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SOME d-TRANSFORM FUNCTORS BASED ON AN IDEAL

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Abstract. Let R be a commutative Noetherian ring, $\mathfrak{b}$ be an ideal of R, M an R-module and let d be a non-negative integer. We introduce a general d-transform functor $T_{d,\mathfrak{b}}(M, \text{-})$ and its right derived functors $T^i_{d,\mathfrak{b}}(M, \text{-})$, $i \in \mathbb{N}_0$, on the category of R-modules and study their various properties. The connection of these functors with some kind of generalized local cohomology functors $H^i_{d,\mathfrak{b}}(M, \text{-})$ is discussed. When both M and N are finitely generated, some finiteness results on $T^i_{d,\mathfrak{b}}(M, N)$ and $H^i_{d,\mathfrak{b}}(M, N)$ are concluded. Then, we study how the depth and dimension of certain subsets of $\operatorname{Spec}(R)$ affect the behavior and vanishing of these modules.

1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring with a non-zero identity, and M and N are two R-modules. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R, and let d be a non-negative integer. The basic theory of the $\mathfrak{a}$-transform functor $D_{\mathfrak{a}}(\text{-}) := \varinjlim_{n\in\mathbb{N}} \operatorname{Hom}R(\mathfrak{a}^n, \text{-})$ and its connections with the local cohomology functor have appeared and been studied in [6] (see also [5, 10]). For an R-module X, the R-module $D\mathfrak{a}(X)$ is an important algebraic tool in studying ordinary local cohomology modules $H^i_{\mathfrak{a}}(M)$.

In [15], the authors defined a new functor $T_d(\text{-})$ and its right derived functors and studied its connection with the functor $U_d(\text{-})$ and with the cohomology functors $H^i_d(\text{-})$, $i \in \mathbb{N}_0$, defined in [3]. In this paper, we study to what extent these results are valid for generalized local cohomology. To be precise, let $\mathcal{I}(R)$ be the set of all ideals of R, and put $\Sigma_d = \mathfrak{a} \in \mathcal{I}(R) : \dim(R/\mathfrak{a}) \le d$.

We define the set $\Gamma_{d,\mathfrak{b}}(X)$ consisting of all elements $x \in X$ such that $\mathfrak{a}x \subseteq \mathfrak{b}x$ for some $\mathfrak{a} \in \Sigma_d$. Then $\Gamma_{d,\mathfrak{b}}(\text{-})$ is an R-linear left exact functor on the category of R-modules $\mathcal{C}(R)$, and we consider its ith ($i \ge 0$) right derived functor, denoted by $H^i_{d,\mathfrak{b}}(\text{-})$.

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Now, for two modules M and N, we put $H^i_{d,\mathfrak{b}}(M,N) := H^i(\Gamma_{d,\mathfrak{b}}(\mathrm{Hom}_R(M,\mathbf{E}^N)))$, where $i \geq 0$ and $\mathbf{E}^N$ is any injective resolution of N.

To give a more concrete description of the aim of the paper, we consider three other sets $\tilde{W}(d,\mathfrak{b}) = \{I \in \mathcal{I}(R) : \exists \mathfrak{a} \in \Sigma_d, \mathfrak{a} \subseteq I + \mathfrak{b}\}$, $W(d,\mathfrak{b}) := \{\mathfrak{p} \in \mathrm{Spec}(R) | \exists \mathfrak{a} \in \Sigma_d$, $\mathfrak{a} \subseteq \mathfrak{p} + \mathfrak{b}\}$ and $S^*_k(M,N) := \{\mathfrak{p} \in \mathrm{Supp}(M) | \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) + \dim(R/\mathfrak{p}) \leq k\}$, $k \in \mathbb{N}_0$ (cf. [12, 14]). Then, with the reverse inclusion, $\tilde{W}(d,\mathfrak{b})$ (and also Σ_d) is a system of ideals in R in the sense of [6, p. 21]. Now we define the d-transform functor $T_{d,\mathfrak{b}}(M,\text{-})$ supported in $\mathfrak{b}$ by $T_{d,\mathfrak{b}}(M,N) = \varinjlim_{\mathfrak{a} \in \tilde{W}(d,\mathfrak{b})} \mathrm{Hom}(\mathfrak{a}M,N)$, for all $N \in \mathcal{C}(R)$.

Evidently, $T_{d,\mathfrak{b}}(M,\text{-})$ is R-linear and left exact, and its right derived functors $T^i_{d,\mathfrak{b}}(M,\text{-})$, $i \in \mathbb{N}_0$, form a strongly connected sequence in the sense of [6, Definition 1.3.1]. It is convenient to mention that, using standard arguments of homological algebra [11, Theorem 6.68] (Axioms for covariant Ext), we have $H^i_{d,\mathfrak{b}}(M,N) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(d,\mathfrak{b})} \mathrm{Ext}^i_R(M/\mathfrak{a}M,N)$. From this fact and using [11, Lemma 5.30], it is easily seen that for each $i \in \mathbb{N}_0$ we have $\Gamma_{d,\mathfrak{b}}(H^i_{d,\mathfrak{b}}(M,N)) = H^i_{d,\mathfrak{b}}(M,N)$.

The aim of this paper is to study the structure of the modules $T^i_{d,\mathfrak{b}}(M,N)$ and their connections with $H^i_{d,\mathfrak{b}}(M,N)$, $i \in \mathbb{N}0$. In Section 2, we present some basic auxiliary facts concerning the modules $\Gamma d, \mathfrak{b}(M)$, $\Gamma_{d,\mathfrak{b}}(M,N)$, $H^i_{d,\mathfrak{b}}(M,N)$, and $T^i_{d,\mathfrak{b}}(M,N)$. It is shown, for example, that when M is finitely generated, or when $\Gamma_{d,\mathfrak{b}}(N) = N$, these modules exhibit interesting properties and coincide with previously known structures. In Section 3, we assume that M is finitely generated and obtain further isomorphism results, which are collected in Theorem 3.1. It is concluded that whenever both M and N are finitely generated, the set of associated primes of $T^i_{d,\mathfrak{b}}(M,N)$ is finite. In Section 4, we investigate how the dimensions of the sets $W(d,\mathfrak{b})$ and $S^*d, \mathfrak{b}(M,N)$ affect the vanishing of the modules $H^id, \mathfrak{b}(M,N)$ and $T^i_{d,\mathfrak{b}}(M,N)$. For any unexplained notation, we refer the reader to [6, 12].

2. Preliminaries

In this section we provide some elementary facts concerning the modules $\Gamma_{d,\mathfrak{b}}(M)$, $\Gamma_{d,\mathfrak{b}}(M,N)$, $H^i_{d,\mathfrak{b}}(M,N)$ and $T^i_{d,\mathfrak{b}}(M,N)$. It is seen that like $\Gamma_\mathfrak{a}(\text{-})$ and $\Gamma_d(\text{-})$, the functor $\Gamma_{d,\mathfrak{b}}(\text{-})$ keeps the injective property. We collect the properties in the following lemma. For an R-module X, the projective dimension of X will be denoted by $\mathrm{pd}(X)$.

Proposition 2.1. *Let M and N be two R-modules. The following statements hold.*

(i) If M is an injective R-module, then $\Gamma_{d,\mathfrak{b}}(M)$ is also an injective R-module.

(ii) If M is finitely generated, then $H^i_{d,\mathfrak{b}}(M,N) \cong H^i(\mathrm{Hom}_R(M,\Gamma_{d,\mathfrak{b}}(\mathbf{E}^N))$.

(iii) If M is finitely generated and $\Gamma_{d,\mathfrak{b}}(N) = N$, then $H^i_{d,\mathfrak{b}}(M,N) \cong \mathrm{Ext}^i_R(M,N)$ for all $i \geq 0$. In particular $H^i_{d,\mathfrak{b}}(M,\Gamma_{d,\mathfrak{b}}(N)) \cong \mathrm{Ext}^i_R(M,\Gamma_{d,\mathfrak{b}}(N))$, for $i \geq 0$.

(iv) If $\mathrm{pd}(M) = p$, then $H^i_{d,\mathfrak{b}}(M,N) \cong H^i_{d,\mathfrak{b}}(M,N/\Gamma_{d,\mathfrak{b}}(N))$, for all $i > p$.

(v) For each $i \in \mathbb{N}_0$, $H^i_{d,\mathfrak{b}}(M,N) \cong \varinjlim_{\mathfrak{a} \in \tilde{\mathrm{W}}(d,J)} H^i_\mathfrak{a}(M,N)$.

(vi) For each $i \in \mathbb{N}_0$, $H^i_{d,\mathfrak{b}}(M,N) \cong \varinjlim_{\mathfrak{a}\in\Sigma_d} H^i_{\mathfrak{a},\mathfrak{b}}(M,N)$, *where* $H^i_{\mathfrak{a},\mathfrak{b}}(M,N)$ *is the* i^{th} *generalized local cohomology relative to* $(\mathfrak{a},\mathfrak{b})$, *studied in [9, 13].*

(vii) $T_{d,\mathfrak{b}}(M,N) \cong \varinjlim_{\mathfrak{a}\in\Sigma_d} D_{\mathfrak{a},\mathfrak{b}}(M,N)$, *where* $D_{\mathfrak{a},\mathfrak{b}}(M,N) = \varinjlim_{\mathfrak{c}\in\tilde{\mathrm{W}}(\mathfrak{a},\mathfrak{b})} D_{\mathfrak{c}}(M,N)$.

(viii) There exists a long exact sequence of R-modules

$$0 \to \Gamma_{d,\mathfrak{b}}(M,N) \to \mathrm{Hom}_R(M,N) \to T_{d,\mathfrak{b}}(M,N) \to H^1_{d,\mathfrak{b}}(M,N) \to \mathrm{Ext}^1_R(M,N)$$
$$\to T^1_{d,\mathfrak{b}}(M,N) \to H^2_{d,\mathfrak{b}}(M,N) \to \cdots .$$

(ix) If M is projective or N is injective, then we have the sequence

$$0 \to \Gamma_{d,\mathfrak{b}}(M,N) \to \mathrm{Hom}_R(M,N) \to T_{d,\mathfrak{b}}(M,N) \to H^1_{d,\mathfrak{b}}(M,N) \to 0.$$

(x) If $\mathrm{pd}(M) = p$, *then* $T^i_{d,\mathfrak{b}}(M,N) \cong H^{i+1}_{d,\mathfrak{b}}(M,N)$, *for all* $i > p$.

(xi) For each $i \in \mathbb{N}_0$, $T^i_{d,\mathfrak{b}}(M,\text{-}) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Ext}^i_R(\mathfrak{a}M,\text{-})$.

Proof. (i) Note that by a similar argument as in [12, Theorem 3.2], we have $\Gamma_{d,\mathfrak{b}}(M) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \Gamma_{\mathfrak{a}}(M)$. As the direct limit of injective modules is injective (note that R is Noetherian), this gives the claim.

To prove (ii), by our definition we have

$$\begin{aligned} H^0_{d,\mathfrak{b}}(M,N) = \Gamma_{d,\mathfrak{b}}(\mathrm{Hom}_R(M,N)) &\cong \varinjlim_{\mathfrak{a}\in\tilde{\mathrm{W}}(d,\mathfrak{b})} \Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,N)) \\ &\cong \varinjlim_{\mathfrak{a}\in\tilde{\mathrm{W}}(d,\mathfrak{b})} \mathrm{Hom}(M,\Gamma_{\mathfrak{a}}(N)) \cong \mathrm{Hom}(M, \varinjlim_{\mathfrak{a}\in\tilde{\mathrm{W}}(d,\mathfrak{b})} \Gamma_{\mathfrak{a}}(N)) \cong \mathrm{Hom}(M,\Gamma_{d,\mathfrak{b}}(N)), \end{aligned}$$

which is the 0^{th}-term of the connected sequence $H^i(\mathrm{Hom}_R(M,\Gamma_{d,\mathfrak{b}}(\mathbf{E}^{\cdot})))$, $i \in \mathbb{N}_0$, evaluated in N (to see why the second isomorphism holds true, see for example [9, Proposition 2.2]) with $\mathfrak{b} = 0$, and for the third isomorphism see [1, Proposition 7.7]. Now, as both connected sequences $H^i_{d,\mathfrak{b}}(M,\text{-})$ and $H^i(\mathrm{Hom}_R(M,\Gamma_{d,\mathfrak{b}}(\mathbf{E}^{\cdot})))$ are zero in injective modules, by the standard argument of homology theory (see [11, Theorem 6.68]) the result follows.

(iii) By a similar argument as in [14, Lemma 2.1(3)], there exists an injective resulotion of N, say $\mathbf{E}^N$, such that $\Gamma_{d,\mathfrak{b}}(\mathbf{E}^N) = \mathbf{E}^N$. By part (ii), we see that

$$H^i_{d,\mathfrak{b}}(M,N) \cong H^i(\mathrm{Hom}_R(M,\Gamma_{d,\mathfrak{b}}(\mathbf{E}^N))) \cong H^i(\mathrm{Hom}_R(M,\mathbf{E}^N)) \cong \mathrm{Ext}^i_R(M,N).$$

In particular, as $\Gamma_{d,\mathfrak{b}}(\Gamma_{d,\mathfrak{b}}(N)) = \Gamma_{d,\mathfrak{b}}(N)$, the desired isomorphism is clear.

(iv) From the short exact sequence $0 \to \Gamma_{d,\mathfrak{b}}(N) \to N \to N/\Gamma_{d,\mathfrak{b}}(N) \to 0$, we obtain the exact sequence

$$H^i_{d,\mathfrak{b}}(M,\Gamma_{d,\mathfrak{b}}(N)) \to H^i_{d,\mathfrak{b}}(M,N) \to H^i_{d,\mathfrak{b}}(M,N/\Gamma_{d,\mathfrak{b}}(N)) \to H^{i+1}_{d,\mathfrak{b}}(M,\Gamma_{d,\mathfrak{b}}(N)),$$

for all $i \geq 0$. Now, since $\Gamma_{d,\mathfrak{b}}(\Gamma_{d,\mathfrak{b}}(N)) = \Gamma_{d,\mathfrak{b}}(N)$ and $\mathrm{Ext}^i(M,\Gamma_{d,\mathfrak{b}}(N)) = 0$, for all $i > p$, we get the result by part (iii).

(v) This follows from

$$\Gamma_{d,\mathfrak{b}}(M,N) = \Gamma_{d,\mathfrak{b}}(\mathrm{Hom}_R(M,N)) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,J)} \Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,N)) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,J)} \Gamma_{\mathfrak{a}}(M,N)),$$

and the standard homology arguments.

(vi) First, for an R-module X, we show that $\Gamma_{d,\mathfrak{b}}(X) \cong \varinjlim_{\mathfrak{a}\in\Sigma_d} \Gamma_{\mathfrak{a},\mathfrak{b}}(X)$. As Σ_d is a system of ideals, it is enough to show that $\Gamma_{d,\mathfrak{b}}(X) = \cup_{\mathfrak{a}\in\Sigma_d}\Gamma_{\mathfrak{a},\mathfrak{b}}(X)$. To do this, let $x \in \Gamma_{d,\mathfrak{b}}(X)$, Then there exists $\mathfrak{a} \in \Sigma_d$ such that $\mathfrak{a}x \subseteq \mathfrak{b}x$. Hence $x \in \Gamma_{\mathfrak{a},\mathfrak{b}}(X)$ and so $x \in \cap_{\mathfrak{a}\in\Sigma_d}\Gamma_{\mathfrak{a},\mathfrak{b}}(X)$. For the opposite inclusion, we just note that for each $\mathfrak{a} \in \Sigma_d$ and each $i \in \mathbb{N}_0$, $\mathfrak{a}^i \in \Sigma_d$. Now, since both $H^i_{d,\mathfrak{b}}(M, \text{-})$ and $\varinjlim_{\mathfrak{a}\in\Sigma_d} H^i_{\mathfrak{a},\mathfrak{b}}(M, \text{-})$, $i \in \mathbb{N}_0$, are strongly connected the result follows again by [11, Theorem 6.68].

(vii) Using the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Gamma_{d,\mathfrak{b}}(M,N) & \longrightarrow & \operatorname{Hom}_R(M,N) & \longrightarrow & T_{d,\mathfrak{b}}(M,N) & \longrightarrow & H^1_{d,\mathfrak{b}}(M,N) & \longrightarrow & \operatorname{Ext}^1_R(M,N) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \varinjlim_{\mathfrak{a}\in\Sigma_d} \Gamma_{\mathfrak{a},\mathfrak{b}}(M,N) & \rightarrow & \operatorname{Hom}_R(M,N) & \rightarrow & \varinjlim_{\mathfrak{a}\in\Sigma_d} D_{\mathfrak{a},\mathfrak{b}}(M,N) & \rightarrow & \varinjlim_{\mathfrak{a}\in\Sigma_d} H^1_{\mathfrak{a},\mathfrak{b}}(M,N) & \rightarrow & \operatorname{Ext}^1_R(M,N)
\end{array}$$

with exact rows, this follows by (vi) and the Five Lemma.

(viii) Let $\mathfrak{a}$ and $\mathfrak{c}$ be two ideals in $\tilde{\mathrm{W}}(d,\mathfrak{b})$ such that $\mathfrak{a} \subseteq \mathfrak{c}$. From the commutative diagram of R-modules and R-homomorphisms with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{a}M & \longrightarrow & M & \longrightarrow & M/\mathfrak{a}M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{c}M & \longrightarrow & M & \longrightarrow & M/\mathfrak{c}M & \longrightarrow & 0,
\end{array}$$

we obtain the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \rightarrow & \operatorname{Hom}_R(M/\mathfrak{c}M,N) & \rightarrow & \operatorname{Hom}_R(M,N) & \rightarrow & \operatorname{Hom}_R(\mathfrak{c}M,N) & \rightarrow & \operatorname{Ext}^1_R(M/\mathfrak{c}M,N) & \rightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \operatorname{Hom}_R(M/\mathfrak{a}M,N) & \rightarrow & \operatorname{Hom}_R(M,N) & \rightarrow & \operatorname{Hom}_R(\mathfrak{a}M,N) & \rightarrow & \operatorname{Ext}^1_R(M/\mathfrak{a}M,N) & \rightarrow & \cdots
\end{array}$$

with exact rows. Now, passing to the direct limit over $\tilde{\mathrm{W}}(d,\mathfrak{b})$ and using standard homology arguments gives the desired sequence.

(ix) Since in either case $\operatorname{Ext}^1_R(M,N) = 0$, this follows by part (viii).

(x) Apply part (viii) and use the fact that $\operatorname{Ext}^i_R(M,N) = 0$, for all $i > p$.

(xi) Let $T(\text{-}) = T_{d,\mathfrak{b}}(M, \text{-})$ and $T^i(\text{-}) = \varinjlim_{\mathfrak{a}\in\tilde{\mathrm{W}}(d,\mathfrak{b})} \operatorname{Ext}^i_R(\mathfrak{a}M, \text{-})$. Then, $T^i(\text{-})_{i\in\mathbb{N}_0}$ is a negative strongly connected sequence of covariant functors and $T^\circ(\text{-})$ is naturally equivalent to $T(\text{-})$. Since $T^i(E) = 0$, for all $i \geq 1$ and all injective R-module E, the result follows by [11, Theorem 6.68]. □

3. Finiteness results

In this section we generaly assume that M is finitely generated and study basic properties of the functor $T_{d,\mathfrak{b}}(M, \text{-})$ and its right derived functors $T^i_{d,\mathfrak{b}}(M, \text{-})$. The end of this section contains a study of the sets $\operatorname{Ass}(T_{d,\mathfrak{b}}(M,N))$ and $\operatorname{Ass}(T^i_{d,\mathfrak{b}}(M,N))$.

THEOREM 3.1. *Let M be a finitely generated R-module and N be an R-module. The following statements hold.*
(i) If $\Gamma_{d,\mathfrak{b}}(N) = N$, then $T^i_{d,\mathfrak{b}}(M,N) = 0$ for all $i \geq 0$.

(ii) $T^i_{d,\mathfrak{b}}(M,N) \cong T^i_{d,\mathfrak{b}}(M,N/\Gamma_{d,\mathfrak{b}}(N)) \cong T^i_{d,\mathfrak{b}}(M,T_{d,\mathfrak{b}}(N))$ for all $i \geq 0$.

(iii) $T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \cong T_{d,\mathfrak{b}}(M,N)$.

(iv) $T_{d,\mathfrak{b}}(\mathrm{Hom}_R(M,N)) \cong \mathrm{Hom}_R(M,T_{d,\mathfrak{b}}(N))$.

(v) $T_{d,\mathfrak{a}}(T_{d,\mathfrak{b}}(M,N)) = T_{d,\mathfrak{b}}(T_{d,\mathfrak{a}}(M,N))$.

(vi) $\Gamma_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) = H^1_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) = 0$.

Proof. (i) By Proposition 2.1 (viii) we have the long exact sequence

$$0 \to \Gamma_{d,\mathfrak{b}}(M,N) \to \mathrm{Hom}_R(M,N) \to T_{d,\mathfrak{b}}(M,N) \to H^1_{d,\mathfrak{b}}(M,N) \to \mathrm{Ext}^1_R(M,N).$$

Thus by Proposition 2.1 (iii) we conclude that $T_{d,\mathfrak{b}}(M,N) = 0$. The proof will be complete if we show $T^i_{d,\mathfrak{b}}(M,N) = 0$ for all $i \geq 1$. Since $\Gamma_{d,\mathfrak{b}}(N) = N$, there is an injective resolution $\mathbf{E}^N$ of N such that $\Gamma_{d,\mathfrak{b}}(E^i) = E^i$ for all $i \geq 0$. So as we did for N, we will have $T_{d,\mathfrak{b}}(M,E^i) = 0$ for all $i \geq 0$. Therefore $T^i_{d,\mathfrak{b}}(M,N) = 0$ for all $i \geq 0$.

(ii) From the short exact sequence $0 \to \Gamma_{d,\mathfrak{b}}(N) \to N \to N/\Gamma_{d,\mathfrak{b}}(N) \to 0$, we obtain the long exact sequence

$$\begin{gathered} 0 \to T_{d,\mathfrak{b}}(M,\Gamma_{d,\mathfrak{b}}(N)) \to T_{d,\mathfrak{b}}(M,N) \to T_{d,\mathfrak{b}}(M,N/\Gamma_{d,\mathfrak{b}}(N)) \to \\ \cdots \to T^i_{d,\mathfrak{b}}(M,N) \to T^i_{d,\mathfrak{b}}(M,N/\Gamma_{d,\mathfrak{b}}(N)) \to T^{i+1}_{d,\mathfrak{b}}(M,T_{d,\mathfrak{b}}(N)) \to \cdots. \end{gathered}$$

Now, using part (i), we have $T^i_{d,\mathfrak{b}}(M,\Gamma_{d,\mathfrak{b}}(N)) = 0$, for all $i \geq 0$, and thus the first isomorphism holds true.

For the second isomorphism, using Proposition 2.1 (ix) with $M = R$, the short exact sequence $0 \to N/\Gamma_{d,\mathfrak{b}}(N) \to T_{d,\mathfrak{b}}(N) \to H^1_{d,\mathfrak{b}}(N) \to 0$, induces the exact sequence

$$\begin{gathered} 0 \to T_{d,\mathfrak{b}}(M,N/\Gamma_{d,\mathfrak{b}}(N)) \to T_{d,\mathfrak{b}}(M,T_{d,\mathfrak{b}}(N)) \to T_{d,\mathfrak{b}}(M,H^1_{d,\mathfrak{b}}(N)) \to \\ \cdots \to T^i_{d,\mathfrak{b}}(M,N/\Gamma_{d,\mathfrak{b}}(N)) \to T^i_{d,\mathfrak{b}}(M,T_{d,\mathfrak{b}}(N)) \to T^i_{d,\mathfrak{b}}(M,H^1_{d,\mathfrak{b}}(N)) \to \cdots. \end{gathered}$$

Since $T^i_{d,\mathfrak{b}}(M,H^1_{d,\mathfrak{b}}(N)) = 0$, for all $i \geq 0$, we are done.

(iii) We have

$$\begin{aligned} T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) &= \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{a},T_{d,\mathfrak{b}}(M,N)) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{a}, \varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{c}M,N)) \\ &\cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} (\varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{a},\mathrm{Hom}_R(\mathfrak{c}M,N))) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} (\varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{c},\mathrm{Hom}_R(\mathfrak{a}M,N))) \\ &\cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} (\varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{c}\otimes_R \mathfrak{a}M,N)) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} (\varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{a}\otimes_R \mathfrak{c}M,N)) \\ &\cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} (\varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{a},\mathrm{Hom}_R(\mathfrak{c}M,N))) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{a}, \varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{c}M,N)) \\ &\cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{a},T_{d,\mathfrak{b}}(M,N)) \cong T_{d,\mathfrak{b}}(M,T_{d,\mathfrak{b}}(N)) \cong T_{d,\mathfrak{b}}(M,N), \end{aligned}$$

where in the second and seventh isomorphisms we use again [1, Proposition 7.7].

(iv) We have

$$T_{d,\mathfrak{b}}(\mathrm{Hom}_R(M,N)) = \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{a}, \mathrm{Hom}_R(M,N)) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{a}\otimes_R M, N)$$

$$\cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(M, \mathrm{Hom}_R(\mathfrak{a},N)) \cong \mathrm{Hom}_R(M, \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{a},N)) \cong \mathrm{Hom}_R(M, T_{d,\mathfrak{b}}(N)),$$

which is the desired isomorphism.

(v) We have

$$T_{d,\mathfrak{a}}(T_{d,\mathfrak{b}}(M,N)) = \varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{a})} \mathrm{Hom}_R(\mathfrak{c}, T_{d,\mathfrak{b}}(M,N))$$

$$\cong \varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{a})} \mathrm{Hom}_R(\mathfrak{c}, \varinjlim_{\mathfrak{d}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{d}M,N)) \cong \varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{a})} (\varinjlim_{\mathfrak{d}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{c}, \mathrm{Hom}_R(\mathfrak{d}M,N)))$$

$$\cong \varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{a})} (\varinjlim_{\mathfrak{d}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{c}\otimes_R \mathfrak{d}M,N)) \cong \varinjlim_{\mathfrak{d}\in\tilde{W}(d,\mathfrak{b})} (\varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{a})} \mathrm{Hom}_R(\mathfrak{d}\otimes_R \mathfrak{c}M,N))$$

$$\cong \varinjlim_{\mathfrak{d}\in\tilde{W}(d,\mathfrak{b})} (\varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{a})} \mathrm{Hom}_R(\mathfrak{d}, \mathrm{Hom}_R(\mathfrak{c}M,N))) \cong \varinjlim_{\mathfrak{d}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{d}, \varinjlim_{\mathfrak{c}\in\tilde{W}(d,\mathfrak{a})} \mathrm{Hom}_R(\mathfrak{c}M,N))$$

$$\cong \varinjlim_{\mathfrak{d}\in\tilde{W}(d,\mathfrak{b})} \mathrm{Hom}_R(\mathfrak{d}, T_{d,\mathfrak{a}}(M,N)) \cong T_{d,\mathfrak{b}}(T_{d,\mathfrak{a}}(M,N)),$$

and the claim is proved.

(vi) Replacing M with R and N with $T_{d,\mathfrak{b}}(M,N)$ in Proposition 2.1 (ix), we have the exact sequence of

$$0 \to \Gamma_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \to T_{d,\mathfrak{b}}(M,N) \to T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \to H^1_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \to 0.$$

Now since by part (iii), $T_{d,\mathfrak{b}}(M,N) \cong T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N))$, we obtain the result. □

In general it is a subtle problem to calculate the modules $T^i_{d,\mathfrak{b}}(M,N)$ for arbitrary R, M and N (see for example [6, 12.4.7 Example]). In the following example we calculate these modules for the case $R = M = N = \mathbb{Z}[x]$, the ring of polynomials over the integers.

Example 3.2. Let $R = \mathbb{Z}[x] = M = N$. We calculate $T^i_{d,\mathfrak{b}}(R,R)$ for $d = 1,2$ and for each ideal $\mathfrak{b}$ of $\mathbb{Z}[x]$. Note that R is Noetherian ring of Krull dimension 2. Thus, for each $d \geq 2$, we have $\Sigma_d = \{\mathfrak{a} \in \mathcal{I}(R) | \dim(R/\mathfrak{a}) \leq d\} = \mathcal{I}(R)$ and so $\Gamma_{d,\mathfrak{b}}(R) = \{f \in R | \mathfrak{a}f \subseteq \mathfrak{b}f \text{ for some } \mathfrak{a} \in \Sigma_d\} = R$. Thus, by Theorem 3.1 (i), $T^i_{d,\mathfrak{b}}(R,R) = 0$, for each ideal $\mathfrak{b}$ and each $i \geq 0$. It follows, by Proposition 2.1 (viii), that $H^i_{d,\mathfrak{b}}(R,R) = 0$ for each for each ideal $\mathfrak{b}$ and each $i \geq 1$. For $d = 1$, $\Sigma_1 = \{\mathfrak{a} \in \mathcal{I}(R) | \dim(R/\mathfrak{a}) \leq 1\} = \mathcal{I}(R) \setminus \{0\}$. Now, we consider two cases $\mathfrak{b} = 0$ and $\mathfrak{b} \neq 0$.

The case $\mathfrak{b} = 0$: It is easily seen that $\widetilde{W}(1,(0)) = \mathcal{I}(R) \setminus \{(0)\}$ and we compute $T_{1,(0)}(R,R)$. By definition, $T_{1,(0)}(R,R) = \varinjlim_{0\neq\mathfrak{a}\in\mathcal{I}(R)} \mathrm{Hom}_R(\mathfrak{a},R)$.

We now show that $\mathrm{Hom}_R(\mathfrak{a},R) \cong \mathfrak{a}^{-1}$ for all $\mathfrak{a} \neq (0)$, where $\mathfrak{a}^{-1} = \{u \in \mathbb{Q}(x) | u\mathfrak{a} \subseteq R\}$, where $\mathbb{Q}(x) = \mathbb{Z}[x]_{\{0\}}$ is the fractions field of $\mathbb{Z}[x]$. To do this, fix $a(x) \in \mathfrak{a}$ and consider the map $\phi : \mathrm{Hom}_R(\mathfrak{a},R) \to \mathfrak{a}^{-1}$ defined by

$$\phi(f) = \frac{f(a(x))}{a(x)} \quad \text{for each } f \in \mathrm{Hom}_R(\mathfrak{a},R).$$

Then, for an arbitrary element $b(x) \in \mathfrak{a}$, we have

$$b(x)\frac{f(a(x))}{a(x)} = \frac{a(x)}{a(x)}f(b(x)) = f(b(x)) \in R,$$

and

$$a(x)f(b(x)) = f(a(x)b(x)) = b(x)f(a(x))$$

so that $\frac{f(a(x))}{a(x)} = \frac{f(a(x))}{a(x)}$. These, together, give that $\phi(f) \in \mathfrak{a}^{-1}$ and that ϕ is well-defined. To see that ϕ is a monomorphism, let $\phi(f) = 0$ for some $f \in \mathrm{Hom}_R(\mathfrak{a}, R)$. Then $f(a(x)) = 0$, and thus $f(b(x)) = b \cdot \phi(f) = 0$ for all $b \in \mathfrak{a}$, by what we observed above. Hence, $f = 0$. For surjectivity of ϕ, let $u(x) \in \mathfrak{a}^{-1}$. Define $f : \mathfrak{a} \to R$ by $f(b(x)) = u(x)b(x)$. Then, $\phi(f) = \frac{f(a(x))}{a(x)} = \frac{u(x)a(x)}{a(x)} = u$ and we conclude that ϕ is an isomorphism. Therefore,

$$T_{1,(0)}(R, R) \cong \varinjlim_{0\neq\mathfrak{a}\in\mathcal{I}(R)} \mathfrak{a}^{-1} = \bigcup_{0\neq\mathfrak{a}\in\mathcal{I}(R)} \mathfrak{a}^{-1} = \mathbb{Q}(x),$$

where the first equality comes from [2, Exersise 17], and the second one is an easy observation. In this case, as $\Gamma_{1,(0)}(R, R) = \Gamma_{1,(0)}(\mathrm{Hom}_R(R, R)) = \Gamma_{1,(0)}(R) = 0$, we obtain again by Proposition 2.1 (viii) that $H^1_{1,(0)}(R, R) = \mathbb{Q}(x)/\mathbb{Z}[x]$.

The case $\mathfrak{b} \neq 0$: Since $\mathfrak{b} \in \Sigma_1$ we easily conclude that $\Gamma_{1,\mathfrak{b}}(R) = R$. Thus by Theorem 3.1 (i), $T^i_{1,\mathfrak{b}}(R, R) = 0$ for all $i \geq 0$.

Theorem 3.3. *Let M be finitely generated and let $f : N \to N'$ be a homomorphism of R-modules such that $\Gamma_{d,\mathfrak{b}}(\mathrm{Ker}f) = \mathrm{Ker}f$ and $\Gamma_{d,\mathfrak{b}}(\mathrm{Coker}f) = \mathrm{Coker}f$. Then $T_{d,\mathfrak{b}}(M, N) \cong T_{d,\mathfrak{b}}(M, N')$.*

Proof. From the short exact sequences

$$0 \to \mathrm{Ker}f \to N \to \mathrm{Im}f \to 0, \quad 0 \to \mathrm{Im}f \to N' \to \mathrm{Coker}f \to 0$$

we obtain the exact sequence

$$0 \to T_{d,\mathfrak{b}}(M, \mathrm{Ker}f) \to T_{d,\mathfrak{b}}(M, N) \to T_{d,\mathfrak{b}}(M, \mathrm{Im}f) \to T^1_{d,\mathfrak{b}}(M, \mathrm{Ker}f))$$

$$0 \to T_{d,\mathfrak{b}}(M, \mathrm{Im}f) \to T_{d,\mathfrak{b}}(M, N') \to T_{d,\mathfrak{b}}(M, \mathrm{Coker}f) \to T^1_{d,\mathfrak{b}}(M, \mathrm{Im}f).$$

Using Theorem 3.1 (i), $T_{d,\mathfrak{b}}(M, \mathrm{Ker}f) = T_{d,\mathfrak{b}}(M, \mathrm{Coker}f) = 0$, and so $T_{d,\mathfrak{b}}(M, N) \cong T_{d,\mathfrak{b}}(M, \mathrm{Im}f)$ and $T_{d,\mathfrak{b}}(M, \mathrm{Im}f) \cong T_{d,\mathfrak{b}}(M, N')$. Hence $T_{d,\mathfrak{b}}(M, N) \cong T_{d,\mathfrak{b}}(M, N')$. □

Theorem 3.4. *Let M be a finitely generated R-module and N be an R-module. Then for each $i \in \mathbb{N}_0$ $T^i_{d,\mathfrak{b}}(M, N) \cong T^i_{d,\mathfrak{b}}(\mathrm{Hom}_R(M, N))$.*

Proof. For the case $i = 0$, consider the exact sequence

$$0 \longrightarrow \Gamma_{d,\mathfrak{b}}(M, N) \xrightarrow{\alpha} \mathrm{Hom}_R(M, N) \xrightarrow{f} T_{d,\mathfrak{b}}(M, N) \xrightarrow{\beta} H^1_{d,\mathfrak{b}}(M, N).$$

We have $\mathrm{Ker}f = \mathrm{Im}\alpha$ and $\mathrm{Coker}f = \mathrm{Im}\beta$. Since $\Gamma_{d,\mathfrak{b}}(\mathrm{Im}\alpha) = \mathrm{Im}\alpha$ and $\Gamma_{d,\mathfrak{b}}(\mathrm{Im}\beta) = \mathrm{Im}\beta$, then using Theorem 3.3, $T_{d,\mathfrak{b}}(R, \mathrm{Hom}_R(M, N)) \cong T_{d,\mathfrak{b}}(R, T_{d,\mathfrak{b}}(M, N))$. Thus $T_{d,\mathfrak{b}}(\mathrm{Hom}_R(M, N)) \cong T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M, N))$.

Now, by Theorem 3.1 (iv) we have $T_{d,\mathfrak{b}}(M, N) \cong T_{d,\mathfrak{b}}(\mathrm{Hom}_R(M, N))$, meaning that $T^0_{d,\mathfrak{b}}(M, N) \cong T^0_{d,\mathfrak{b}}(\mathrm{Hom}_R(M, N))$. Now, we use the standard argument of homology and Proposition 2.1 (xi) to deduce the result. □

Theorem 3.5. *Let M and N be two finitely generated R-modules. Then*

$$\mathrm{Ass}(T_{d,\mathfrak{b}}(M,N)) = \mathrm{Supp}(M) \cap \mathrm{Ass}(N/\Gamma_{d,\mathfrak{b}}(N)).$$

Proof. Since N is finitely generated R-module, by a modification of [15, Theorem 1], we have $\mathrm{Ass}(T_{d,\mathfrak{b}}(N)) = \mathrm{Ass}(N/\Gamma_{d,\mathfrak{b}}(N))$.

Now, by Theorem 3.4 and Theorem 3.1 (iv), we have

$$\begin{aligned}\mathrm{Ass}(T_{d,\mathfrak{b}}(M,N)) &= \mathrm{Ass}(T_{d,\mathfrak{b}}(\mathrm{Hom}_R(M,N))) = \mathrm{Ass}(\mathrm{Hom}_R(M,T_{d,\mathfrak{b}}(N)))\\ &= \mathrm{Supp}(M) \cap \mathrm{Ass}(T_{d,\mathfrak{b}}(N)) = \mathrm{Supp}(M) \cap \mathrm{Ass}(N/\Gamma_{d,\mathfrak{b}}(N)). \qquad \square\end{aligned}$$

In case that M is finitely generated R-module, in Theorem 3.1 (iii), we showed that $T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \cong T_{d,\mathfrak{b}}(M,N)$. Also, in Theorem 3.4, we proved $T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \cong T_{d,\mathfrak{b}}(\mathrm{Hom}_R(M,N))$. In the next theorem we drop the finiteness assumption on M and conclude these two isomorphisms with some other assumptions on M.

Theorem 3.6. *Let M and N be two R-modules. The followings hold:*
(i) If $\mathrm{Ext}^1_R(M,N) = 0$, then $T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \cong T_{d,\mathfrak{b}}(\mathrm{Hom}_R(M,N))$ and

$$H^i_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \cong H^i_{d,\mathfrak{b}}(\mathrm{Hom}_R(M,N)),$$

for each $i \geq 1$.

(ii) If M is a flat R-module, then

$$T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \cong T_{d,\mathfrak{b}}(M,N),$$

and

$$\Gamma_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \cong 0 \cong H^1_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)).$$

Proof. (i) As $\mathrm{Ext}^1_R(M,N) = 0$, by Proposition 2.1 (viii) we have the exact sequence

$$0 \to \frac{\mathrm{Hom}_R(M,N)}{\Gamma_{d,\mathfrak{b}}(M,N)} \to T_{d,\mathfrak{b}}(M,N) \to H^1_{d,\mathfrak{b}}(M,N) \to 0$$

from which the long sequence

$$\begin{aligned}0 &\to T_{d,\mathfrak{b}}\left(\frac{\mathrm{Hom}_R(M,N)}{\Gamma_{d,\mathfrak{b}}(M,N)}\right) \to T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \to T_{d,\mathfrak{b}}(H^1_{d,\mathfrak{b}}(M,N))\\ &\to T^1_{d,\mathfrak{b}}\left(\frac{\mathrm{Hom}_R(M,N)}{\Gamma_{d,\mathfrak{b}}(M,N)}\right) \to \cdots\end{aligned}$$

is obtained. But, we have $\Gamma_{d,\mathfrak{b}}(H^1_{d,\mathfrak{b}}(M,N)) = H^1_{d,\mathfrak{b}}(M,N)$, which in turn by Theorem 3.1 (i), gives $T_{d,\mathfrak{b}}(H^1_{d,\mathfrak{b}}(M,N)) = 0$. So

$$T_{d,\mathfrak{b}}\left(\frac{\mathrm{Hom}_R(M,N)}{\Gamma_{d,\mathfrak{b}}(M,N)}\right) \cong T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)). \tag{1}$$

Also, since $\Gamma_{d,\mathfrak{b}}(M,N) \cong \Gamma_{d,\mathfrak{b}}(\mathrm{Hom}_R(M,N))$ and $T_{d,\mathfrak{b}}(\Gamma_{d,\mathfrak{b}}(M,N)) = 0$, then from the short exact sequence

$$0 \to \Gamma_{d,\mathfrak{b}}(M,N) \to \mathrm{Hom}_R(M,N)) \to \frac{\mathrm{Hom}_R(M,N)}{\Gamma_{d,\mathfrak{b}}(M,N)} \to 0,$$

we have

$$T_{d,\mathfrak{b}}\left(\frac{\mathrm{Hom}_R(M,N)}{\Gamma_{d,\mathfrak{b}}(M,N)}\right) \cong T_{d,\mathfrak{b}}(\mathrm{Hom}_R(M,N)). \tag{2}$$

Comparing (1) and (2), we conclude that $T_{d,\mathfrak{b}}(\operatorname{Hom}_R(M,N)) \cong T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N))$.

(ii) Since M is a flat R-module, by [8, Theorem 7.7], for each ideal $\mathfrak{a}$, $\mathfrak{a}M \cong \mathfrak{a}\otimes_R M$. Thus, we have

$$\begin{aligned} T_{d,\mathfrak{b}}(M,N) &= \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \operatorname{Hom}_R(\mathfrak{a}M,N) \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \operatorname{Hom}_R(\mathfrak{a}\otimes_R M,N) \\ &\cong \varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \operatorname{Hom}_R(\mathfrak{a},\operatorname{Hom}_R(M,N)) \cong T_{d,\mathfrak{b}}(\operatorname{Hom}_R(M,N)) \cong T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)). \end{aligned}$$

To prove the second part of (ii), one simply use the exact sequence

$$0 \to \Gamma_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \to T_{d,\mathfrak{b}}(M,N) \to T_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \to H^1_{d,\mathfrak{b}}(T_{d,\mathfrak{b}}(M,N)) \to 0.$$

□

The next result gives a necessary condition for the functor $T_{d,\mathfrak{b}}(M,\text{-})$ to be exact. We were not able to find a better sufficient condition for the exactness of $T_{d,\mathfrak{b}}(M,\text{-})$.

Theorem 3.7. *Let M be an R-module. If $T_{d,\mathfrak{b}}(M,\text{-})$ is an exact functor, then $H^i_{d,\mathfrak{b}}(M,N) \cong \operatorname{Ext}^i_R(M,N)$, for all $i \geq 2$ and all R-module N. In addition, if $H^i_{d,\mathfrak{b}}(M,N) \cong \operatorname{Ext}^i_R(M,N)$ for each $i \geq 1$ and each finitely generated R-module N, then $T_{d,\mathfrak{b}}(M,\text{-})$ is exact.*

Proof. By the exactness of $T_{d,\mathfrak{b}}(M,\text{-})$ we have $T^i_{d,\mathfrak{b}}(M,\text{-}) = 0$, for all $i \geq 1$. Thus by Proposition 2.1 (xi), $\varinjlim_{\mathfrak{a}\in\tilde{W}(d,\mathfrak{b})} \operatorname{Ext}^i_R(\mathfrak{a}M,\text{-}) = 0$, for all $i \geq 1$. Now, for two elements $\mathfrak{a}$ and $\mathfrak{b}$ of $\tilde{W}(d,\mathfrak{b})$ with $\mathfrak{a}\subseteq\mathfrak{b}$, we have the commutative diagram

$$\begin{array}{ccccccccc} \operatorname{Ext}^i_R(\mathfrak{b}M,N) & \to & \operatorname{Ext}^{i+1}_R(M/\mathfrak{b}M,N) & \to & \operatorname{Ext}^{i+1}_R(M,N) & \to & \operatorname{Ext}^{i+1}_R(\mathfrak{b}M,N) & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \operatorname{Ext}^i_R(\mathfrak{a}M,N) & \to & \operatorname{Ext}^{i+1}_R(M/\mathfrak{a}M,N) & \to & \operatorname{Ext}^{i+1}_R(M,N) & \to & \operatorname{Ext}^{i+1}_R(\mathfrak{a}M,N) & \to & \cdots \end{array}$$

with exact rows for each $i \geq 1$. Then passing to direct limit over $\tilde{W}(d,\mathfrak{b})$ we obtain the exact sequence

$$T^i_{d,\mathfrak{b}}(M,N) \longrightarrow H^{i+1}_{d,\mathfrak{b}}(M,N) \longrightarrow \operatorname{Ext}^{i+1}_R(M,N) \longrightarrow T^{i+1}_{d,\mathfrak{b}}(M,N) \qquad (3)$$

As both end sides are zero for all $i \geq 1$, we get $H^i_{d,\mathfrak{b}}(M,N) \cong \operatorname{Ext}^i_R(M,N)$, for all $i \geq 2$.

In addition, as each R-module can be written as a direct limit of its finitely generated submodules over a suitable directed set, we deduce that $H^i_{d,\mathfrak{b}}(M,N) \cong \operatorname{Ext}^i_R(M,N)$ for each $i \geq 1$. Now, from the exact sequence (3) we get the exactness of $T_{d,\mathfrak{b}}(M,\text{-})$. □

The following theorem and its proof is a counterpart of [10, Theorem 3.5].

Theorem 3.8. *Let M be a finitely generated R-module and N be an R-module. Then, for each $t \geq 0$ we have*

$$\operatorname{Ass}(T^t_{d,\mathfrak{b}}(M,N)) \subseteq \bigcup_{i=1}^{t} \operatorname{Ass}(E^{i,t-i}_{t+2}) \cup \operatorname{Ass}(\operatorname{Hom}_R(M,T^t_{d,\mathfrak{b}}(N))),$$

and $$\mathrm{Supp}(T^t_{d,\mathfrak{b}}(M,N)) \subseteq \bigcup_{i=0}^{t} \mathrm{Supp}(\mathrm{Ext}^i_R(M, T^{t-i}_{d,\mathfrak{b}}(N))).$$

Proof. We note that by [11, Theorem 10.47] there exists a Grothendieck spectral sequence $E_2^{p,q} = \mathrm{Ext}^p_R(M, T^q_{d,\mathfrak{b}}(N)) \underset{p}{\to} T^{p+q}_{d,\mathfrak{b}}(M,N)$, and modifying the arguments of the proof of [10, Theorem 3.5] gives the results. □

Corollary 3.9. *Let M be a finitely generated R-module, N an R-module and t a non-negative integer. If $\mathrm{Supp}(T^i_{d,\mathfrak{b}}(N))$ is a finite set for all $i \le t$, then $\mathrm{Supp}(T^t_{d,\mathfrak{b}}(M,N))$ is also a finite set.*

Proof. We note that by Theorem 3.8
$$\mathrm{Supp}(T^t_{d,\mathfrak{b}}(M,N)) \subseteq \bigcup_{i=0}^{t} \mathrm{Supp}(\mathrm{Ext}^i_R(M, T^{t-i}_{d,\mathfrak{b}}(N))) \subseteq \bigcup_{i=0}^{t} \mathrm{Supp}(T^{t-i}_{d,\mathfrak{b}}(N)),$$
and the result follows. □

4. Using the sets $W(d,\mathfrak{b})$ and $S^*_k(M,N)$

In this section, we study the connection of the vanishing of the modules $T^i_{d,\mathfrak{b}}(M,N)$, $i \in \mathbb{N}_0$, with the dimension of the set
$$W(d,\mathfrak{b}) := \{\mathfrak{p} \in \mathrm{Spec}(R) |\ \exists\ \mathfrak{a} \in \Sigma_d,\ \mathfrak{a} \subseteq \mathfrak{p} + \mathfrak{b}\},$$
and the dimension of the sets
$$S^*_k(M,N) := \{\mathfrak{p} \in \mathrm{Supp}(M) \mid \mathrm{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) + \dim(R/\mathfrak{p}) \le k\},$$
$k \in \mathbb{N}_0$ (see [4,14]).

For simplicity we put $w = \dim\big(W(d,\mathfrak{b})\big) := \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in W(d,\mathfrak{b})\}$, and $s^*_k(M,N) = \dim\big(S^*_k(M,N)\big) := \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in S^*_k(M,N)\}$.

The other aim of this section is to give a low vanishing theorem for the modules $H^i_{d,\mathfrak{b}}(M,N)$ in case that both M and N are finitely generated. Recall from [8, Theorem 16.7] that for an R-module M and an ideal $\mathfrak{a}$ of R with $\mathfrak{a}M \neq M$, the length of a maximal M-sequence contained in $\mathfrak{a}$, denoted by $\mathrm{depth}(\mathfrak{a},M)$, is a well determined least integer n such that $\mathrm{Ext}^n_R(R/\mathfrak{a},M) \neq 0$. It is known that for each ideal $\mathrm{depth}(\mathfrak{a},M) = \mathrm{Min}\{\mathrm{depth}(\mathfrak{p},M) | \ \mathfrak{p} \supseteq \mathfrak{a}\}$. For a local ring $(R,\mathfrak{m})$ and a nonzero finitely generated R-module M, $\mathrm{depth}(\mathfrak{m},M)$ is simply denoted by $\mathrm{depth}(M)$. Note that, as usual (see [8, p. 131]), by $\mathrm{grade}_R(M)$, we mean $\inf\{i \in \mathbb{N}_0 \mid \mathrm{Ext}^i_R(M,R) \neq 0\}$.

Theorem 4.1. *Let M and N be two finitely generated R-modules, $n \ge 2$ and k be two non-negative integers such that $s^*_{n+k}(M,N) \le k$ for all $k < w$. Then $T^i_{d,\mathfrak{b}}(M,N) \cong \mathrm{Ext}^i_R(M,N)$ for all $i < n-1$.*

Proof. Using the exact sequence of Proposition 2.1 (viii), it suffices to prove that $H^i_{d,\mathfrak{b}}(M,N) = 0$ for all $i < n$. Let $\mathfrak{a} \in \tilde{W}(d,\mathfrak{b})$ and $\mathfrak{p} \in \mathrm{V}(\mathfrak{a} + (0 :_R M))$. Then

$\mathfrak{p} \in W(d, \mathfrak{b})$. Put $k := \dim(R/\mathfrak{p}) - 1$. Then $k < w$ and so $\mathfrak{p} \notin \mathrm{S}^*_{n+k}(M, N)$. Hence, we have

$$\begin{aligned} \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) + \dim(R/\mathfrak{p}) > k + n &\Rightarrow \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) + k + 1 > k + n \\ &\Rightarrow \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) + 1 > n \Rightarrow \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) \geq n. \end{aligned}$$

Then $\operatorname{depth}(\mathfrak{a} + (0 :_R M), N) \geq n$ and so by [8, Theorem 16.6], $\operatorname{Ext}^i_R(M/\mathfrak{a}M, N) = 0$ for all $i < n$. But, as mentioned in introduction, $H^i_{d,\mathfrak{b}}(M, N)$ is the direct limit of $\operatorname{Ext}^i_R(M/\mathfrak{a}M, N)$ over the set $\tilde{W}(d, \mathfrak{b})$. Thus, it follows that $H^i_{d,\mathfrak{b}}(M, N) = 0$ for all $i < n$. □

Corollary 4.2. *Let M and N be two finitely generated R-modules. The following statements hold.*
(i) If $\operatorname{pd}(N) < \operatorname{grade}_R(M) < w$, $s^*_{n+k}(M, N)) \leq k$, then $T^i_{d,\mathfrak{b}}(M, N) = 0$ for all $i < \operatorname{grade}_R(M) - \operatorname{pd}(N) - 1$.

(ii) If $(R, \mathfrak{m})$ is local, $\operatorname{depth}(M) < w$ and $s^*_{n+k}(M, N)) \leq \operatorname{depth}(M)$, then $T^i_{d,\mathfrak{b}}(M, N) = 0$ for all $i < \operatorname{depth}(M) - \dim(N) - 1$.

Proof. (i) By [8, Theorem 16.9], we have $\operatorname{Ext}^i_R(M, N) = 0$ for all $i < \operatorname{grade}_R(M) - \operatorname{pd}(N)$ and the result follows by Theorem 4.1.

(ii) Using [8, Theorem 17.1] and Theorem 4.1, we see that $T^i_{d,\mathfrak{b}}(M, N) = 0$ for all $i < \operatorname{depth}(M) - \dim(N) - 1$. □

Corollary 4.3. *Let M and N be two finitely generated R-modules, and let n and k be two non-negative integers such that $s^*_{n+k}(M, N)) \leq k$ for all $k < w$.*
(i) If $\Gamma_{d,\mathfrak{b}}(N) = N$, then $\operatorname{Ext}^i_R(M, N) = 0$ for all $i < n - 1$.

(ii) If $\operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) \neq \operatorname{depth}_{R_\mathfrak{p}}(\Gamma_{d,\mathfrak{b}}(N)_\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Supp}(M)$, then $\operatorname{Ext}^i_R(M, N) \cong \operatorname{Ext}^i_R(M, N/\Gamma_{d,\mathfrak{b}}(N))$ for all $i < n - 1$.

Proof. (i) As $\Gamma_{d,\mathfrak{b}}(N) = N$, by Theorem 3.1 (i) we have $T^i_{d,\mathfrak{b}}(M, N) = 0$ for all $i \geq 0$. On the other hand, by Theorem 4.1, $\operatorname{Ext}(M, N) \cong T^i_{d,\mathfrak{b}}(M, N)$ for all $i < n-1$. These together gives the result.

(ii) Assume that $\mathfrak{p} \in \operatorname{Supp}(M)$. From the short exact sequence

$$0 \to \Gamma_{d,\mathfrak{b}}(N) \to N \to N/\Gamma_{d,\mathfrak{b}}(N) \to 0,$$

and [7, Proposition 1.29] we have

$$\begin{aligned} &\operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) \geq \min\{\operatorname{depth}_{R_\mathfrak{p}}(\Gamma_{d,\mathfrak{b}}(N)_\mathfrak{p}), \operatorname{depth}_{R_\mathfrak{p}}((N/\Gamma_{d,\mathfrak{b}}(N))_\mathfrak{p})\}, \\ &\operatorname{depth}_{R_\mathfrak{p}}(\Gamma_{d,\mathfrak{b}}(N)_\mathfrak{p}) \geq \min\{\operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}), \operatorname{depth}_{R_\mathfrak{p}}((N/\Gamma_{d,\mathfrak{b}}(N))_\mathfrak{p}) + 1\}, \\ &\operatorname{depth}_{R_\mathfrak{p}}((N/\Gamma_{d,\mathfrak{b}}(N))_\mathfrak{p}) \geq \min\{\operatorname{depth}_{R_\mathfrak{p}}(\Gamma_{d,\mathfrak{b}}(N)_\mathfrak{p}) - 1\ ,\ \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p})\}. \end{aligned}$$

Now, according to the our assumption in the statement of the corollary and these inequalities, we obtain $\operatorname{depth}_{R_\mathfrak{p}}((N/\Gamma_{d,\mathfrak{b}}(N))_\mathfrak{p}) \leq \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p})$. Hence, we have

$$\begin{aligned} &\operatorname{depth}_{R_\mathfrak{p}}((N/\Gamma_{d,\mathfrak{b}}(N))_\mathfrak{p}) + \dim(R/\mathfrak{p}) \leq \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) + \dim(R/\mathfrak{p}) \\ &\Rightarrow s^*_{n+k}(M, N/\Gamma_{d,\mathfrak{b}}(N)) \subseteq s^*_{n+k}(M, N) \Rightarrow s^*_{n+k}(M, N/\Gamma_{d,\mathfrak{b}}(N))) \leq s^*_{n+k}(M, N)) < k. \end{aligned} \tag{4}$$

Since $N/\Gamma_{d,\mathfrak{b}}(N)$ is finitely generated R-module, then in view of (4) and using Theorem 4.1, we get that $T^i_{d,\mathfrak{b}}(M,N) \cong \mathrm{Ext}^i_R(M, N/\Gamma_{d,\mathfrak{b}}(N))$ and $T^i_{d,\mathfrak{b}}(M,N) \cong \mathrm{Ext}^i_R(M,N)$ for all $i < n-1$. Comparing these and using Theorem 3.1 (ii), the proof of the claim is complete. □

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