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THE BOREL-HSIANG-QUILLEN LOCALIZATION THEOREM FOR COMPACT GROUP ACTIONS ON COMPACT SPACES

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Abstract. In this article, we will study the equivariant cohomology theory for actions of compact group (not necessarily Lie group) on compact spaces. We will present a somewhat more general and shorter proof of the localization theorem, known as the Borel-Hsiang-Quillen localization theorem, which was generalized by Özkurt and Onat to actions of finite-dimensional compact groups on compact connected spaces. In particular, we will apply this to the problem of the existence of equivariant maps between topological transformation groups.

1. Introduction

It is well known and a fundamental fact that an action of a topological group on a topological space naturally gives rise to two spaces: the orbit space and the fixed point space. A significant challenge in the field of cohomology theory concerning actions of topological transformation groups pertains to the relationship between the cohomological structure of a given space and the cohomological structures of its orbit space and fixed point set. One of the important tools in the theory is Borel's equivariant cohomology theory. In this context, the research focuses on methods of determining the equivariant cohomology algebra using the cohomological structure of the orbit space or the fixed point set of the given action. For instance, if an action of a group G on a space X is free, then the equivariant cohomology algebra $H_G^*(X)$ is isomorphic to the cohomology algebra of the orbit space X/G of the action. More precisely, if G acts freely on X, then the first projection $X_G \longrightarrow X/G$ is a fibration with a fiber E_G, which is contractible, so $H^*(X/G) \longrightarrow H^*(X_G)$ is an isomorphism. More generally, the relation between the cohomological algebra structure of the fixed point X^G of a given G-space X and the equivariant cohomology algebraic structure $H^*(X_G)$ is given by the following Borel-Hsiang-Quillen localization theorem. On the

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other hand, a criterion for the existence of fixed points of an action can be given in terms of the algebraic structure of the equivariant cohomology (see Corollary 3.3). This is only one of the many important consequences of the localization theorem of Borel-Hsiang-Quillen. The equivariant cohomology for actions of compact Lie groups has been the subject of extensive research; however, comparatively few studies have focused on actions of compact topological groups. Montgomery [15] has proved that any locally compact group which acts effectively on a manifold is finite-dimensional. However it still remains unknown whether or not such a group must be a Lie group (known as the Hilbert-Smith conjecture). So such topological group actions are still actively studied.

THEOREM 1.1. *Suppose that G is a compact Lie group, and that S is a multiplicative closed subset of $H^*(B_G;k)$. Then the localized restriction homomorphism $S^{-1}H^*_G(X;k) \to S^{-1}H^*_G\left(X^S;k\right)$ is an isomorphism of $H^*\left(B_G;k\right)$-modules whenever X is a G-space satisfying one of the following conditions:*

a. X is a compact space [10].

b. X is a paracompact space of finite cohomological dimension, and X has finitely many connective orbit types (briefly FMCOT) [1, 10].

c. X is a finitistic (always paracompact) space which has FMCOT [9].

If X is compact, the assumption that the number of orbit types is finite is not needed.

As can be seen from the above theorem, generalizations of the theorem are made on the topological spaces on which the topological group acts. The generalization of the theorem for topological groups was studied by Özkurt and Onat [18]. The proof of this theorem is contingent on the existence of a slice in the actions of compact Lie groups [5, Chapter II]. The existence of slice in compact group actions is not yet established, but recently, Biller [2] offered a partial solution to the issue. One way to overcome this difficulty is to extend the theorem to actions that are not Lie groups by reducing compact group actions to compact Lie group actions and comparing the cohomological structure of the given space with the cohomological structure of the space on which the compact Lie group acts (for details see [18]). The present article aims to furnish more direct proof of the theorem.

Another important consequence of the localization theorem is the following Borsuk-Ulam type theorem, proved by Clapp and Puppe [7]. The problem of the existence of an equivariant map between G-spaces is a subject that continues to be a major focus in the field of algebraic topology. We will not discuss these theorems at length here. We refer the interested reader to [11, 16, 21]. In particular, many publications related to the Borsuk-Ulam theorem are listed in [21].

THEOREM 1.2. *Let X be a connected G-space. Suppose that Y is a compact space or a paracompact space of finite cohomology dimension or a finitistic space, and Y has no fixed-point. If one of the following conditions is satisfied, then there exists no equivariant map $X \to Y$.*

a) G is a p-torus (i.e. $G = \mathbb{Z}_p^k$*) and*

$$H^i(X;\mathbb{Z}/p) = \{0\} \text{ for all } 0 < i < n,$$
$$H^j(Y;\mathbb{Z}/p) = \{0\} \text{ for all } j \geq n.$$

b) G is a torus (i.e. $G = (\mathbb{S}^1)^m$*),*

$$H^i(X;\mathbb{Q}) = \{0\} \text{ for all } 0 < i < n,$$
$$H^j(Y;\mathbb{Q}) = \{0\} \text{ for all } j \geq n,$$

and Y has FMCOT.

Clapp and Puppe stated this theorem only for G-CW complexes of finite dimension, but the theorem is also realized for spaces satisfying Borel's fixed point criterion (see [10, p.45] or [9]).

The purpose of this note is to present a simpler proof than theirs of the following theorem proved by Özkurt and Onat [18] for a finite-dimensional compact group G and a connected G-space X.

Theorem 1.3. *Assume G is a compact group which acts on a compact space X. Let S be a multiplicative closed subset of the center of $H^*(B_G;k)$. Then the homomorphism $S^{-1}H_G^*(X;k) \to S^{-1}H_G^*(X^S;k)$ is an isomorphism of $H^*(B_G;k)$-modules.*

We also give the following results, which are some of the important consequences of the above theorem.

Theorem 1.4. *Let G be a compact connected abelian group (called pro-torus) acting on a connected space X, and a compact space Y with no fixed points. Suppose that for some integer $n \geq 1$, $H^i(X;k) = \{0\}$ for all $0 < i < n$ and $H^j(Y;k) = \{0\}$ for all $j \geq n$. Then there exists no equivariant map $X \to Y$.*

2. Preliminaries

By $H^n(X;\Lambda)$, we mean the cohomology, in the sense of the theory of sheaves, of the space X with coefficients in the constant sheaf associated to a given ring Λ. When X is a paracompact space, the family of supports is taken as the family of all closed subsets of X. For much more information on the sheaf cohomology we refer the reader to the artifact of Bredon [6]. We also define $H^*(X;\Lambda) = \bigoplus_{n=0}^{\infty} H^n(X;\Lambda)$, which is an algebra with the cup product $\cup$.

In this article, the letter k will be used to denote a field of characteristic zero, while the symbol G will be used to denote a topological group with unit element e. Any notation and terminology not explained here can be found in [1,3,5,10]. In this study, all topological groups and topological spaces are assumed to be Hausdorff, and all maps between these are assumed to be continuous.

A topological transformation group is a triple $\langle G, X, \theta\rangle$ with a topological group G, a topological space X, and an action θ of G on X. A continuous map $\theta : G \times X \longrightarrow X$ is an action if and only if the following conditions are satisfied:

a. $\theta(e, x) = x$ for all $x \in X$.

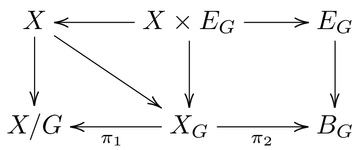
b. $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $g, h \in G$ and $x \in X$.

For brevity, we will write gx for $\theta(g, x)$. A topological space X together with a given θ action will be called G-space.

A continuous map $f : X \longrightarrow Y$ between G-spaces with the property $f(gx) = gf(x)$ for all $x \in X$, $g \in G$ is called an equivariant map or G-map. In this case, a G-map $f : X \longrightarrow Y$ induces the map $\overline{f} : X/G \longrightarrow Y/G$ defined by $\overline{f}(G(x)) = G(f(x))$. A G-map which is also homeomorphism is called G-homoemorphism. The isotropy subgroup of x is the subgroup $G_x = \{g \in G : gx = x\}$ of G. If X is a Hausdorff space, it is a closed subgroup. The G-subspace $G(x) = \{gx \in X : g \in G\}$ of X is called the orbit of $x \in X$. Let us observe that $G(x)$ is G-homeomorphic to the homogenous G-space G/G_x if X is a Hausdorff space and G is a compact group. If $G_x = G$ (or equivalently $G(x) = \{x\}$), then the point $x \in X$ is called the fixed point of the action. The fixed point set of the action will be denoted by X^G, i.e. $X^G = \{x \in X : gx = x \text{ for all } g \in G\}$. The orbit space $X/G = \{G(x) : x \in X\}$ has the topology induced by the quotient map $X \to X/G$, $x \mapsto G(x)$.

The use of the equivariant cohomology, introduced by Borel [3], is a very useful method in examining the cohomological structures of the fixed point space and the orbit space in a topological transformation group. Now let's recall this cohomology theory. It is well-known that there is a universal G-bundle $E_G \to B_G$ for every topological group G [13], where G acts freely on E_G and $B_G = E_G/G$, called the classifying space of G. For any G-space X, there is an action of G on the product space $X \times E_G$ given by $g(x, e) = (gx, ge)$. The orbit space $(X \times E_G)/G$ will be denoted by $X \times_G E_G$ or simply by X_G. Then the equivariant cohomology $H_G^*(X; k)$ of the G-space X is defined by the usual cohomology $H^*(X_G; k)$.

The canonical projections $X \times E_G \to X$ and $X \times E_G \to E_G$ are G-equivariant, and thus they induce the maps $\pi_1 : X_G \to X/G$ and $\pi_2 : X_G \to B_G$. The next commutative diagram is also obtained.



Note that π_2 is a fibration with fiber X and structural group G, but π_1 is generally not a fibration. The fiber of $X_G \to X/G$ over a point $x' = G(x)$ is homeomorphic to E_G/G_x.

Remark 2.1. If G is a compact group acting on any space X, then $H^*(B_{G_x}; k) \cong H^*(E_G/G_x; k)$. In fact, since G_x acts freely on E_{G_x}, then the projection $E_G \times E_{G_x} \longrightarrow E_{G_x}$ on the second factor gives rise to a map $(E_G \times E_{G_x})/G_x \longrightarrow E_{G_x}/G_x = B_{G_x}$, which is a fibration with typical fiber E_G. Since E_G is a contractible space, we conclude that $H^*(B_{G_x}; k) \cong H^*((E_G \times E_{G_x})/G_x; k)$.

Similarly, for the projection $(E_G \times E_{G_x})/G_x \longrightarrow E_G/G_x$ on the first factor, we have $H^*(E_G/G_x; k) \cong H^*((E_G \times E_{G_x})/G_x; k)$. Thus we obtain that $H^*(E_G/G_x; k)$

$\cong H^*(B_{G_x};k)$.

Let X be any G-space and set $R = H^*(B_G;k)$, then $H_G^*(X;k)$ is an R-module via the ring homomorphism $\pi_2^* : H^*(B_G;k) \to H_G^*(X;k)$. For $a \in R$ and $x \in H_G^*(X;k)$, the module (algebra) multiplication is given by $ax := \pi_2^*(a) \cup x$, where $\cup$ is the cup product in $H_G^*(X;k)$. Moreover any G-map $f : X \to Y$ induces the homomorphism $f^* : H_G^*(Y;k) \to H_G^*(X;k)$, which is an R-module homomorphism [10, 19].

In particular, if G is a compact group, then $H^*(B_{G_x};k)$ is an R-module via the map $i_x^* : H^*(B_G;k) \to H^*(B_{G_x};k)$ induced by the map $i_x : E_G/G_x \to E_G/G = B_G$. For a multiplicative closed subset S of the center of R (i.e. $1 \in S$, $ab = ba$ for all $a \in R, b \in S$, and $ab \in S$ for all $a, b \in S$), define $X^S = \{x \in X : i_x^*(s) \neq 0 \text{ for all } s \in S\}$. Note that if X is a paracompact space, then X^S is a closed invariant subset of X by the tautness property of the sheaf cohomology (see [1]). Clearly, the restriction homomorphism $H_G^*(X) \to H_G^*\left(X^S\right)$, induced by the inclusion $X^S \subset X$, is an R-module homomorphism. Hence the localized restriction map $S^{-1}H_G^*(X;k) \to S^{-1}H_G^*\left(X^S;k\right)$ is an R-module homomorphism.

Let us recall some well-known facts in order to prove the localization theorem.

Every element of positive degree in the Čech cohomology ring (isomorphic to the sheaf cohomology for any sheaf on a paracompact Huasdorff space) of a compact space is nilpotent. More generally the following lemma has been proved by Deo et al. [9]. Recall that every compact space is finitistic.

Lemma 2.2. *Assume that X is a finitistic space and $\mathcal{F}$ is a sheaf of k-modules on X. Then for any $\alpha \in H^i(X, \mathcal{F})$ where $i > 0$, there exists a positive integer n_0 so that $\alpha^n = 0 \in H^*(X, \mathcal{F})$ for all $n > n_0$.*

Recall that to any map $f : X \longrightarrow Y$ and sheaf $\mathcal{F}$ on X is associated a Leray spectral sequence $E_2^{p,q} = H^p(Y; \mathcal{H}^q(f)) \Longrightarrow H^{p+q}(X;\mathcal{F})$ where the sheaf $\mathcal{H}^q(f)$ is the sheaf associated to the presheaf $U \mapsto H^q\left(f^{-1}(U);\mathcal{F}\right)$ on Y, its stalks at y is $\mathcal{H}^q(f)_y = \varinjlim_{U \ni y} H^q\left(f^{-1}(U);\mathcal{F}\right)$ [19].

Suppose that F is the decreasing filtration of $H^*(X;k)$ such that

$$F^pH^*(X;k)/F^{p+1}H^*(X;k) = E_\infty^{p,q}.$$

Now, assume that X is any space, Y is a finitistic space, and $f : X \to Y$ is a map. We have the following lemma from [9] for any element $s \in F^1H^*(X;k) \subset H^*(X;k)$.

Lemma 2.3 ([9]). *There is a positive integer n_0 such that $s^{n_0} = 0 \in H^*(X;k)$.*

3. Main results

Using Hsiang's techniques [10, Theorem.III.1′], the localization theorem was generalized for finitistic spaces by Deo et al. [9]. The theorem was first proved in [18] by Özkurt and Onat for the actions of finite-dimensional compact groups on compact spaces. Although not explicitly stated by the authors there, the compact space X

needs to be connected because this hypothesis is necessary for comparing the cohomology groups of the fiber, total and base spaces of two fibrations (Zeeman's comparison theorem [12, p.82]). There, given the action of a finite-dimensional compact connected group G on a connected compact space X, the authors reduce this action to the action of a compact Lie group G/N on a compact space X/N for a totally disconnected closed normal subgroup N of G, and then they prove their main theorem [18, Theorem 3.7] by showing that the cohomology of X/N is isomorphic to the cohomology of X and that the cohomology of the classifying space of G is isomorphic to the cohomology of the classifying space of G/N. They then generalize the connected case to the compact case [18, Theorem 3.8]. Here we provide a direct proof for the actions of compact groups (not necessarily finite-dimensional) on compact spaces (not necessarily connected). In this sense, our theorem is somewhat more general and its proof is briefer than that of Özkurt and Onat.

We will now prove Theorem 1.3 using Deo et al.'s techniques.

Proof (Proof of Theorem 1.3). First we will prove the case $X^S = \emptyset$. For this, it is enough to show that there exists $s \in S$ such that $\pi_2^* (s) = 0$, where $\pi_2^* : H^* (B_G; k) \to H_G^* (X; k)$.

There exists the Leray spectral sequence associated to the map $\pi_1 : X_G \to X/G$, in such a way that $E_2^{p,q} = H^p (X/G; \mathcal{H}^q (\pi_2))$ and there exists a decreasing filtration F^p of $H_G^* (X; k)$ such that $E_\infty^{p,q} = F^p H_G^* (X; k) / F^{p+1} H_G^* (X; k)$ for all p, q. Besides, we obtain that the stalk at the point $G(x) \in X/G$ of the Leray sheaf $\mathcal{H}^q (\pi_1)$ is isomorphic to $H^q (B_{G_x}; k)$ (see [19, p.553]). Since $X^S = \emptyset$, then there exists an $s \in S$ such that s maps to zero under $H^* (B_G; k) \to H^* (B_{G_x}; k)$. Therefore, s maps to zero in $E_2^{0,*} = H^0 (X/G, \mathcal{H}^* (\pi_1))$ and in $E_\infty^{0,*}$. Since the next sequence is exact, we obtain that $\pi_2^* (s) \in F^1 H_G^* (X; k)$:

$$0 \to F^1 H_G^* (X; k) \to F^0 H_G^* (X; k) = H_G^* (X; k) \to E_\infty^{0,*} \to 0.$$

As X/G is compact, we can conclude from Lemma 2.3, some positive power n_0 of $\pi_2^* (s)$ vanishes in $H_G^* (X; k)$, i.e., $(\pi_2^* (s))^{n_0} = \pi_2^* (s^{n_0}) = 0$ and $S^{-1} H_G^* (X; k) = 0$ ($s^{n_0} \in S$).

Now, suppose that $X^S \neq \emptyset$. Let U be a closed invariant neighbourhood of X^S in X and V be the complement of its interior. Then $V^S = (U \cap V)^S = \emptyset$. Furthermore, X_G is a paracompact space ([1, Section 3.2]). Thus we get the following Mayer-Vietoris long exact sequence for the sheaf cohomology [6, Chapter II, Section 13.].

$$\cdots \to H_G^i (X; k) \to H_G^i (U; k) \oplus H_G^i (V; k) \to H_G^i (U \cap V; k) \to \cdots$$

Since the localization is exact functor $S^{-1} H_G^* (X; k) \to S^{-1} H_G^* (U; k)$ is an isomorphism by the first part of the proof. By the tautness property of the sheaf cohomology, since X^S is a closed invariant subspace of X, then $H_G^* \left(X^S; k\right) \cong \varinjlim_N H_G^* (N; k)$, where, in the direct limit, N ranges over all closed invariant neighborhoods of X^S directed downwards by inclusion (see, e.g. [19]). The result is obtained from the fact that the localization commutes with direct limits. □

Remark 3.1. Let us determine when the set X^S is equal to the fixed point set X^G of the action. It is a well-established result that if a pro-torus (i.e. connected abelian compact group) G acts on a space X, then $X^S = X^G$ for the multiplicative closed subset $S = H^*(B_G;k) - \{0\} \subset H^*(B_G;k)$ (see [18]).

This gives the following generalization of Borel's fixed point theorem.

Corollary 3.2 (A. Borel). *Let G be a pro-torus which acts on a compact space X. Then the homomorphism $S^{-1}H_G^*(X;k) \to S^{-1}H_G^*\left(X^G;k\right)$ is an isomorphism for $S = H^*(B_G;k) - \{0\} \subset H^*(B_G;k)$.*

The following was proved in [10, p. 45].

Corollary 3.3 (Borel Fixed Point Criterion). *Let G be a pro-torus which acts on a compact space X. Then $\pi_2^* : H^*(B_G;k) \to H_G^*(X;k)$ is monomorphism iff the fixed point set $X^G \neq \emptyset$.*

We will now show that the fixed point set of actions of finite-dimensional pro-tori on compact acyclic spaces is also an acyclic space, which is one of the important results of the theorem. A topological space X is called k-acyclic space (or acyclic over k) if $H^*(X;k) = H^*(pt;k)$.

Remark 3.4. Since G acts trivially on X^G, then $X_G^G = \left(X^G \times E_G\right)/G = X^G \times B_G$.

To apply the Künneth theorem [6, Chapter IV, 7.6] to this product, we need to overcome some technical problems, namely the cohomology of X^G or B_G must be finitely generated free module in each dimension. Now assume that G is a finite-dimensional compact group. Here by the term dimension, we mean the covering dimension of the topological space. Then one can find a compact totally disconnected normal subgroup N of G such that G/N is a compact Lie group, also $H^*\left(B_{G/N};\mathbb{Q}\right) \longrightarrow H^*(B_G;\mathbb{Q})$ is an isomorphism (for details, see [18]). It is well known the cohomology of the classifying space of a compact Lie group is finite-dimensional [3, Chapter IV], so $H^*(B_G;\mathbb{Q})$ is finite-dimensional. Therefore, by Künneth theorem, we have that $H_G^*\left(X^G;k\right) = H^*\left(X^G;k\right) \otimes_k H^*(B_G;k)$.

This gives the following.

Corollary 3.5. *Let G be a finite-dimensional pro-torus which acts on a compact space X. If X is a k-acylic space, then X^G is also k-acylic space.*

Proof. From the Serre spectral sequence (or Vietoris-Begle mapping theorem [20, p.344]) of $X \to X_G \to B_G$, we obtain that $H^*(B_G;k) \cong H_G^*(X;k)$. Since

$$S^{-1}H_G^*(X;k) \cong H_G^*(X;k) \otimes_{H^*(B_G;k)} S^{-1}H^*(B_G;k) \cong S^{-1}H^*(B_G;k)$$

and

$$S^{-1}H^*(B_G;k) \cong S^{-1}H_G^*(X;k) \cong S^{-1}H_G^*\left(X^G;k\right) \cong H^*\left(X^G;k\right) \otimes_k S^{-1}H^*(B_G;k)$$

we conclude that X^G is a k-acyclic space. □

Using Corollary 3.3 it is not hard to obtain the following.

Theorem 3.6. *Suppose that G is a pro-torus which acts on a connected space X and a compact space Y. Then if the following contiditions hold, there exists no equivariant map $X \to Y$.*

a. The fixed point set Y^G is empty.

b. $H^i(X;k) = \{0\}$ for all $0 < i < n$.

c. $H^j(Y;k) = \{0\}$ for all $j \geq n$.

Proof. Clapp and Puppe's proof [7, Theorem 6.4] is exactly valid using Corollary 3.3. See also [16, Theorem 3.10] for proof. □

Notice that if Y has a fixed-point, then there exists always an equivariant map $f : X \to Y$. Moreover, if $f : X \to Y$ is an equivariant map and X has a fixed-point, then Y has a fixed-point as well.

The following proposition will now be demonstrated: if an action of a finite-dimensional pro-torus on a compact space satisfies the condition of being totally non-homologous to zero, then a fixed point exists for said action. Let G be a topological group acting on a topological space X. Then X is said to be totally non-homologous to zero (TNHZ) in X_G with respect to $H^*(-;R)$ if the inclusion map $i : X \hookrightarrow X_G$ induces a surjection in the cohomology $i^* : H^*_G(X;R) \longrightarrow H^*(X;R)$.

We will need the following well-known lemma, proved in [4] and [14, p. 126].

Lemma 3.7. *If a G-space X is TNHZ in X_G, then the map $\pi_2 : X_G \longrightarrow B_G$ induces a monomorphism $\pi_2^* : H^*(B_G;k) \to H^*_G(X;k)$.*

We obtain the following result (see [17, Corollary 3.5]).

Theorem 3.8. *Suppose that G is a finite-dimensional pro-torus which acts on a compact space X. If X is TNHZ in X_G, then the fixed point set X^G is non-empty.*

Proof. It is clear from Corollary 3.3 and Lemma 3.7. □

Bredon [5, p.425] (see also [1, p.204]) provided an action of a torus G on a space X such that X is not a TNHZ in X_G with respect to $H^*(-;\mathbb{Q})$, but the fixed point set X^G is non-empty.

The author has not yet determined whether Theorem 1.3 is satisfied when the compact space X is replaced by a finitistic space or a paracompact space of finite cohomological dimension. It is important to note that for any finite-dimensional compact group G, the orbit space X/G of a finitistic space X may not be finitistic, but this is true for compact Lie group G [8]. For an action of a finite-dimensional compact group G on a paracompact space X of finite cohomological dimension on $\mathbb{Q}$, the author does not know whether the orbit space X/G also has finite cohomological dimension on $\mathbb{Q}$. It is evident that these two research topics are of significant importance and require further investigation.

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