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ASYMPTOTIC ANALYSIS OF POSITIVE SOLUTIONS FOR A POLYHARMONIC PROBLEM OUTSIDE THE UNIT BALL

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Abstract. This paper deals with the existence and asymptotic analysis of positive continuous solutions for the following nonlinear polyharmonic boundary value problem:

$$\begin{cases} (-\Delta)^m u = b(x)u^p, & \text{in } D, \\ \lim\limits_{|x|\to 1} \left(|x|^2-1\right)^{1-m} u(x) = 0, \\ \lim\limits_{|x|\to \infty} \left(|x|^2-1\right)^{1-m} u(x) = 0. \end{cases}$$

Here m is an integer greater than 2, $p \in (-1,1)$, D is the complementary of the closed unit ball of $\mathbb{R}^n$ with $n > 2m$, and the function b is nonnegative and continuous on D, satisfying some appropriate assumptions related to Karamata regular variation theory.

1. Introduction

Let m be an integer greater than 2, $D := \{x \in \mathbb{R}^n; |x| > 1\}$ be the complementary of the closed unit ball of $\mathbb{R}^n$ with $n > 2m$. In this paper, we deal with the following higher order elliptic equation

$$(-\Delta)^m u = b(x)u^p, \quad \text{in } D, \tag{1}$$

subject to the boundary conditions

$$\lim_{|x|\to 1} \left(|x|^2-1\right)^{1-m} u(x) = \lim_{|x|\to \infty} \left(|x|^2-1\right)^{1-m} u(x) = 0. \tag{2}$$

Where $p \in (-1,1)$ and the nonlinearity b satisfies a suitable condition relying to Karamata regular variation theory.

The topic of higher order differential equations, known as polyharmonic equations, has recently received considerable attention [3, 12, 16, 18]. Such equations appear naturally in physics and engineering. Indeed, many phenomena from the theory of

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plates, the theory of elasticity, and the creeping flow of a viscous fluid can be modeled by polyharmonic differential equations [2, 8, 17]. The investigation of higher order differential equations is also important in many areas of mathematics, such as free boundary problems [1], conformal geometry involving Q-curvature and the Paneitz operator [6,7]. Besides, one more motivation to address polyharmonic equations is to see whether the results available in the elliptic case ($m = 1$) can also be proved for any $m \geqslant 2$.

For $m = 1$, the non-existence, existence and asymptotic analysis of positive solutions for equation (1) in both bounded and unbounded domains of $\mathbb{R}^n (n \geqslant 2)$, subject to various boundary conditions, have been extensively studied; see, for instance, [4, 14, 15].

In [15], the authors have considered the elliptic counterpart of problem (1)-(2), which is given by

$$\begin{cases} -\Delta u = b(x)u^p, \text{ in } D, \\ \lim\limits_{|x|\to 1} u(x) = \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases} \tag{3}$$

By using Karamata regular variation theory and the sub-supresolutions method, the authors in [15] have shown that for $p < 1$ and b satisfying a suitable assumption related to slowly varying functions, problem (3) admits a unique classical positive solution with a specific asymptotic behavior.

In this paper our objective is to expand the result established in [15, Theorem 1.5], to address problem (1)-(2). To achieve this and present our principal result, we must introduce the Karamata classes $\mathcal{K}_0$ and $\mathcal{K}_\infty$, which play a crucial role in our analysis of asymptotic behavior.

Definition 1.1. (i) The class $\mathcal{K}_0$ is the set of all functions L_0 defined on $(0, a]$ for some $a > 0$, by

$$L_0(t) := c \exp\left(\int_t^a \frac{k(s)}{s} ds \right),$$

where $c > 0$ and k is a continuous function on $[0, a]$, with $k(0) = 0$.

(ii) The class $\mathcal{K}_\infty$ is the set of all functions L_∞ defined on $[1, \infty)$ by

$$L_\infty(t) := c \exp\left(\int_1^t \frac{k(s)}{s} ds \right),$$

where $c > 0$ and k is a continuous function on $[1, \infty)$ such that $\lim_{t\to\infty} k(t) = 0$.

Remark 1.2. (i) A function $L_0 \in \mathcal{K}_0$ if and only if L_0 is a positive function in $C^1((0, a])$, for some $a > 0$ satisfying $\lim\limits_{t\to 0^+} \frac{tL_0'(t)}{L_0(t)} = 0$.

(ii) A function L_∞ is in $\mathcal{K}_\infty$ if and only if L_∞ is a positive function in $C^1([1, \infty))$ satisfying $\lim\limits_{t\to\infty} \frac{tL_\infty'(t)}{L_\infty(t)} = 0$.

(iii) Let M be a function defined on $[1, \infty)$. The map $t \mapsto M(t)$ belongs to $\mathcal{K}_\infty$ if and only if the map $t \mapsto M(\frac{1}{t})$, defined on $(0, 1]$, belongs to $\mathcal{K}_0$.

We emphasize that the functions in the class $\mathcal{K}_0$ (resp. $\mathcal{K}_\infty$) are slowly varying near 0 (resp. ∞). The concept of slowly varying functions was originally introduced by Jovan Karamata [13] and has proven to be extremely useful in the analysis of qualitative properties and the asymptotic behavior of positive solutions for differential equations and problems, due to the pioneering works of Cîrstea and Rădulescu [10, 11]. Subsequently, these early studies have been followed by several investigations to explore the asymptotic analysis of positive solutions for a wide variety of boundary value problems in both bounded and unbounded domains. See, for example, the papers [4, 9, 12] and the references cited therein.

To simplify our statements, we introduce some notations. For $x \in D$, we set $\rho(x) = 1 - \frac{1}{|x|}$. We denote by $\partial D := \{x \in \mathbb{R}^n; |x| = 1\}$ the boundary of D and by $\overline{D} := D \cup \partial D$ the closure of D. Moreover, for two nonnegative functions f and g defined on a set S, we will use the notation $f \approx g$ to indicate the existence of a positive constant $c > 0$ such that for all $x \in S$, $c^{-1}g(x) \leq f(x) \leq c\,g(x)$. Also, we define $\mathbf{L}^1_{Loc}(D)$ as the set of real measurable functions that are locally integrable within D, $C(\overline{D} \cup \infty)$ as the collection of continuous functions on $\overline{D}$ that have a finite limit at ∂D and ∞. The collection $C_0(D)$ is the subset of $C(\overline{D} \cup \infty)$ consisting of functions vanishing continuously at ∂D and ∞. Throughout the paper we adopt the following assumption:

(**H**) b is a nonnegative continuous function on D, satisfying for $x \in D$,

$$b(x) \approx \big(\rho(x)\big)^{-\mu}|x|^{-\lambda}L_0(\rho(x))L_\infty(|x|),$$

where $\mu \leqslant m(1+p)+1-p$, $\lambda \geqslant 2m$, $L_0 \in \mathcal{K}_0$ defined in $(0, a]$, $a>1$, $L_\infty \in \mathcal{K}_\infty$ such that:

$$\int_0^a t^{m(1+p)-p-\mu}L_0(t)\mathrm{d}t < \infty \quad \text{and} \quad \int_1^\infty t^{2m-1-\lambda}L_\infty(t)\mathrm{d}t < \infty.$$

To provide an example of a function b satisfying (H), we can take the function defined on D by:

$$b(x) = \big(\rho(x)\big)^{-\mu}|x|^{-\lambda}\exp\left(\sqrt{\ln(\frac{2}{\rho(x)})}\right)\ln^{-\nu}(2|x|),$$

where $\mu < m(1+p)+1-p$, $\lambda \geqslant 2m$ and $\nu > 1$.

Now we are ready to present our main result.

Theorem 1.3. *Let $p \in (-1, 1)$ and assume (**H**). Then problem* (1)-(2) *has a positive continuous solution u satisfying, for $x \in D$,*

$$u(x) \approx \frac{(\rho(x))^{\min(\frac{2m-\mu}{1-p}, m)}}{|x|^{\min(\frac{\lambda-2m}{1-p}, n-2m)}}\, F_{L_0,\mu,p}(\rho(x))\, G_{L_\infty,\lambda,p}(|x|),$$

where $F_{L_0,\mu,p}$ is the function defined on $(0, 1]$, by

$$F_{L_0,\mu,p}(t) := \begin{cases} 1, & \text{if } \mu < m(1+p), \\ \left(\int_t^a \frac{L_0(s)}{s}ds\right)^{\frac{1}{1-p}}, & \text{if } \mu = m(1+p), \\ (L_0(t))^{\frac{1}{1-p}}, & \text{if } m(1+p) < \mu < m(1+p)+1-p, \\ \left(\int_0^t \frac{L_0(s)}{s}ds\right)^{\frac{1}{1-p}}, & \text{if } \mu = m(1+p)+1-p, \end{cases} \tag{4}$$

and $G_{L_\infty,\lambda,p}$ *is defined on* $[1,\infty)$, *by*

$$G_{L_\infty,\lambda,p}(t) := \begin{cases} \left(\int_t^\infty \frac{L_\infty(s)}{s}ds\right)^{\frac{1}{1-p}}, & \text{if} \quad \lambda = 2m, \\ (L_\infty(t))^{\frac{1}{1-p}}, & \text{if} \quad 2m < \lambda < n - p(n-2m), \\ \left(\int_1^{t+1} \frac{L_\infty(s)}{s}ds\right)^{\frac{1}{1-p}}, & \text{if} \quad \lambda = n - p(n-2m), \\ 1, & \text{if} \quad \lambda > n - p(n-2m). \end{cases} \tag{5}$$

The remainder of the paper is structured as follows. In Section 2, we present some already known results on functions in $\mathcal{K}_0$ and $\mathcal{K}_\infty$. In Section 3, we provide some preliminary results related to potential theory tools associated to the operator $(-\Delta)^m$ on D under Dirichlet boundary conditions. We also derive estimates on some potential functions. Section 4 is devoted to the proof of Theorem 1.3. The last section is reserved for an illustrative example of our main result.

2. Properties of the Karamata classes $\mathcal{K}_0$ and $\mathcal{K}_\infty$

We collect in this paragraph some fundamental properties of functions belonging to the Karamata classes $\mathcal{K}_0$ and $\mathcal{K}_\infty$. We refer the interested reader to [19].

Lemma 2.1. *(i) Let* $\sigma \in \mathbb{R}$, $M,\ N \in \mathcal{K}_0$ *(resp.* $\mathcal{K}_\infty$*). Then the functions* $M+N$, MN *and* M^σ *belong to the class* $\mathcal{K}_0$ *(resp.* $\mathcal{K}_\infty$*).*

(ii) Let $\alpha > 0$ *and* $M \in \mathcal{K}_0$*(resp.* $\mathcal{K}_\infty$*). Then we have* $\lim_{t\to 0^+} t^\alpha M(t) = 0$ *(resp.* $\lim\limits_{t\to\infty} t^{-\alpha}M(t) = 0$*).*

(iii) Let $\eta > 0$ *and* $M \in \mathcal{K}_\infty$ *then we have* $M(t+\eta) \approx M(t)$, *for* $t \geqslant 1$.

Lemma 2.2 (Karamata's Theorem). *(a) Let* $\gamma \in \mathbb{R}$ *and* $L_0 \in \mathcal{K}_0$ *defined on* $(0,a]$, $a > 0$. *Then we have the following:*

- *(i) If* $\gamma > -1$, *then* $\int_0^a t^\gamma L_0(t)dt$ *converges and* $\int_0^t s^\gamma L_0(s)ds \underset{t\to 0^+}{\sim} \frac{t^{1+\gamma}L_0(t)}{1+\gamma}$.
- *(ii) If* $\gamma < -1$, *then* $\int_0^a t^\gamma L_0(t)dt$ *diverges and* $\int_t^a s^\gamma L_0(s)ds \underset{t\to 0^+}{\sim} -\frac{t^{1+\gamma}L_0(t)}{1+\gamma}$.

(b) Let $\gamma \in \mathbb{R}$ *and* $L_\infty \in \mathcal{K}_\infty$. *Then we have the following:*

- *(i) If* $\gamma < -1$, *then* $\int_1^\infty t^\gamma L_\infty(t)dt$ *converges and* $\int_t^\infty s^\gamma L_\infty(s)ds \underset{t\to\infty}{\sim} -\frac{t^{1+\gamma}L_\infty(t)}{1+\gamma}$.
- *(ii) If* $\gamma > -1$, *then* $\int_1^\infty t^\gamma L_\infty(t)dt$ *diverges and* $\int_1^t s^\gamma L_\infty(s)ds \underset{t\to\infty}{\sim} \frac{t^{1+\gamma}L_\infty(t)}{1+\gamma}$.

Lemma 2.3. *Let* $L_0 \in \mathcal{K}_0$ *defined on* $(0,a]$, $a > 0$, *then we have*

$$\lim_{t\to 0^+} \frac{L_0(t)}{\int_t^a \frac{L_0(s)}{s}ds} = 0 \quad \text{and} \quad t \longmapsto \int_t^a \frac{L_0(s)}{s}ds \in \mathcal{K}_0.$$

If further, $\int_0^a \frac{L_0(s)}{s}ds$ *converges, then*

$$\lim_{t\to 0^+} \frac{L_0(t)}{\int_0^t \frac{L_0(s)}{s}ds} = 0 \text{ and } t \longmapsto \int_0^t \frac{L_0(s)}{s}ds \in \mathcal{K}_0.$$

Lemma 2.4. *Let* $L_\infty \in \mathcal{K}_\infty$*, then we have*

$$\lim_{t\to\infty} \frac{L_\infty(t)}{\int_1^t \frac{L_\infty(s)}{s}ds} = 0 \quad \text{and} \quad t \mapsto \int_1^{t+1} \frac{L_\infty(s)}{s}ds \in \mathcal{K}_\infty.$$

If further, $\int_1^\infty \frac{L_\infty(s)}{s}ds$ *converges, then*

$$\lim_{t\to\infty} \frac{L_\infty(t)}{\int_t^\infty \frac{L_\infty(s)}{s}ds} = 0 \quad \text{and} \quad t \mapsto \int_t^\infty \frac{L_\infty(s)}{s}ds \in \mathcal{K}_\infty.$$

3. Potential theory tools

3.1 Green's function $G_{m,n}$

Let $m \geqslant 1$, $n \geqslant 2$ and $G_{m,n}$ be the Green's function of $(-\Delta)^m$ on D with Dirichlet boundary conditions $\frac{\partial^j u}{\partial \nu^j} = 0$, $j \in \{0, \ldots, m-1\}$, where $\dfrac{\partial}{\partial \nu}$ is the outward normal derivative. The explicit expression of $G_{m,n}$ is given on $D \times D$ (see [3]):

$$G_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{\frac{[x,y]}{|x-y|}} \frac{(t^2-1)^{m-1}}{t^{n-1}}\mathrm{d}t,$$

where $k_{m,n} > 0$ and for x, y in D, $[x,y]^2 = |x-y|^2 + (|x|^2-1)(|y|^2-1)$.

In what follows, we refer to $V_{m,n}f$ the m-potential of a Borel nonnegative measurable function f on D defined by

$$V_{m,n}f(x) = \int_D G_{m,n}(x,z)f(z)\mathrm{d}z, \quad x \in D.$$

Recall that if $f \in \mathbf{L}^1_{Loc}(D)$ and $V_{m,n}f \in \mathbf{L}^1_{Loc}(D)$, then we have, $(-\Delta)^m (V_{m,n}f) = f$, in the distributional sense.

3.2 Kato class $K^\infty_{m,n}$

We recall the definition and some properties of the Kato class $K^\infty_{m,n}$ with $m \geqslant 1$ and $n \geqslant 2$. We give, in particular, a characterization of radial functions belonging to $K^\infty_{m,n}$. For more details, we refer to [3].

Definition 3.1. A Borel measurable function q defined on D belongs to the Kato class $K^\infty_{m,n}$ if the following hypotheses are fulfilled:

$$\lim_{r\to 0}\left(\sup_{x\in D}\int_{D\cap B(x,r)} \left(\frac{\rho(z)}{\rho(x)}\right)^m G_{m,n}(x,z)|q(z)|\mathrm{d}z\right) = 0,$$

$$\lim_{M\to\infty}\left(\sup_{x\in D}\int_{|z|\geqslant M}\left(\frac{\rho(z)}{\rho(x)}\right)^m G_{m,n}(x,z)|q(z)|\mathrm{d}z\right)=0.$$

PROPOSITION 3.2. *Let q be a radial function on D and $\nu=\max(0,2m-n)$. Then the following assertions are equivalent:*
(i) $q\in K^{\infty}_{m,n}$;

$$\text{(ii)}\quad\begin{cases}\displaystyle\int_1^\infty r^{\nu}(r-1)^{2m-1}|q(r)|\mathrm{d}r<\infty, & \text{if } n\neq 2m,\\[2ex] \displaystyle\int_1^\infty r(r-1)^{2(m-1)}\log(r)|q(r)|\mathrm{d}r<\infty, & \text{if } n=2m.\end{cases}$$

PROPOSITION 3.3. *Let q be a nonnegative function in $K^{\infty}_{m,n}$. The family of functions*

$$\left\{\frac{1}{(|.|^2-1)^{m-1}}\int_D G_{m,n}(.,y)(|y|^2-1)^{m-1}g(y)\mathrm{d}y,\ |g|\leqslant q\right\}$$

is uniformly bounded and equicontinuous on $\overline{D}\cup\{\infty\}$. Consequently, it is relatively compact in $C(\overline{D}\cup\{\infty\})$.

3.3 Estimates of some potential functions

Let $n>2m\geqslant 2$. We are going to give estimates on the potential functions $V_{m,n}(b\theta^p)$, where $p\in(-1,1)$, b is a function satisfying (**H**) and θ is the function defined on D by

$$\theta(x):=\frac{(\rho(x))^{\min(\frac{2m-\mu}{1-p},m)}}{|x|^{\min(\frac{\lambda-2m}{1-p},n-2m)}}F_{L_0,\mu,p}(\rho(x))\,G_{L_\infty,\lambda,p}(|x|),\tag{6}$$

with $F_{L_0,\mu,p}$ and $G_{L_\infty,\lambda,p}$ are respectively given by (4) and (5).

These estimates will be useful in the proof of our main result stated in Theorem 1.3.

PROPOSITION 3.4. *We consider the function f defined on D by*

$$f(x)=\big(\rho(x)\big)^{-\beta}|x|^{-\alpha}M_0(\rho(x))M_\infty(|x|),$$

where $\beta\leqslant m+1$, $\alpha\geqslant 2m$, $M_0\in\mathcal{K}_0$, defined in $(0,a]$, $a>1$ and $M_\infty\in\mathcal{K}_\infty$, such that

$$\int_0^a t^{m-\beta}M_0(t)\mathrm{d}t<\infty\quad\text{and}\quad\int_1^\infty t^{2m-1-\alpha}M_\infty(t)\mathrm{d}t<\infty.$$

Then, we have for $x\in D$

$$V_{m,n}f(x)\approx\frac{(\rho(x))^{\min(m,2m-\beta)}}{|x|^{\min(n,\alpha)-2m}}F_{M_0,\beta,0}(\rho(x))G_{M_\infty,\alpha,0}(|x|).$$

Where $F_{M_0,\beta,0}$, $G_{M_\infty,\alpha,0}$ are respectively given by (4) and (5).

Proof. Let $B=\{x\in\mathbb{R}^n;|x|<1\}$ denote the unit ball of $\mathbb{R}^n$, and let $G^B_{m,n}$ be the Green function of the polyharmonic operator $(-\Delta)^m$ on B with Dirichlet boundary conditions $\frac{\partial^j u}{\partial\nu^j}=0$, for $j\in\{0,\dots,m-1\}$. It is well known that for each $x,y\in D$,

$$G_{m,n}(x,y)=|x|^{2m-n}|y|^{2m-n}G^B_{m,n}(x^*,y^*),\tag{7}$$

where $x^* = \frac{x}{|x|^2}$ represents the Kelvin transformation from D to $B\backslash\{0\}$. Using (7), we obtain for $x \in D$,

$$V_{m,n}f(x) = \int_D |x|^{2m-n}|y|^{2m-n}G^B_{m,n}(x^*,y^*)\big(\rho(y)\big)^{-\beta}|y|^{-\alpha}M_0(\rho(y))M_\infty(|y|)\mathrm{d}y.$$

By the change of variable $z = y^*$, we obtain that

$$V_{m,n}f(x) = |x|^{2m-n}\int_B G^B_{m,n}(x^*,z)(\delta(z))^{-\beta}M_0(\delta(z))|z|^{-n-2m+\alpha}M_\infty(\frac{1}{|z|})\mathrm{d}z,$$

where $\delta(z) = 1-|z|$.

We denote by N_0 the function defined on $(0,1]$ by $N_0(t) = M_\infty(\frac{1}{t})$, then we get that for $x \in D$, $V_{m,n}f(x) = |x|^{2m-n}V^B_{m,n}h(x^*)$, where for $z \in B$, $h(z) = \big(\delta(z)\big)^{-\beta}M_0(\delta(z))|z|^{-(n+2m-\alpha)}N_0(|z|)$.
We put $\gamma = n+2m-\alpha$ and $\nu = \beta$. We note that $\gamma \leqslant n$, $\nu \leqslant m+1$, $M_0, N_0 \in \mathcal{K}_0$ satisfying

$$\int_0^1 s^{n-\gamma-1}N_0(s)\mathrm{d}s < \infty \text{ and } \int_0^1 s^{m-\nu}M_0(s)\mathrm{d}s < \infty.$$

Hence by applying [5, Proposition 2.15], we obtain that for $x \in D$,

$$V_{m,n}f(x) \approx |x|^{2m-n}|x^*|^{\min(0,2m-\gamma)}(\delta(x^*))^{\min(m,2m-\nu)}\tilde{M}_{0,}(\delta(x^*))\tilde{N}_0(|x^*|),$$

where the functions $\tilde{N}_0$ and $\tilde{M}_0$ are defined on $(0,1)$ respectively by

$$\tilde{M}_0(t) = \begin{cases} 1, & \text{if } \nu < m,\\ \displaystyle\int_t^a \frac{M_0(s)}{s}\mathrm{d}s, & \text{if } \nu = m,\\ M_0(t), & \text{if } m < \nu < m+1,\\ \displaystyle\int_0^t \frac{M_0(s)}{s}\mathrm{d}s, & \text{if } \nu = m+1,\end{cases}$$

and

$$\tilde{N}_0(t) = \begin{cases} 1, & \text{if } \gamma < 2m,\\ \displaystyle\int_t^1 \frac{N_0(s)}{s}\mathrm{d}s, & \text{if } \gamma = 2m,\\ N_0(t), & \text{if } 2m < \gamma < n,\\ \displaystyle\int_0^t \frac{N_0(s)}{s}\mathrm{d}s, & \text{if } \gamma = n.\end{cases}$$

A straightforward computation gives that for $x \in D$,

$$V_{m,n}f(x) \approx \frac{(\rho(x))^{\min(m,2m-\beta)}}{|x|^{\min(n,\alpha)-2m}}F_{M_0,\beta,0}(\rho(x))G_{M_\infty,\alpha,0}(|x|).$$

This completes the proof. □

The following proposition plays a key role in this paper.

Proposition 3.5. *Let $p \in (-1,1)$, b a function satisfying (**H**) and θ be the function given by* (6). *Then for $x \in D$, we have $V_{m,n}(b\theta^p)(x) \approx \theta(x)$.*

Proof. Let $p \in (-1,1)$ and b be a function satisfying (**H**). From hypothesis (**H**) and (6) we obtain that for $x \in D$,

$$(b\theta^p)(x) \approx (\rho(x))^{-\mu+p\min(\frac{2m-\mu}{1-p},m)}(L_0F^p_{L_0,\mu,p})(\rho(x))|x|^{-\lambda-p\min(\frac{\lambda-2m}{1-p},n-2m)}(L_\infty G^p_{L_\infty,\lambda,p})(|x|)$$

$$:= \big(\rho(x)\big)^{-\beta}|x|^{-\alpha}M_0(\rho(x))M_\infty(|x|).$$

Here, $\beta = \mu - p\min(\frac{2m-\mu}{1-p},m)$, $\alpha = \lambda + p\min(\frac{\lambda-2m}{1-p},n-2m)$, and $M_0 = L_0F^p_{L_0,\mu,p}$, $M_\infty = L_\infty G^p_{L_\infty,\lambda,p}$.

We can easily see that $\beta \leqslant m+1$, $\alpha \geqslant 2m$, $M_0 \in \mathcal{K}_0$ and $M_\infty \in \mathcal{K}_\infty$ satisfying

$$\int_0^a t^{m-\beta}M_0(t)\mathrm{d}t < \infty \quad \text{and} \quad \int_1^\infty t^{2m-1-\alpha}M_\infty(t)\mathrm{d}t < \infty.$$

Applying Proposition 3.4 and noting that $\min(2m-\beta,m) = \min\left(\frac{2m-\mu}{1-p},m\right)$ and $\min(n,\alpha) - 2m = \min\left(\frac{\lambda-2m}{1-p},n-2m\right)$, we obtain by elementary calculus the desired result. □

4. Proof of Theorem 1.3

Let b be a function satisfying (**H**) and let θ be the function given in (6). For $x \in D$, we set $\varphi(x) = (|x|^2-1)^{1-m}\theta(x)$. We remark that $\varphi \in C_0(D)$. By Proposition 3.5, there exists $M > 1$ such that for each $x \in D$,

$$\frac{1}{M}\theta(x) \le V_{m,n}(b\theta^p)(x) \le M\theta(x). \tag{8}$$

This implies that for $x \in D$,

$$\frac{1}{M}\varphi(x) \le \big(|x|^2-1\big)^{1-m}V_{m,n}(b\theta^p)(x) \le M\varphi(x). \tag{9}$$

We shall use a fixed point argument to construct a solution of problem (1)-(2). With this aim, we put $C = M^{\frac{1}{1-|p|}}$ and we consider the closed convex set given by

$$\Lambda = \Big\{v \in C(\overline{D}\cup\{\infty\});\ \frac{1}{C}\varphi \le v \le C\varphi\Big\}.$$

We define the operator T on Λ as follows

$$Tv(x) = \int_D \left(\frac{|y|^2-1}{|x|^2-1}\right)^{m-1} G_{m,n}(x,y)b(y)\big(|y|^2-1\big)^{(p-1)(m-1)}v^p(y)\mathrm{d}y, x \in D.$$

Let $v \in \Lambda$. For $y \in D$, $\big(|y|^2-1\big)^{(p-1)(m-1)}b(y)v^p(y) \leqslant C^{|p|}q(y)$, where $q(y) = \big(|y|^2-1\big)^{1-m}b(y)\theta^p(y)$. By the virtue of Proposition 3.2, we get that the function q is in $K^\infty_{m,n}$. Hence, Proposition 3.3 implies that $T\Lambda$ is relatively compact in $C(\overline{D}\cup\{\infty\})$.

Next we shall prove that $T\Lambda \subset \Lambda$. Let $v \in \Lambda$ then we have,

$$\frac{1}{C^{|p|}}\varphi^p(x) \le v^p(x) \le C^{|p|}\varphi^p(x), \quad \text{for} \quad x \in D. \tag{10}$$

It follows from (9) and (10) that $\frac{1}{C^{|p|}}\frac{1}{M}\varphi \le Tv \le C^{|p|}M\varphi$. Since $MC^{|p|} = C$ and

$T\Lambda \subset C(\overline{D} \cup \{\infty\})$, we deduce that $T\Lambda \subset \Lambda$.

Now, we will establish that T is continuous with reference to the uniform norm on $C(\overline{D} \cup \{\infty\})$ given by $\|u\|_\infty = \sup_{x \in \overline{D} \cup \{\infty\}} |u(x)|$.

We consider a sequence $(v_k)_k$ in Λ which converges uniformly to a function v in Λ. For $k \in \mathbb{N}$, we have for $x \in D$,

$$|Tv_k(x) - Tv(x)| \leq \frac{1}{\big(|x|^2-1\big)^{m-1}} \int_D G_{m,n}(x,y) b(y) \big(|y|^2 - 1\big)^{p(m-1)} \big|v_k^p(y) - v^p(y)\big| \mathrm{d}y.$$

Since for $x, y \in D$,

$$G_{m,n}(x,y) b(y) \big(|y|^2 - 1\big)^{p(m-1)} \big|v_k^p(y) - v^p(y)\big| \leq 2C^{|p|} G_{m,n}(x,y) b(y) \theta^p(y),$$

we obtain from (8), that for $x \in D$,

$$\frac{1}{\big(|x|^2-1\big)^{m-1}} \int_D G_{m,n}(x,y) b(y) \big(|y|^2 - 1\big)^{p(m-1)} \big|v_k^p(y) - v^p(y)\big| \mathrm{d}y < \infty.$$

Hence, by the dominated convergence theorem we get that for $x \in D$, $|Tv_k(x) - Tv(x)| \to 0$ as $k \to \infty$.

Due to the fact that $T\Lambda$ is relatively compact in $C(\overline{D} \cup \{\infty\})$, then we obtain the uniform convergence. Therefore, T is a compact mapping from Λ to itself. So by the Schauder fixed point theorem, there exists a function $v \in \Lambda$ such that $Tv = v$.

Put $u(x) = \big(|x|^2 - 1\big)^{m-1} v(x)$, $x \in D$. We clearly have that $u(x) \approx \theta(x)$, for $x \in D$. On the other hand, one can easily that

$$u \in C_0(D), \tag{11}$$

and satisfies on D the integral equation

$$u = V_{m,n}(bu^p). \tag{12}$$

Using hypothesis (**H**), (11) and (12), we obtain that the functions bu^p and $V_{m,n}(bu^p)$ are in $\mathbf{L^1_{loc}}(D)$. Hence, u satisfies in the distributional sense,

$$(-\Delta)^m u = (-\Delta)^m V_{m,n}(bu^p) = bu^p, \quad \text{in } D.$$

The fact that $u \in C_0(D)$ implies that $\lim_{|x| \to \infty} (|x|^2 - 1)^{1-m} u(x) = 0$.

Finally, we obviously have

$$0 \leq \lim_{|x| \to 1} (|x|^2 - 1)^{1-m} u(x) \leq M \lim_{|x| \to 1} (|x|^2 - 1)^{1-m} \theta(x) = 0.$$

This completes the proof.

5. Example

Let $p \in (-1, 1)$. We consider b the function defined on D by

$$b(x) = \big(\rho(x)\big)^{-\mu} |x|^{-\lambda} \log\Big(\frac{4}{\rho(x)}\Big) \log^{-\alpha}(2|x|),$$

where the real numbers μ, λ and α satisfy one of the following conditions:

- $\mu < m(1+p)+1-p$, $\lambda > 2m$ and $\alpha \in \mathbb{R}$;
- $\mu < m(1+p)+1-p$, $\lambda = 2m$ and $\alpha > 1$.

One can easily see that hypothesis (**H**) is fulfilled. Then by Theorem 1.3, the problem (1)-(2) has a positive continuous solution u satisfying for $x \in D$, $u(x) \approx \phi(\rho(x))\psi(|x|)$, where

$$\phi(\rho(x)) = \begin{cases} (\rho(x))^m, & \text{if } \mu < m(1+p), \\ (\rho(x))^m \big(\log(\frac{4}{\rho(x)})\big)^{\frac{2}{1-p}}, & \text{if } \mu = m(1+p), \\ (\rho(x))^{\frac{2m-\mu}{1-p}} \big(\log(\frac{4}{\rho(x)})\big)^{\frac{1}{1-p}}, & \text{if } m(1+p) < \mu < m(1+p)+1-p, \end{cases}$$

and

$$\psi(|x|) = \begin{cases} \big(\log(2|x|)\big)^{\frac{1-\alpha}{1-p}}, & \text{if } \lambda = 2m \text{ and } \alpha > 1, \\ |x|^{\frac{2m-\lambda}{1-p}} \big(\log(2|x|)\big)^{\frac{-\alpha}{1-p}}, & \text{if } 2m < \lambda < n-p(n-2m) \text{ and } \alpha \in \mathbb{R}, \\ |x|^{2m-n} \big(\log(2|x|)\big)^{\frac{1-\alpha}{1-p}}, & \text{if } \lambda = n-p(n-2m) \text{ and } \alpha < 1, \\ |x|^{2m-n} \big(\log(\log(2|x|))\big)^{\frac{1}{1-p}}, & \text{if } \lambda = n-p(n-2m) \text{ and } \alpha = 1, \\ |x|^{2m-n}, & \text{if } \lambda = n-p(n-2m) \text{ and } \alpha > 1, \\ & \text{or } \lambda > n-p(n-2m) \text{ and } \alpha \in \mathbb{R}. \end{cases}$$

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