

CERTAIN GENERALIZED LIMITS DEFINED BY IDEALS AND σ -MEAN IN GRADUAL NORMED SPACES

Archana Sharma and Vijay Kumar

Abstract. The purpose of this communication is to define and illustrate the notions of $\mathcal{J}(S_\sigma : \mathcal{G})$ -summable, $\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})$ -summable and $\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})$ -summable sequences in gradual normed linear spaces (briefly abbreviated as *GNLP*) and to demonstrate some connections between their corresponding sequence spaces.

1. Introduction and preliminaries

In this section we collect some basic information of statistical convergence, ideal convergence, and *GNLS*.

Fast [2] presented a method of sum for divergent sequences using the density of subsets of $\mathbb{N}$, called as statistical convergence. The idea is later explored by several authors including [1, 4–7, 9, 11–14]. “A sequence $t = (t_k)$ of numbers is said to converge statistically to a number t_0 if for each $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |t_k - t_0| \geq \varepsilon\}|_c = 0$$

where $|\cdot|_c$ denotes the cardinality of the enclosed set. In this case, we will write $S - \lim_{k \rightarrow \infty} t_k = t_0$ or simply $t_k \rightarrow t_0(S)$ as $k \rightarrow \infty$.”

Let $\mathbb{Z}^+$ denote the set of positive integers, l_∞ , the space of all bounded real sequences of numbers, and $\sigma : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ is a map. “A continuous linear functional φ on l_∞ is called an σ -mean or invariant mean if it satisfies the following:

- (i) $\varphi(t) \geq 0$ where $t = (t_n)$ with $t_n \geq 0$ for all n ;
- (ii) $\varphi(t) = 1$ if $t = (t_n)$ with $t_n = 1$ for all n and
- (iii) $\varphi((t_{\sigma^m(n)})) = \varphi(t)$ if $t = (t_n)$ is in l_∞ .”

2020 Mathematics Subject Classification: 40A35, 40A05, 03E72

Keywords and phrases: Statistical convergence; σ -mean; $\mathcal{J}$ -convergence; linear spaces of gradual norm.

The map φ is injective and $\sigma^m(n) \neq n$, $\forall m$ and n . If the map is defined by $\sigma : n \rightarrow n + 1$, then σ -mean is usually called a Banach limit.

Savaş and Nuray [11] used σ -mean to define an interesting generalization of statistical convergence with the help of lacunary sequences as follows: “Let $\vartheta = (\vartheta_r)$ be a lacunary sequence, i.e., a sequence of positive integers satisfying $\vartheta_0 = 0$, $h_r = \vartheta_r - \vartheta_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Denote $I_r = (\vartheta_{r-1}, \vartheta_r]$ and the ratio $q_r = \vartheta_r / \vartheta_{r-1}$. A sequence $t = (t_k)$ of numbers is said to be lacunary statistically convergent to a number t_0 if for each $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |t_{\sigma^k(n)} - t_0| \geq \varepsilon\}|_c = 0.$$

In this case, we write $S_\sigma(\vartheta) - \lim_{k \rightarrow \infty} t_k = t_0$ or $t_k \rightarrow t_0(S_\sigma(\vartheta))$ as $k \rightarrow \infty$.”

Kostryrko et al [8] gave another generalization of statistical convergence using ideals. “For any nonempty set X , let $\mathcal{P}(X)$ denote the power set of X . A family of subsets $\mathcal{J} \subset \mathcal{P}(X)$ is said to be ideal in X if and only if

- (i) for each $T_1, T_2 \in \mathcal{J}$, $T_1 \cup T_2 \in \mathcal{J}$;
- (ii) for each $T_1 \in \mathcal{J}$ and $T_2 \subseteq T_1$ implies $T_2 \in \mathcal{J}$.

An ideal $\mathcal{J}$ is called nontrivial if $\mathcal{J} \neq \emptyset$ and $X \notin \mathcal{J}$. The filter $\mathcal{F} = \mathcal{F}(\mathcal{J}) = \{X - A : A \in \mathcal{J}\}$ is called the filter associated with the ideal $\mathcal{J}$. A non-trivial ideal $\mathcal{J} \subset \mathcal{P}(X)$ is called an admissible ideal in X if and only if $\{\{x\} : x \in X\} \in \mathcal{J}$. For a non-trivial admissible ideal $\mathcal{J} \subset \mathcal{P}(\mathbb{N})$, A real sequence $t = (t_k)$ is said to be $\mathcal{J}$ -convergent to t_0 if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in \mathbb{N} : |t_k - t_0| \leq \varepsilon\} \in \mathcal{J}$. In this case, we write $\mathcal{J} - \lim_{k \rightarrow \infty} t_k = t_0$.”

On the other side, Fortin et al. [3] gave the concept of gradual numbers to deal with complex uncertainty and vagueness in complex data. They considered fuzzy intervals as crisp sets of entities instead of fuzzy sets. These numbers have wide applications in the examination of fuzzy intervals and fuzzy optimization theory. “A gradual real number $\tilde{n}$ is an assignment function $A_{\tilde{n}} : (0, 1] \rightarrow \mathbb{R}$. Let $\mathcal{G}(\mathbb{R})$ denote the set of all gradual real numbers. In addition, any $\tilde{n} \in \mathcal{G}(\mathbb{R})$ is said to be positive if $A_{\tilde{n}}(\eta) \geq 0$ for $\eta \in (0, 1]$. Let $\mathcal{G}^*(\mathbb{R})$ denote the set of such numbers. For $\tilde{n}_1, \tilde{n}_2 \in \mathcal{G}(\mathbb{R})$, $\tilde{n}_1 \circ \tilde{n}_2 \in \mathcal{G}(\mathbb{R})$ is defined by $A_{\tilde{n}_1 \circ \tilde{n}_2}(\eta) = A_{\tilde{n}_1}(\eta) \circ A_{\tilde{n}_2}(\eta)$ for $\eta \in (0, 1]$; where $A_{\tilde{n}_1 \circ \tilde{n}_2}$ denotes the assignment function of $\tilde{n}_1 \circ \tilde{n}_2 \in \mathcal{G}(\mathbb{R})$. Similarly, $\tilde{n}_1 + \tilde{n}_2$ and $c\tilde{n}_1$ for $\tilde{n}_1, \tilde{n}_2 \in \mathcal{G}(\mathbb{R})$ and $c \in \mathbb{R}$ are defined by $A_{\tilde{n}_1 + \tilde{n}_2}(\eta) = A_{\tilde{n}_1}(\eta) + A_{\tilde{n}_2}(\eta)$ for $\eta \in (0, 1]$ and $A_{c\tilde{n}_1}(\eta) = cA_{\tilde{n}_1}(\eta)$ for $\eta \in (0, 1]$. Moreover, if $A_{\tilde{l}}(\eta) = l$ for $\eta \in (0, 1]$, then $\tilde{l}$ is called the constant gradual real number.” Using the above operations on $\mathcal{G}(\mathbb{R})$ Sadeqi and Azari [10] defined the gradual normed linear space as follows.

DEFINITION 1.1. Let $\mathcal{X}$ be a real vector space. The function $\|\cdot\|_{\mathcal{G}} : \mathcal{X} \rightarrow \mathcal{G}^*(\mathbb{R})$ is said to be a gradual norm on $\mathcal{X}$, if for every $\eta \in (0, 1]$, following three properties are satisfied for any $x, y \in \mathcal{X}$

$$(\mathcal{G}_1) \quad A_{\|x\|_{\mathcal{G}}}(\eta) = A_{\tilde{0}}(\eta) \text{ iff } x = \theta, \text{ the zero vector of } \mathcal{X};$$

$$(\mathcal{G}_2) \quad A_{\|\lambda x\|_{\mathcal{G}}}(\eta) = |\lambda| A_{\|x\|_{\mathcal{G}}}(\eta) \text{ for any } \lambda \in \mathbb{R};$$

$$(\mathcal{G}_3) \quad A_{\|x+y\|_{\mathcal{G}}}(\eta) \leq A_{\|x\|_{\mathcal{G}}}(\eta) + A_{\|y\|_{\mathcal{G}}}(\eta).$$

The pair $(\mathcal{X}, \|\cdot\|_{\mathcal{G}})$ is called a gradual normed linear space or briefly *GNLS*.

Let $\mathcal{X} = \mathbb{R}^n$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $0 < \eta \leq 1$. If $\|\cdot\|_{\mathcal{G}}$ on $\mathbb{R}^n$ is defined by

$$A_{\|x\|_{\mathcal{G}}}(\eta) = e^{\eta} \sum_{j=1}^n |x_j|$$

then Sadeqi and Azari [10] have shown that $(\mathcal{X}, \|\cdot\|_{\mathcal{G}})$ is a *GNLS*. “A sequence $w = (t_k)$ in *GNLS*: $(\mathcal{X}, \|\cdot\|_{\mathcal{G}})$ is said to be convergent to t_0 w.r.t the gradual norm $\|\cdot\|_{\mathcal{G}}$, if for every $\eta \in (0, 1]$ and $\varepsilon > 0$, there exist $m \in \mathbb{N}$ s.t $A_{\|t_k - t_0\|_{\mathcal{G}}}(\eta) < \varepsilon, \forall k \geq m$. In this case, we write $\mathcal{G} - \lim_{k \rightarrow \infty} t_k = t_0$.”

“A sequence $t = (t_k)$ in *GNLS*: $(\mathcal{X}, \|\cdot\|_{\mathcal{G}})$ is said to be Cauchy w.r.t gradual norm $\|\cdot\|_{\mathcal{G}}$, if for every $\eta \in (0, 1]$ and $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ s.t $A_{\|t_k - t_m\|_{\mathcal{G}}}(\eta) < \varepsilon, \forall k, m \geq m_0$.”

In this paper, we incorporate certain generalized summability methods, called $\mathcal{J}(S_{\sigma} : \mathcal{G})$ -summability, $\mathcal{J}(S_{\sigma} : \vartheta : \mathcal{G})$ -summability, and $\mathcal{J}(N_{\sigma} : \vartheta : \mathcal{G})$ -summability in *GNLS* and study some relations between the corresponding sequence spaces.

2. Spaces of $S_{\sigma}(\mathcal{J} : \mathcal{G})$ summable and $S_{\sigma}(\mathcal{J} : \vartheta : \mathcal{G})$ summable sequences

Throughout this communication, unless otherwise specified, $(\Omega, \|\cdot\|_{\mathcal{G}})$ be any arbitrary *GNLS*, σ denotes an invariant mean or σ -mean, $\vartheta = (\vartheta_r)$ be any lacunary sequence and $\mathcal{J} \subseteq \mathcal{P}(\mathbb{N})$ be an admissible ideal.

DEFINITION 2.1. A sequence $\omega = (\omega_k)$ in $(\Omega, \|\cdot\|_{\mathcal{G}})$ is said to be $\mathcal{J}$ -sigma statistical summable or $\mathcal{J}(S_{\sigma} : \mathcal{G})$ -summable to ω_0 w.r.t $\|\cdot\|_{\mathcal{G}}$ if for $\varepsilon > 0, \gamma > 0$ and $0 < \zeta \leq 1$

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ 0 \leq k \leq n : \mathcal{A}_{\|\omega_{\sigma k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \right\} \right| \geq \gamma \right\} \in \mathcal{J}$$

uniformly in $m = 1, 2, \dots$

We denote it by $\mathcal{J}(S_{\sigma} : \mathcal{G}) - \lim_{k \rightarrow \infty} \omega_k = \omega_0$ or simply $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(S_{\sigma} : \mathcal{G})$) as $k \rightarrow \infty$. Let $[\mathcal{J}(S_{\sigma} : \mathcal{G})] = \left\{ \omega = (\omega_k) \in \Omega : \mathcal{J}(S_{\sigma} : \mathcal{G}) - \lim_{k \rightarrow \infty} \omega_k \text{ exists} \right\}$ uniformly in $m = 1, 2, \dots$

DEFINITION 2.2. A sequence $\omega = (\omega_k)$ in $(\Omega, \|\cdot\|_{\mathcal{G}})$ is said to be $\mathcal{J}$ -lacunary sigma statistical summable or $\mathcal{J}(S_{\sigma} : \vartheta : \mathcal{G})$ -summable to ω_0 w.r.t $\|\cdot\|_{\mathcal{G}}$ if for $\varepsilon > 0, \gamma > 0$ and $0 < \zeta \leq 1$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \mathcal{A}_{\|\omega_{\sigma k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \right\} \right| \geq \gamma \right\} \in \mathcal{J}$$

uniformly in $m = 1, 2, \dots$

We denote it by $\mathcal{J}(S_{\sigma} : \vartheta : \mathcal{G}) - \lim_{k \rightarrow \infty} \omega_k = \omega_0$ or simply $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(S_{\sigma} : \vartheta : \mathcal{G})$) as $k \rightarrow \infty$. Let $[\mathcal{J}(S_{\sigma} : \vartheta : \mathcal{G})] = \left\{ \omega = (\omega_k) \in \Omega : \mathcal{J}(S_{\sigma} : \vartheta : \mathcal{G}) - \lim_{k \rightarrow \infty} \omega_k \text{ exists} \right\}$ uniformly in $m = 1, 2, \dots$

DEFINITION 2.3. A sequence $\omega = (\omega_k)$ in $(\Omega, \|\cdot\|_{\mathcal{G}})$ is said to be $\mathcal{J}$ -lacunary strongly σ -summable or $\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})$ -summable to ω_0 w.r.t $\|\cdot\|_{\mathcal{G}}$ if for $\varepsilon > 0$, $\gamma > 0$ and $0 < \zeta \leq 1$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \right\} \in \mathcal{J}$$

uniformly $m = 1, 2, \dots$.

We denote it by, $\mathcal{J}(N_\sigma : \vartheta : \mathcal{G}) - \lim_{k \rightarrow \infty} \omega_k = \omega_0$ or simply $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})$) as $k \rightarrow \infty$. Let

$$[\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})] = \left\{ \omega = (\omega_k) \in \Omega : \mathcal{J}(N_\sigma : \vartheta : \mathcal{G}) - \lim_{k \rightarrow \infty} \omega_k \text{ exists, uniformly } m = 1, 2, \dots \right\}.$$

We now give the following example.

EXAMPLE 2.4. Let $\Omega = \mathbb{R}^m$ and if $\|\cdot\|_{\mathcal{G}} : \Omega \rightarrow G^*$ is defined by

$$\mathcal{A}_{\|t\|_{\mathcal{G}}}(\eta) = e^\eta \sum_{i=1}^m |t_i|$$

where $t = (t_1, t_2, \dots, t_m) \in \mathcal{X}$ and $0 < \eta \leq 1$, then $(\mathbb{R}^m, \|\cdot\|_{\mathcal{G}})$ is a *GNLS*.

Let $\vartheta = (\vartheta_r)$ be a lacunary sequence. Define a sequence by

$$\omega = \begin{cases} \omega_k = (0, 0, \dots, 1), (0, 0, \dots, 2) \dots (0, 0, \dots, \lfloor \sqrt{h_r} \rfloor) & \text{for } k = \sigma^n, \\ \omega_k = \theta & \text{otherwise,} \end{cases}$$

where $n = \vartheta_{r-1} + 1, \vartheta_{r-1} + 2, \dots, \vartheta_{r-1} + \lfloor \sqrt{h_r} \rfloor$, $m \geq 1$, $\theta = (0, 0, \dots, 0)$ and $\lfloor p \rfloor$ denotes the greatest integer of the number p . By the structure of the sequence it is clear that $\omega = (\omega_k)$ is not bounded w.r.t. the gradual norm $\|\cdot\|_{\mathcal{G}}$ and for $\varepsilon > 0$ and $0 < \zeta \leq 1$

$$\frac{1}{h_r} |\{k \in I_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \theta\|_{\mathcal{G}}}(\zeta) \geq \varepsilon\}|_c \leq \frac{\lfloor \sqrt{h_r} \rfloor}{h_r} \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

Thus, for $\gamma > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \theta\|_{\mathcal{G}}}(\zeta) \geq \varepsilon\}|_c \geq \gamma \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{\lfloor \sqrt{h_r} \rfloor}{h_r} \geq \gamma \right\},$$

and is in $\mathcal{J}$ as $\lim_{r \rightarrow \infty} \frac{\lfloor \sqrt{h_r} \rfloor}{h_r} = 0$. This shows that $\omega_k \rightarrow \theta$ ($\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})$). Further,

$$\frac{1}{h_r} \sum_{k \in I_r} \mathcal{A}_{\|\omega_{\sigma^k(m)} - \theta\|_{\mathcal{G}}}(\zeta) = \frac{1}{h_r} \left(\frac{\lfloor \sqrt{h_r} \rfloor (\lfloor \sqrt{h_r} \rfloor + 1)}{2} \right) \rightarrow \frac{1}{2} \quad \text{as } r \rightarrow \infty; \quad (1)$$

so if we choose $\varepsilon = \frac{1}{4}$, then by (1)

$$\begin{aligned} \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathcal{A}_{\|\omega_{\sigma^k(m)} - \theta\|_{\mathcal{G}}}(\zeta) \geq \frac{1}{4} \right\} &= \left\{ r \in \mathbb{N} : \lfloor \sqrt{h_r} \rfloor (\lfloor \sqrt{h_r} \rfloor + 1) h_r \geq \frac{1}{2} \right\} \\ &= \{r_0, r_0 + 1, r_0 + 2, \dots\} \in \mathcal{F}(\mathcal{J}) \end{aligned}$$

by admissibility of $\mathcal{J}$. This shows that $\omega_k \nrightarrow \omega_0$ ($\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})$).

THEOREM 2.5. $[\mathcal{J}(S_\sigma : \mathcal{G})]$, $[\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})]$ and $[\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})]$ are linear spaces.

Proof. We prove the result for $[\mathcal{J}(S_\sigma : \mathcal{G})]$ and the other two cases can be proved similarly. Let $\omega = (\omega_k), \tau = (\tau_k) \in [\mathcal{J}(S_\sigma : \mathcal{G})]$ and α, β be any two nonzero scalars (as otherwise, i.e., if $\alpha = \beta = 0$ or one of α and β is zero, the case follows easily). There exists ω_0 and $\tau_0 \in \Omega$ s.t for each $\varepsilon > 0$, $\gamma > 0$ and $0 < \zeta \leq 1$

$$T_1 = \left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ 0 \leq k \leq n : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_G}(\zeta) \geq \frac{\varepsilon}{|\alpha|} \right\} \right|_c \geq \frac{\gamma}{2} \right\} \in \mathcal{J}$$

$$T_2 = \left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ 0 \leq k \leq n : \mathcal{A}_{\|\tau_{\sigma^k(m)} - \tau_0\|_G}(\zeta) \geq \frac{\varepsilon}{|\beta|} \right\} \right|_c \geq \frac{\gamma}{2} \right\} \in \mathcal{J};$$

uniformly in $m = 1, 2, \dots$. Let

$$T = \left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ 0 \leq k \leq n : \mathcal{A}_{\|(\alpha \omega_{\sigma^k(m)} + \beta \tau_{\sigma^k(m)}) - (\alpha \omega_0 + \beta \tau_0)\|_G}(\zeta) \geq \varepsilon \right\} \right|_c \geq \gamma \right\},$$

then $T \subseteq T_1 \cup T_2$ and so in $\mathcal{J}$. This shows that $[\mathcal{J}(S_\sigma : \mathcal{G})]$ is linear space. $\square$

The next theorem permits the following relationships in the spaces $[\mathcal{J}(S_\sigma : \mathcal{G})]$, $[\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})]$ and $[\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})]$.

THEOREM 2.6. For any sequence $\omega = (\omega_k)$ in Ω , we have the following.

(i) $[\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})] \subset [\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})]$ and the inclusion is proper.

(ii) In addition, if $\omega = (\omega_k)$ is bounded in Ω w.r.t. the gradual norm $\|\cdot\|_G$ and $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})$), then $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})$), i.e., $[\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})] \subseteq [\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})]$.

(iii) $[\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})] \cap l_\infty(\mathcal{G}) = [\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})] \cap l_\infty(\mathcal{G})$, where $l_\infty(\mathcal{G})$ denotes the space of bounded sequences in Ω w.r.t. $\|\cdot\|_G$.

Proof. (i) Assume $\omega = (\omega_k)$ in Ω be s.t $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})$) as $k \rightarrow \infty$. For $\varepsilon > 0$ and $0 < \zeta \leq 1$

$$\begin{aligned} \sum_{k \in I_r} \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_G}(\zeta) &= \sum_{\substack{k \in I_r \\ \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_G}(\zeta) \geq \varepsilon}} \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_G}(\zeta) + \sum_{\substack{k \in I_r \\ \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_G}(\zeta) < \varepsilon}} \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_G}(\zeta) \\ &\geq \sum_{\substack{k \in I_r \\ \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_G}(\zeta) \geq \varepsilon}} \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_G}(\zeta) \\ &\geq \varepsilon \left| \left\{ k \in I_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_G}(\zeta) \geq \varepsilon \right\} \right|. \end{aligned}$$

This gives for each $\gamma > 0$

$$\begin{aligned} &\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_G}(\zeta) \geq \varepsilon \right\} \right|_c \geq \gamma \right\} \\ &\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_G}(\zeta) \geq \varepsilon_1, \varepsilon_1 = \varepsilon \gamma \right\}. \end{aligned}$$

Since $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})$), the later set is in $\mathcal{J}$, which implies the former set also in $\mathcal{J}$ and therefore $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})$) as $k \rightarrow \infty$.

To see the inclusion is strict, please see Example 2.4 above.

(ii) Let $\omega = (\omega_k)$ is bounded w.r.t. $\|\cdot\|_{\mathcal{G}}$ and $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})$). Let $M > 0$ s.t for for all k, m and $0 < \zeta \leq 1$

$$\mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \leq M. \quad (2)$$

For $\varepsilon > 0, \gamma > 0$

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) &= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \frac{\varepsilon}{2}}} \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \\ &\quad + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) < \frac{\varepsilon}{2}}} \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \\ &\leq \frac{M}{h_r} \left| \left\{ k \in I_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \frac{\varepsilon}{2} \right\} \right|_c + \frac{\varepsilon}{2} \quad \text{by (2)} \end{aligned}$$

and so

$$\begin{aligned} &\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \right\} \\ &\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \right\} \right|_c \geq \gamma \right\} \in \mathcal{J} \left(\gamma = \frac{\varepsilon}{2M} \right). \end{aligned}$$

This shows that $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(N_\sigma : \vartheta : \mathcal{G})$) as $k \rightarrow \infty$.

(iii) This is a direct implication of (i) and (ii). $\square$

We now consider the relationships between the spaces $[\mathcal{J}(S_\sigma : \mathcal{G})]$, $[\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})]$.

THEOREM 2.7. *If the lacunary sequence $\vartheta = (\vartheta_r)$ satisfies the condition $\liminf_{r \rightarrow \infty} q_r > 1$, then $[\mathcal{J}(S_\sigma : \mathcal{G})] \subseteq [\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})]$.*

Proof. Let $\omega = (\omega_k)$ be in $[\mathcal{J}(S_\sigma : \mathcal{G})]$ s.t $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(S_\sigma : \mathcal{G})$). Since $\liminf_{r \rightarrow \infty} q_r > 1$ so there exists $M > 0$ s.t $q_r \geq 1 + M$ for sufficiently large r and therefore we have

$$\frac{h_r}{\vartheta_r} \geq \frac{M}{M+1}$$

Further, for $\varepsilon > 0$ and $0 < \zeta \leq 1$, for sufficiently large r and uniformly $m = 1, 2, 3, \dots$

$$\begin{aligned} \frac{1}{\vartheta_r} \left| \left\{ k \leq \vartheta_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \right\} \right|_c &\geq \frac{1}{\vartheta_r} \left| \left\{ k \in I_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \right\} \right|_c \\ &= \frac{h_r}{\vartheta_r} \left(\frac{1}{h_r} \left| \left\{ k \in I_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \right\} \right|_c \right) \\ &\geq \frac{M}{M+1} \left(\frac{1}{h_r} \left| \left\{ k \in I_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \right\} \right|_c \right). \end{aligned}$$

This implies that for $\gamma > 0$, we have the containment

$$\begin{aligned} & \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \{ k \in I_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \} \right|_c \geq \gamma \right\} \\ & \subseteq \left\{ r \in \mathbb{N} : \frac{1}{\vartheta_r} \left| \{ k \leq \vartheta_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \} \right|_c \geq \frac{\gamma M}{M+1} \right\}. \end{aligned}$$

Since $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(S_\sigma : \mathcal{G})$), so the later set belongs to $\mathcal{J}$ and therefore by second property of the ideal $\mathcal{J}$, the former set belongs to $\mathcal{J}$. Hence, $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})$) i.e., $\omega = (\omega_k) \in [\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})]$. $\square$

In the next theorem, we assume that the lacunary sequence $\vartheta = (\vartheta_r)$ satisfies the following condition: For any set $T \in \mathcal{F}(\mathcal{J})$

$$\bigcup \left\{ n \in \mathbb{N} : \vartheta_{r-1} < n \leq \vartheta_r, r \in T \right\} \in \mathcal{F}(\mathcal{J}). \quad (3)$$

THEOREM 2.8. *If the lacunary sequence $\vartheta = (\vartheta_r)$ satisfies condition $\limsup_{r \rightarrow \infty} q_r < \infty$ and (3), then $[\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})] \subseteq [\mathcal{J}(S_\sigma : \mathcal{G})]$.*

Proof. Let $\omega = (\omega_k) \in [\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})]$ with $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})$) as $k \rightarrow \infty$. As $\limsup_{r \rightarrow \infty} q_r < \infty$, so we can find $M > 0$ s.t $q_r < M$ for all r .

For any $\varepsilon > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$ and $0 < \zeta \leq 1$, define the sets:

$$\begin{aligned} T_1 &= \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \{ k \in I_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \} \right|_c < \gamma_1 \right\} \quad \text{and} \\ T_2 &= \left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \} \right|_c < \gamma_2 \right\}. \end{aligned}$$

Since $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})$) we have $\emptyset \neq T_1 \in \mathcal{F}(\mathcal{J})$.

For all $j \in T_1$, if we define

$$K_j = \frac{1}{h_j} \left| \{ k \in I_j : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \} \right|_c < \gamma_1.$$

then for any $n \in \mathbb{N}$ s.t $\vartheta_{r-1} < n < \vartheta_r$, we can write

$$\begin{aligned} & \frac{1}{n} \left| \{ k \leq n : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \} \right|_c \leq \frac{1}{\vartheta_{r-1}} \left| \{ k \leq \vartheta_r : \mathcal{A}_{\|\omega_{\sigma^k(m)} - \omega_0\|_{\mathcal{G}}}(\zeta) \geq \varepsilon \} \right|_c \\ &= \left(\frac{\vartheta_1}{\vartheta_{r-1}} \right) K_1 + \left(\frac{\vartheta_2 - \vartheta_1}{\vartheta_{r-1}} \right) K_2 \cdots + \left(\frac{\vartheta_r}{\vartheta_{r-1}} \right) K_r \\ &\leq \left(\frac{\vartheta_r}{\vartheta_{r-1}} \right) \sup_{r \in T_1} K_r = q_r \left\{ \sup_{r \in T_1} K_r \right\} < M\gamma_1. \end{aligned}$$

So if we select $\gamma_2 = \frac{\gamma_1}{M}$, then by the facts that

$$\bigcup \left\{ n \in \mathbb{N} : \vartheta_{r-1} < n \leq \vartheta_r, r \in T_1 \right\} \subseteq T_2 \quad \text{and} \quad T_1 \in \mathcal{F}(\mathcal{J}),$$

we have $T_2 \in \mathcal{F}(\mathcal{J})$. This implies that $T_2^C \in \mathcal{J}$ and therefore, we have $\omega_k \rightarrow \omega_0$ ($\mathcal{J}(S_\sigma : \mathcal{G})$). Thus, $[\mathcal{J}(S_\sigma : \vartheta : \mathcal{G})] \subseteq [\mathcal{J}(S_\sigma : \mathcal{G})]$. $\square$

3. Conclusion

The gradual norm follows a more flexible structure than the classical norm and is used to model complex problems involving gradual or smooth variations. In the present work, we define generalized sums in gradual normed spaces, which can be applied to convergence-related problems where the usual norm is inadequate and a generalized norm, such as the gradual norm, is more appropriate.

ACKNOWLEDGEMENT. We would like to thank the reviewers of the paper for their valuable comments and suggestions.

REFERENCES

- [1] J. Connor, *The statistical and strong p -Cesaro convergence of sequences*, Analysis, **8(2)** (1988), 47–63.
- [2] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244.
- [3] J. Fortin, D. Dubois, H. Fargier, *Gradual numbers and their application to fuzzy interval analysis*, IEEE Trans. Fuzzy Syst., **16(2)** (2008), 388–402.
- [4] J. A. Fridy, *On statistical convergence*, Analysis, **4** (1985), 301–313.
- [5] J. A. Fridy, *Lacunary statistical summability*, J. Math. Anal. Appl., **173(2)** (1993), 497–504.
- [6] J. A. Fridy, C. Orhan, *Lacunary statistical convergence*, Pacific J. Math., **160(1)** (1993), 43–51.
- [7] R. Goswami, B. C. Tripathy, *On I -statistical convergence in G -metric spaces*, Southeast Asian J. Math. Math. Sci., **21(1)** (2025), 23–32.
- [8] P. Kostyrko, T. Salat, W. Wilczyński, *I -convergence*, Real Anal. Exchange, **26** (2000), 669–686.
- [9] M. Mursaleen, *λ -statistical convergence*, Math. Slovaca, **50(1)** (2000), 111–115.
- [10] I. Sadeqi, F. Y. Azari, *Gradual normed linear space*, Iran J. Fuzzy Syst., **8(5)** (2011), 131–139.
- [11] E. Savaş, F. Nuray, *On sigma-statistically convergence and lacunary sigma-statistically convergence*, Math. Slovaca, **43(3)** (1993), 309–315.
- [12] B. Tamuli, B. C. Tripathy, *Generalized difference lacunary weak convergence of sequences*, Sahand Commun. Math. Anal., **21(2)** (2023), 195–206.
- [13] B. Tamuli, B. C. Tripathy, *Lacunary weak convergence of sequences defined by Orlicz function*, J. Appl. Anal., **31(1)** (2024), doi:10.1515/jaa-2024-0027.
- [14] S. Bera, B. Tamuli, B. C. Tripathy, *Lacunary statistical convergence sequences defined by D -Orlicz function*, Ann. Univ. Craiova Math. Comput. Sci. Ser., **52(1)** (2025), 202–212.

(received 13.05.2025; in revised form 13.04.2026; available online 12.10.2026)

Department of Mathematics, University Institute of Sciences, Chandigarh University, India
E-mail: dr.archanasharma1022@gmail.com
 ORCID iD: <https://orcid.org/0009-0000-4810-6611>

Department of Mathematics, University Institute of Sciences, Chandigarh University, India
E-mail: kaushikvjy@gmail.com
 ORCID iD: <https://orcid.org/0000-0002-8839-0597>