q(\cdot)}(u), u\rangle$. Then

$$\begin{aligned}&|\langle \Psi'_{q(\cdot)}(u_n), u_n\rangle - \langle \Psi'_{q(\cdot)}(u_n), u\rangle|\\ &\leq |\Psi'_{q(\cdot)}(u_n), u_n\rangle - \langle \Psi'_{q(\cdot)}(u), u\rangle| + |\langle \Psi'_{q(\cdot)}(u_n), u\rangle - \langle \Psi'_{q(\cdot)}(u), u\rangle|.\end{aligned}$$

It follows that

$$\begin{aligned}&|\langle \Psi'(u_n), u_n\rangle - \langle \Psi'_{q(\cdot)}(u_n), u\rangle|\\ &\leq |\langle \Psi'_{q(\cdot)}(u_n), u_n\rangle - \langle \Psi'_{q(\cdot)}(u), u\rangle| + \|\Psi'_{q(\cdot)}(u_n) - \Psi'_{q(\cdot)}(u)\|_*\|u\|.\end{aligned}$$

This implies that $\langle \Psi'_{q(\cdot)}(u_n), u_n\rangle - \langle \Psi'_{q(\cdot)}(u_n), u\rangle \to 0$ as $n \to \infty$. Combining with the above equalities, we obtain

$$\limsup_{n\to+\infty} \big\langle \Theta(u_n), u_n - u\big\rangle \leq \frac{K\ell}{\langle \Psi'_{q(\cdot)}(u), u\rangle} \limsup_{n\to\infty}(\langle \Psi_{q(\cdot)}(u_n), u_n\rangle - \langle \Psi'_{q(\cdot)}(u_n), u\rangle).$$

We deduce $\limsup_{n\to\infty} \big\langle \Theta(u_n), u_n - u\big\rangle \leq 0$. Lemma 3.6 yields the strong convergence $u_n \to u$ in X. □

4. Existence of eigencurves sequences propres

Set $\Gamma_j = \{A \subset \mathcal{H} : A \text{ is symmetric, compact and } \gamma(A) \geq j\}$, where $\gamma(A) = j$ is the Krasnoselskii genus of the set A, i.e., the smallest integer j, such that there exists an odd continuous map from A to $\mathbb{R}^j \setminus \{0\}$.

We recall some useful properties of the Krasnoselskii genus proved by Szulkin [18].

Lemma 4.1. *Let X be a real Banach space and A, B be symmetric subsets of $X \setminus \{0\}$ which are closed in X. Then*
(a) If there exists an odd continuous mapping $f : A \to B$, then $\gamma(A) \leq \gamma(B)$.

(b) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.

(c) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

(d) If $\gamma(B) < +\infty$, then $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.

(e) If A is compact, then $\gamma(A) < +\infty$ and there exists a neighborhood N of A, N is a symmetric subset of $X \setminus \{0\}$, closed in X such that $\gamma(N) = \gamma(A)$.

(f) If N is a symmetric and bounded neighborhood of the origin in $\mathbb{R}^k$ and if A is homeomorphic to the boundary of N by an odd homeomorphism, then $\gamma(A) = k$.

(g) If X_0 is a subspace of X of codimension k and if $\gamma(A) > k$ then $A \cap X_0 \neq \phi$.

Let us now state the first main result of this paper using the Ljusternick–Schnirelmann theory.

Theorem 4.2. *For any integer $j \in \mathbb{N}^*$, $\mu_j(\lambda) = \inf_{A \in \Gamma_j} \max_{u \in A} \Phi_\lambda(u)$ is a critical value of Φ_λ restricted on $\mathcal{M}$. More precisely, there exists $u_j \in K$ such that $\mu_j(\lambda) = \Phi_\lambda(u_j) = \sup_{u \in A} \Phi_\lambda(u)$, and $u_j(\lambda)$ is an eigenfunctin of* (12) *associated to the positive eigenvalue $(\lambda, \mu_j(\lambda))$. Moreover, $\mu_j(\lambda) \to \infty$, as $j \to \infty$.*

Proof. **Setup 1.** We prove that for any $j \in \mathbb{N}^*$, $\Gamma_j \neq \emptyset$.

Since the Sobolev space X is separable . Therefore there exists sequence of functions v_1, v_2, …, v_j lineary dence in $W_0^{2,p(\cdot)}(\Omega)$ such that

$$\begin{cases} \operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \emptyset & \text{if } i \neq j, \\ \operatorname{meas}(\operatorname{supp}(v_i)) > 0 & \text{for } i \in \{1,\, 2,\, \ldots,\, j\}. \end{cases}$$

Let X_j be the vector subspace of $C_0^\infty(\Omega)$ spanned by $\{v_1, v_2, \ldots, v_j\}$. Then, $\dim F_j = j$ and note that $X_j \subset L^{p(\cdot)}(\Omega)$ because $X_j \subset X \subset L^{p(\cdot)}(\Omega)$. Since X_j is a finite dimensional space the norm $\|\cdot\|_X$ and $|.|_{p(\cdot)}$ are equivalent on F_j. Consequentially, the map

$$v \mapsto |v|_{p(.)} := \inf \left\{ \gamma > 0 : \int_\Omega \left| \frac{v(x)}{\gamma} \right|^{p(x)} dx \right\},$$

defines a norm on X_j. Putting $S := \{w \in X_j : |v|_{p(.)} = 1\}$ the unit sphere of X_j. Let us introduce the functional $h : \mathbb{R}^+ \times X_j \longrightarrow \mathbb{R}$, $(\tau, v) \mapsto h(\tau,, v) = \Psi_{q(\cdot)}(\tau v)$. Remarking that

- $h(0, v) = 0$.
- $h(\tau, u)$ is non decreasing with respect to s. Moreover, for $\tau > 1$ we have $h(\tau, v) \geq \tau^{q^-} \Psi_{q(\cdot)}(v)$, and thus $\lim_{\tau \to +\infty} h(\tau, v) = +\infty$. Therefore, for every fixed $v \in S$, there exists a unique value $\tau = \tau(v) > 0$ such that $h(\tau(v), v) = 1$.

On the other hand, since

$$\frac{\partial h}{\partial \tau}(\tau(v),v)=\left(\int_\Omega \tau\vartheta(x)|u|^{q(x)}dx\right)^r\int_\Omega (\tau(v))^{q(x)-1}\vartheta(x)|u|^{q(x)}dx$$
$$\geq \frac{q^-}{\tau(v)}h(s(v),v)=\frac{q^-}{\tau(v)}>0.$$

The implicit function theorem implies that the map $v\mapsto\tau(v)$ is continuous and even by uniqueness.

Now, we define the following continuous and odd maping from S to compact $A_j:=\mathcal{H}\cap F_j$ by $g: S\longrightarrow A_j,\ v\mapsto h(v)=\tau(v).v$, it follows by the property (f) of Lemma 4.1, that $\gamma(A_j)\geq j$. Then $A_j\in\Gamma_j$.

Setup 2. We claim that $\lambda_j\to\infty$ as $j\to\infty$. Since $W_0^{2,p(\cdot)}(\Omega)$ is separable, there exists $(e_k,e_n^*)_{k,n}$ a bi-orthogonal system such that

- $(e_k)_k$ are linearly dense in X.
- $(e_n^*)_n$ are total for the dual X^*.

For $k\in\mathbb{N}^*$, set $F_k=\operatorname{span}\{e_1,\dots,e_k\}$ and $F_k^\perp=\operatorname{span}\{e_{k+1},e_{k+2},\dots\}$. By (g) of Lemma 4.1, we have for any $A\in\Gamma_k$, $A\cap F_{k-1}^\perp\neq\emptyset$. Thus $t_k=\inf_{A\in\Gamma_k}\sup_{u\in A\cap F_{k-1}^\perp}\Phi(u)\to\infty$ as $k\to\infty$. Indeed, if not, for large k there exists $u_k\in F_{k-1}^\perp$ with $\Psi_{q(\cdot)}(u)=1$ such that $t_k\leq\Phi_\lambda(u_k)\leq M$ for some $M>0$ independent of k. Thus by inequality (16), we have

$$\frac{L}{p^+}\left(1-\frac{\lambda}{R}\right)||u_k||_X^{p^-}-C^{q^-}\|u_k\|_X^{q^-}\leq M.$$

This implies that $(u_k)_k$ is bounded in X. For a subsequence of $\{u_k\}$ if necessary, we can assume that $\{u_k\}$ converges weakly in X and strongly in $L^{p(\cdot)}(\Omega)$. By our choice of $F_{k-1}^\perp$, we have $u_k\rightharpoonup 0$ weakly in X because $\langle e_n^*,e_k\rangle=0$, for any $k>n$. this contradicts the fact that $\Psi_{q(\cdot)}(u)=\int_\Omega\frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\,dx=1$ for all k. Since $\lambda_k\geq t_k$.

Setup 3. From the auxiliary results proved in **Setup 1.** and **Setup 2.** and by applying the Ljusternik-Schnireleman theory the proof of Theorem 4.2 is achieved. □

Corollary 4.3. *we have the following statements:*

(i) $\mu_1(\lambda)=\inf\left\{\dfrac{\int_\Omega\frac{1}{p(x)}\Lambda(|\Delta u|^{p(x)})\,dx-\lambda\int_\Omega\frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\,dx}{\frac{1}{r+1}\left[\int_\Omega\frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\right]^{r+1}dx};\ u\in X\setminus\{0\}\right\}$;

(ii) $0<\mu_1(\lambda)\leq\mu_2(\lambda)\leq\dots\leq\mu_n(\lambda)\to+\infty$.

Proof. (i) For $u\in\mathcal{M}$, set $A_1=\{u,-u\}$. It is clear that $\gamma(A_1)=1$, Φ_λ is even and $\Phi_\lambda(u)=\max_{A_1}\Phi_\lambda\geq\inf_{A\in\Gamma_1}\max_{u\in A}\Phi_\lambda(u)$. Thus $\inf_{u\in\mathcal{H}}\Phi_\lambda(u)\geq\inf_{A\in\Gamma_1}\max_{u\in A}\Phi_\lambda(u)=\mu_1(\lambda)$. On the other hand, for all $A\in\Gamma_1$ and $u\in A$, we have $\sup_{u\in A}\Phi_\lambda\geq\Phi_\lambda(u)\geq\inf_{u\in\mathcal{H}}\Phi_\lambda(u)$. It follows that $\inf_{A\in\Gamma_1}\max_A\Phi_\lambda=\mu_1(\lambda)\geq\inf_{u\in\mathcal{H}}\Phi_\lambda(u)$. Then

$$\mu_1(\lambda)=\inf\left\{\frac{\int_\Omega\frac{1}{p(x)}\Lambda(|\Delta u|^{p(x)})\,dx-\lambda\int_\Omega\frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\,dx}{\frac{1}{r+1}\left[\int_\Omega\frac{\vartheta(x)}{q(x)}|u|^{q(x)}dx\right]^{r+1}dx};\ u\in X\setminus\{0\}\right\};$$

For all $i \geq j$, we have $\Gamma_i \subset \Gamma_j$ and in view of the definition of $\lambda_i, i \in \mathbb{N}^*$, we get $\mu_i(\lambda) \geq \mu_j(\lambda)$. As regards $\mu_n(\lambda) \to \infty$, it has been proved in Theorem 4.2. □

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AMNEA Group, Department of Mathematics, Faculty of Sciences and Technologies (FST), Moulay Ismail University of Meknes, BP 509, Boutalamine, 52000 Errachidia, Morocco

E-mail: morchid_driss@yahoo.fr

ORCID iD: https://orcid.org/0000-0002-9953-5982