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A NECESSARY CONDITION FOR THE EXISTENCE OF A FIXED POINT IN COMPACT GROUP ACTIONS ON PARACOMPACT SPACES

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Abstract. It is well known that if the Borel fibration of a compact abelian connected group action on a compact space is totally nonhomologous to zero, then the action must have fixed points. But this is usually not true. In this article, we will give an example that are both TNHZ and the action has no fixed point. Further, we will examine paracompact spaces with TNHZ for compact abelian connected group actions without fixed points, and provide certain necessary conditions for the existence of such actions.

1. Introduction

The continuous action of a topological group on a given topological space gives rise to two distinct spaces, known as the orbit space and the fixed point set. Since these spaces are generally used to determine the equivariant cohomology of the space being acted upon, and thus sometimes the cohomology of the whole space, determining these spaces based on cohomology becomes a central problem. While particularly strong results exist for equivariant cohomology theory, especially concerning actions of abelian groups, there also exist non-abelian versions of many classical results such as the localization theorem [2,13]. Nevertheless, topological groups, without assuming a smooth structure play a significant role in various mathematical contexts, such as the Hilbert-Smith conjecture.

The primary challenge in extending theorems from Lie group actions to topological group actions lies in effectively addressing these unconventional groups. Recently Onat [18–20] and Özkurt, Onat [21, 22] have published various studies related to the Borel-based cohomology theory of compact groups using the approximation method through Lie groups via projective limit, mainly focusing on finite-dimensional compact non-Lie group actions. In these studies, the non-emptiness of the fixed point set and the cohomological algebraic structure of this set holds a significant place.

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Locally compact groups are well-known to admit an "approximation" by Lie groups. Specifically, if G is a locally compact group with a finite number of components, it can be demonstrated that there exists an arbitrarily small compact normal subgroup N of G such that the quotient group G/N is a Lie group. This result was originally established by Yamabe [26] and can also be found in the book of Montgomery and Zippin [17]. If G is a compact (respectively abelian, connected) group of finite dimension, then it is possible to find a totally disconnected compact subgroup N such that G/N is a compact (respectively abelian, connected) Lie group (see for details [23]).

For the dimension definition of a compact group we refer the reader to [22]. If G is a compact group of n-dimension, then $G = \varprojlim_{N \in \mathcal{N}} G/N$ (i.e. G is the projective limit of compact Lie groups of n-dimension), where $\mathcal{N}$ is a filter basis of compact totally disconnected subgroups of G such that $\bigcap \mathcal{N} = \{1\}$ and G/N is a compact Lie group of n-dimension for each $N \in \mathcal{N}$ [12]. It is also well known fact that for a projective system of compact topological groups its projective limit is a compact group. A one-dimensional compact connected abelian (non-Lie) group is called solenoid.

It is a known fact that the fixed point set for TNHZ actions of connected abelian compact Lie groups on compact or paracompact spaces of finite cohomological dimension are not empty, and the cohomological dimension over rationals is equal to the cohomological dimension of the space. This also holds for compact connected abelian non-Lie group action on finite-dimensional compact space (see Onat [20]). On the other hand, in the non-abelian case, fixed point sets of TNHZ actions may be empty sets.

Upon considering the orbit space, it is established that various local (as well as global) topological and cohomological properties inherent to the space X is passed down to the orbit space X/G. For instance, if X possesses compactness, local compactness, paracompactness, local connectedness, or normality, then the corresponding orbit space X/G also exhibits these properties when the acting group G is compact.

2. Preliminaries

Throughout the article, it will be assumed that all topological spaces are Hausdorff, and all groups are compact unless explicitly stated otherwise. Additionally, the utilized cohomology is the Alexander-Spanier cohomology (agree with the Sheaf cohomology and the Čech cohomology for paracompact spaces [8]) with rational coefficients. The present article concerns itself with the equivariant cohomology theory of A. Borel and its application to the theory of topological transformation groups.

Borel [4] introduced a technique for examining the cohomology of G-spaces, which has since become a fundamental tool in the field of topological transformation groups.

It was proved by Milnor [16] that there is a contractible space E_G on which each topological group G acts freely. In fact Milnor only proved that it was weakly contractible. It was shown to be contractible by Dold [10]. The quotient space E_G/G

is denoted by B_G, called the classifying space of G, and the projection $E_G \longrightarrow B_G$ is a G-principal fiber bundle.

Assume that G acts (not necessarily freely) on a topological space X. Then G acts freely on $X \times E_G$ given by $g\,(x,e) = (gx, ge)$ for $g \in G, (x,e) \in X \times E_G$. The orbit space $(X \times E_G)/G$ is called the Borel construction, denoted by X_G. The fibration $X \xrightarrow{i} X_G \longrightarrow B_G$ is called the Borel fibration [4].

Definition 2.1. A G-space X is said to be the totally non-homologous to zero (TNHZ) in $X_G \longrightarrow B_G$ if $i^* : H^*(X_G) \longrightarrow H^*(X)$ is surjective.

Remark 2.2. It is well-known as a consequence of the Localization theorem (it holds not only to actions of compact Lie group actions but also holds for actions of finite-dimensional compact groups; see [22]) that the action of torus groups on compact or paracompact spaces with finite cohomological dimension being TNHZ is equivalent to the cohomological dimension of the fixed-point set matching the cohomological dimension of the space. This situation also pertains to the actions of a connected abelian compact group of finite-dimension (see [20]).

If the condition of the group is abelian, as stated in Remark 2.2, is removed, then there may not be fixed points in the actions of TNHZ. A rather well-known example can be given as follows. Let G be a connected compact Lie group, and let T be its maximal torus. Then there is the transitive action of G on the homogeneous space G/T. Hence the fixed point set $F(G, G/T)$ is empty. On the other hand, because $H^{odd}(G/T; \mathbb{Q}) = 0$, then the Leray-Serre spectral sequence of the Borel fibration $G/T \longrightarrow (G/T)_G \longrightarrow B_G$ collapses, so the induced map $H^*_G(G/T) \longrightarrow H^*(G/T)$ is surjective.

Definition 2.3. A G-space X is said to has finitely many connective orbit types (FMCOT) if the set $\left\{\left[G_x^0\right] : x \in X\right\}$ is finite, where $\left[G_x^0\right]$ stands for the conjugate class of the identity component of the isotropy subgroup G_x in G.

If $G = \mathbb{S}^1$, then, clearly, any G-space has FMCOT.

Remark 2.4. Since $(G/N)_{N(x)} = NG_x/N$ ([12, Proposition 10.31] or [9]) and by [12, Lemma 9.18] $(G/N)^0_{N(x)} = \left(NG_x^0\right)/N$ for every $x \in X$. It follows that the action of G/N on X/N has FMCOT when an action of G on X has FMCOT.

Lemma 2.5 ([2, Lemma 4.2.1]). *If G is a torus and X is any G-space has FMCOT, then there is a subcircle $\mathbb{S}^1 \subset G$ such that their fixed point sets are the same, that is $F\left(\mathbb{S}^1, X\right) = F(G, X)$.*

Consequently, under the additional assumption of FMCOT, the demonstration of the results concerning fixed points of torus actions is only applicable to circle actions.

Lemma 2.6 ([14, 20]). *Suppose that G is a connected compact group, N is a totally disconnected normal compact subgroup of G, and X be a G-space. Then $F(G,X) \approx F(G/N, X/N)$.*

Lemma 2.7 ([14,19]). *If G is a compact, totally disconnected group, then $H^*(B_G;\mathbb{Q}) = \mathbb{Q}$.*

Theorem 2.8 ([6,15]). *Let N be a compact, totally disconnected group which acts on a compact space X. The homomorphism $H^*(X/N;\mathbb{Q}) \longrightarrow (H^*(X;\mathbb{Q}))^N$ induced by the orbit map $X \longrightarrow X/N$, is an isomorphism.*

If G is a connected compact group that acts on a compact space X, the induced action of G on $H^*(X)$ is trivial [8, Corollary 11.11]. If G is a connected compact group acting on a compact space X, and N is a compact totally disconnected subgroup of G, then G/N acts on X/N, and $H^*(X/N;\mathbb{Q}) \cong H^*(X;\mathbb{Q})$. Consequently, numerous issues pertaining to the cohomological properties of fixed point sets (or orbit spaces) for the actions of compact connected groups of finite dimension are reduced to problems concerning compact connected Lie group actions.

The proof of the following theorem can be found in [25, p. 344].

Theorem 2.9 (Vietoris-Begle mapping theorem). *Let $f : X \longrightarrow Y$ be a surjective closed map between paracompact spaces and let G be an abelian group. Suppose that there is some $n \geq 0$ such that $\widetilde{H}^q\left(f^{-1}(y);G\right) = 0$ for all $y \in Y$ and $q < n$. Then the induced homomorphism $f^q : \widetilde{H}^q(Y;G) \longrightarrow \widetilde{H}^q(X;G)$ is an isomorphism for $q < n$ and a monomorphism for $q = n$.*

3. Main results

The theory of the actions of compact abelian Lie groups on compact or paracompact spaces of finite cohomological dimension, and with the TNHZ property are well-known. The fixed point set plays an important role in understanding such actions. We want to extend this theory to non-Lie group actions. Therefore, as a motivation, let us first provide examples of compact abelian non-Lie group actions with the TNHZ property.

One way to define compact non-Lie group action is to consider the projective limit of the actions of compact Lie groups. To be more specific: If $\{G_\alpha, \varphi_\alpha^\beta, I\}$ is a projective system of compact groups (with continuous group homomorphisms φ_α^β), and $\{X_\alpha, f_\alpha^\beta, I\}$ is a projective system of compact spaces (with continuous maps f_α^β) such that each G_α acts on X_α, and each map f_α^β is φ_α^β-equivariant (i.e. $f_\alpha^\beta(g_\beta x_\beta) = \varphi_\alpha^\beta(g_\beta) f_\alpha^\beta(x_\beta)$ for all $g_\beta \in G_\beta, x_\beta \in X_\beta$), then the compact group $\varprojlim G_\alpha$ acts on the compact space $\varprojlim X_\alpha$, where the action is given by $(g_\alpha)(x_\alpha) = (g_\alpha x_\alpha)$ for $(g_\alpha) \in \varprojlim G_\alpha$ and $(x_\alpha) \in \varprojlim X_\alpha$.

It is well-known that if a compact connected abelian Lie group acting on a compact X or paracompact of finite cohomological dimension X with FMCOT, and X is TNHZ in $X_G \longrightarrow B_G$, then $H^*(B_G;\mathbb{Q}) \longrightarrow H_G^*(X;\mathbb{Q})$ is injective [5, Theorem 14.2], so $F(G,X) \neq \emptyset$ by the Borel's fixed point criterion [13, p.45]. The same argument holds for the actions of connected abelian compact group of finite dimension on compact spaces [20, 22].

Remark 3.1. Quillen [24] demonstrated that if X is a paracompact space, and G is a compact Lie group acting on X, then the cohomological dimension of the orbit space X/G cannot exceed the cohomological dimension of X. As a result of this fact the localization theorem and thus the Borel's fixed point criterion remain valid in the paracompact case.

However, it is not yet known whether Quillen's theorem holds for non-Lie group actions. It may, therefore, be worthwhile to initiate an investigation into the following question: If G is a connected abelian compact non-Lie group of finite dimension, and X be a paracompact and non-compact G-space of finite cohomological dimension, and X is TNHZ in $X_G \longrightarrow B_G$, then is the set of fixed points non-empty?

First, let us give an example of a compact connected abelian (non-Lie) group action on a compact space with a fixed point, which has the TNHZ property.

Example 3.2. Let p be a fixed prime number. Set $G_n = \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and define $\varphi_n^{n+1} : G_{n+1} \longrightarrow G_n, \varphi_n^{n+1}(z) = z^p$. Similarly, set $X_n = \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and define $f_n^{n+1} : X_{n+1} \longrightarrow X_n, f_n^{n+1}(w) = w^p$ for all $n \in \mathbb{N}$. We consider the action of G_n on X_n given by multiplication. It is clear that each bonding map f_n^{n+1} is φ_n^{n+1}-equivariant. So the projective limit, $X = \varprojlim X_n$, of the inverse system $\{X_n, f_n^{n+1}, \mathbb{N}\}$ is a compact $T_p = \varprojlim G_n$-space with one fixed point $(0, 0, \ldots)$. The group T_p is compact connected abelian non-Lie group is called p-adic solenoid. Clearly the kernel of the first limit map, $f_1 : T_p \longrightarrow G_1 = \mathbb{S}^1$, which are p-adic integers is the totally disconnected closed normal subgroup denoted by Z_p. Since the limit maps are surjective there exists an exact sequence $0 \longrightarrow Z_p \longrightarrow T_p \longrightarrow G_1 = \mathbb{S}^1 \longrightarrow 0$. The induced $T_p/Z_p = G_1$ action on $X/Z_p = X_1$ is TNHZ and from the following commutative diagram,

$$\begin{array}{ccc} H^*_{G_1}(X_1; \mathbb{Q}) & \longrightarrow & H^*(X_1; \mathbb{Q}) \\ \downarrow & & \downarrow \cong \\ H^*_{T_p}(X; \mathbb{Q}) & \longrightarrow & H^*(X; \mathbb{Q}) \end{array}$$

we have $H^*_{T_p}(X; \mathbb{Q}) \longrightarrow H^*(X; \mathbb{Q})$ is surjective.

However, if the space X is a paracompact, non-compact and of infinite-cohomological dimension with the TNHZ property may not necessarily have fixed points. Here is an example of such a compact connected abelian Lie group action.

Example 3.3. Let T be a torus and H be a proper subtorus of T. Let E_T be the contractible space on which T acts freely. It is clear that the free space on which H acts can be chosen as E_T. Thus the torus T/H acts freely on $E_T/H = B_H$. Hence we have that the Borel fibration $B_H \longrightarrow (B_H)_{T/H} \longrightarrow B_{T/H}$. In fact we may take for B_T the space $\left(E_T \times E_{T/H}\right)/T = \left(B_H \times E_{T/H}\right)/(T/H) = (B_H)_{T/H}$. Therefore, the Borel fibration $B_H \longrightarrow (B_H)_{T/H} \longrightarrow B_{T/H}$ becomes $B_H \longrightarrow B_T \longrightarrow B_{T/H}$. Since the base, and fibre spaces have only even-dimensional cohomologies, then the Serre spectral sequence for $B_H \longrightarrow B_T \longrightarrow B_{T/H}$ in rational cohomology collapses, so B_H is TNHZ in $B_T \longrightarrow B_{T/H}$. It is well-known that the classifying space B_G of any compact Hausdorff group G is paracompact Hausdorff. Thus we obtain a connected abelian compact Lie group action on a paracompact space such that it is TNHZ without fixed points.

The following lemma provides a generalization of Lemma 2.5 to compact connected abelian group actions.

LEMMA 3.4. *Let G be a compact connected abelian group that acts on any space X with FMCOT. Then there exists a solenoidal subgroup H of G such that $F(H, X) = F(G, X)$.*

Proof. Let $K_1, K_2, \ldots, K_n$ be the connective isotropy subgroups other than G itself. First we will prove that there is a solenoidal subgroup H of G such that $H \cap K_i$ is totally disconnected for $1 \leq i \leq n$. Assume on the contrary, for each solenoidal subgroup H of G, there was a K_i such that $H \cap K_i$ is not totally disconnected. Since the closed subgroups of a solenoidal group are either totally disconnected or H itself then for each H there exists a K_i such that $H \subset K_i$. Therefore, the union of all solenoidal subgroups H of G is in the closed subgroup $K_1 \cup K_2 \cup \cdots \cup K_n$ of G other than G. This is a contradiction because the union of all solenoidal subgroups is dense in G. (Since the union of all circle subgroups of a torus is dense in the torus and each solenoidal subgroup of G is the inverse image of a circle under the quotient morphism $G \longrightarrow G/N$, where G/N is a torus and N is a compact totally disconnected normal subgroup of G.)

Now let's show that $F(H, X) = F(G, X)$ for this H. If there exists a point $x \in F(H, X) - F(G, X)$, then $H \subset G_x^0 \subset G_x \neq G$, and H is connected and $H = H \cap G_x^0$ is totally disconnected. This is a contradiction, so we have $F(H, X) = F(G, X)$. $\square$

THEOREM 3.5. *Let G be a finite-dimensional compact connected abelian group acting on any paracompact, non-compact and of infinite cohomological dimension space X with FMCOT. If $F(G, X) = \emptyset$ and X is TNHZ in $X_G \longrightarrow B_G$, then there exists a solenoidal subgroup H of G such that $H^*(X/H; \mathbb{Q}) \longrightarrow H^*(X; \mathbb{Q})$ is surjective.*

Proof. There exists a solenoidal subgroup H of G such that $F(H, X) = F(G, X) = \emptyset$ by Lemma 3.4. Note that if X is a paracompact H-space, then $X \times E_H$ is a paracompact space by [3, Chap. IX, §4, Ex. 20(d)], so $X_H = (X \times E_H)/H$ is a paracompact space. It is clear that $H = \varprojlim_{N \in \mathcal{N}} HN/N$, where $\mathcal{N}$ is a filter basis of normal compact subgroups of H such that HN/N is a circle group for every $N \in \mathcal{N}$ and $\bigcap \mathcal{N} = 1$. Then, by [11, Theorem 2.2] $H_x = \varprojlim_{N \in \mathcal{N}} H_x N/N = \varprojlim_{N \in \mathcal{N}} (HN/N)_{Nx}$. Since $F(H, X) = \emptyset$, all $(HN/N)_{Nx}$ are finite (cyclic) groups and, thereby H_x is compact abelian totally disconnected group. By Lemma 2.7, we have that $H^*(B_{H_x}; \mathbb{Q}) = \mathbb{Q}$. Then the orbit map $\pi_1 : X_H \longrightarrow X/H$ induces the isomorphism $\pi_1^* : H^*(X/H; \mathbb{Q}) \longrightarrow H_H^*(X; \mathbb{Q})$ by the Vietoris-Begle mapping theorem. Further, it follows from the commutative diagram below that $H_H^*(X;\mathbb{Q}) \longrightarrow H^*(X;\mathbb{Q})$ is surjective.

$$\begin{array}{ccc} H_G^*(X;\mathbb{Q}) & \longrightarrow & H_H^*(X;\mathbb{Q}) \\ \downarrow & \swarrow & \\ H^*(X;\mathbb{Q}) & & \end{array}$$

Thereby, from the following commutative diagram

$$\begin{array}{ccc} H^*(X/H;\mathbb{Q}) & \xrightarrow{\cong} & H_H^*(X;\mathbb{Q}) \\ \downarrow & \swarrow & \\ H^*(X;\mathbb{Q}) & & \end{array}$$

it is clear that $H^*(X/H;\mathbb{Q}) \longrightarrow H^*(X;\mathbb{Q})$ is surjective. □

The fact that G is an abelian group constitutes a significant argument for Lemma 3.4. The following remark provides a partial generalization of Lemma 3.4 for compact non-totally disconnected groups.

Remark 3.6. Let G be a compact non-totally disconnected group acting on a compact space X with fixed point set $F = F(G, X)$ such that G_x is finite for $x \notin F$. It is well-known fact that if $K = G_0$ is the identity component of G then the quotient G/K is totally disconnected. This is true for any topological group (see [23, Section 22] for details).

Since G is a non-totally disconnected group, then K is a nontrivial connected closed subgroup of G. The dimension of K may not be finite. In this case, we can choose a non-trivial (arbitrarily high) finite-dimensional compact connected subgroup of K. Therefore, it may be hypothesised that the dimension of G is finite. Let N be a compact totally disconnected normal subgroup of K such that K/N is a Lie group. Since G_x is finite for $x \notin F$, it clear that $F(K, X) = F(G, X)$.

Suppose now that T is any circle subgroup of K/N. If we assume $x^* = N(x) \in F(T, X/N) - F(K/N, X/N)$ then $T < (K/N)_{x^*} = K_xN/N$ and $x \notin F(K, X) = F(G, X)$ by Lemma 2.6. Thus K_x and K_xN/N are finite, this is a contradiction. So we have $F(G, X) = F(K, X) \approx F(K/N, X/N) = F(T, X/N)$.

In Remark 3.6, we can replace the finiteness of the isotropy subgroups with the triviality of the cohomology algebra of classifying spaces of isotropy subgroups.

Lemma 3.7. *Let G be a non-totally disconnected compact group acting on a space X with fixed point set $F = F(G, X)$ such that $H^*(B_{G_x};\mathbb{Q})$ is trivial for $x \notin F$. Then there exists a solenoidal subgroup H of G such that $F(G, X) = F(H, X)$.*

Proof. As indicated in Remark 3.6, since G is a compact non-totally disconnected group, we can assume G to be of finite dimension.

Let n be the dimension of G and $\mathcal{N}$ is a filter basis of totally disconnected, normal compact subgroups of G such that G/N is a compact Lie group of n -dimension for each $N \in \mathcal{N}$ and $\bigcap \mathcal{N} = \{1\}$.

If $K = G_0$ is the identity component of G then there is an $N \in \mathcal{N}$ such that KN/N is a compact connected Lie group of finite dimension. Thus the compact Lie group, G/N, contains a circle subgroup T. Therefore there exists a solenoidal subgroup H of G such that $T = HN/N$.

If we assume $x \in F(H, X) - F(G, X)$ then we have $x^* = N(x) \in F(T, X/N) - F(G/N, X/N)$ then $T < (G/N)_{x^*} = G_x N/N$. From the Peter Weyl theorem, we can assume $T < (G/N)_{x^*} < Gl(k, \mathbb{C})$ for some $k \in \mathbb{N}$. So we have the non-trivial composition $H^*\left(B_{Gl(k,\mathbb{C})};\mathbb{Q}\right) \longrightarrow H^*\left(B_{(G/N)_{x^*}};\mathbb{Q}\right) \longrightarrow H^*\left(B_T;\mathbb{Q}\right)$ which implies $H^*\left(B_{(G/N)_{x^*}};\mathbb{Q}\right)$ is not trivial. Furthermore, from Remark of [22], we obtain the result that $H^*\left(B_{G_x};\mathbb{Q}\right)$ is not trivial, which leads to a contradiction. So we have $F(G, X) = F(H, X)$. □

Remark 3.8. The actions of T_p mentioned in Example 3.2 can easily be verified to be of the type indicated in Lemma 3.7.

From Lemma 3.7 (similarly from Remark 3.6), we have the following theorem, derived from a proof analogous to that of Theorem 3.5.

Theorem 3.9. *Let G be a non-totally disconnected compact group acting on a paracompact space X such that $H^*\left(B_{G_x};\mathbb{Q}\right)$ is trivial for all $x \in X$ (similarly G_x is finite for $x \notin X$). If $F(G, X) = \emptyset$ and X is TNHZ in $X_G \longrightarrow B_G$, then there exists a solenoidal subgroup H of G_0 such that $H^*\left(X/H;\mathbb{Q}\right) \longrightarrow H^*\left(X;\mathbb{Q}\right)$ is surjective.*

Definition 3.10. A connected topological space X is called Poincaré duality space over $\mathbb{Q}$ if $H^*\left(X;\mathbb{Q}\right)$ is a finite-dimensional vector space over $\mathbb{Q}$ and if $H^i\left(X;\mathbb{Q}\right) = 0$ for $i > n$, and the cup product $H^i\left(X;\mathbb{Q}\right) \times H^{n-i}\left(X;\mathbb{Q}\right) \stackrel{\cup}{\longrightarrow} H^n\left(X;\mathbb{Q}\right) = \mathbb{Q}$ is a non-degenerate bilinear form for all $0 \le i \le n$. The number n is called the formal dimension of X, denoted by $fd\left(X\right)$.

The following corollary implies that a non-totally disconnected compact group can not act freely on a compact Poincaré duality space in a way that satisfies the TNHZ property.

Corollary 3.11. *Let G be a non-totally disconnected compact group which acts on a compact Poincaré duality space X with formal dimension n. If G_x is finite for all $x \in X$, then X can not be TNHZ in $X_G \longrightarrow B_G$.*

While G is also connected and abelian in the Corollary, it is clear that X cannot be TNHZ in $X_G \longrightarrow B_G$, because the fixed point set of the action is empty.

Proof. Suppose that X is TNHZ in $X_G \longrightarrow B_G$, then there exists a solenoidal subgroup H of G_0 such that $H^*(X/H;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ is surjective by Theorem 3.9. Consider the restricted action of H on X. Let M be a totally disconnected subgroup of H such that the quotient H/M is a circle group.

Theorem 2.8 asserts that the orbit map $\pi : X \to X/M$ induces isomorphism $\pi^* : H^*(X/M;\mathbb{Q}) \to H^*(X;\mathbb{Q})$. Hence $H^i\left(X/M;\mathbb{Q}\right) = 0$ for all $i > n$. Then, by [2, Corollary 3.10.12], we have $H^i\left(\left(X/M\right)/\left(H/M\right);\mathbb{Q}\right) = H^i\left(X/H;\mathbb{Q}\right) = 0$ for all $i > n$. Observe that the stabilizers of the action of circle group H/M on X/M $(H/M)_{M(x)} = H_x M/M \simeq H_x/H_x \cap M$ is finite (in other words, the action of H/M on X/M is almost free). Then from the following Gysin exact sequence for the orbit map $X/M \longrightarrow (X/M)/(H/M) \approx X/H$

$$0 \to H^1(X/H;\mathbb{Q}) \to H^1(X/M;\mathbb{Q}) \to H^0(X/H;\mathbb{Q}) \to H^2(X/H;\mathbb{Q}) \to$$
$$\cdots \to H^n(X/M;\mathbb{Q}) \to H^{n-1}(X/H;\mathbb{Q}) \to H^{n+1}(X/H;\mathbb{Q}) \to \cdots$$

we have $H^n(X/H;\mathbb{Q}) \neq 0$ (because $H^*(X/H;\mathbb{Q}) \longrightarrow H^*(X;\mathbb{Q})$ is surjective) and $H^i(X/H;\mathbb{Q}) \cong H^{i+2}(X/H;\mathbb{Q})$ for all $i \geq n$, which leads to a contradiction. □

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