

EXACT OPEN-LOCATING-DOMINATING NUMBERS OF HELM, CYCLE CORONA, AND GEAR GRAPHS

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Abstract. An open-locating-dominating (OLD) set in a graph is a subset of vertices that dominates the graph and distinguishes all pairs of vertices via their open neighborhoods within the set. In this paper, we determine the exact OLD number $\gamma_{\text{OLD}}(G)$ for three families of structured graphs. We first establish that for helm graphs H_n and cycle coronae $C_n \circ K_1$, the parameter equals n for all $n \geq 3$, except for $n = 4$ where $\gamma_{\text{OLD}}(H_4) = 6$. We then focus on gear graphs G_n , where standard degree-based bounds prove insufficient. By modeling the problem via binary sequences and analyzing forbidden local patterns, we prove that for $n \geq 4$, $\gamma_{\text{OLD}}(G_n) = 2\lceil 2n/3 \rceil$, and $\gamma_{\text{OLD}}(G_3) = 5$.

1. Introduction

All graphs considered in this paper are finite, simple, and undirected. For a graph $G = (V, E)$ and a vertex $v \in V$, $N(v)$ denotes the *open neighborhood* of v (the set of neighbors excluding v). A set $S \subseteq V$ is an *open-locating-dominating set* (OLD-set) if it satisfies two conditions:

1. **Total Domination:** For every $v \in V$, $N(v) \cap S \neq \emptyset$. That is, every vertex must be adjacent to at least one vertex in S (a vertex in S does not dominate itself).

2. **Location:** For every pair of distinct vertices $x, y \in V$, $N(x) \cap S \neq N(y) \cap S$.

The *open-locating-dominating number*, denoted $\gamma_{\text{OLD}}(G)$, is the minimum cardinality of such a set. We assume G contains no isolated vertices.

The concept of OLD sets was first introduced by Honkala et al. [6] in the context of identifying codes for hypercubes, and was later generalized to arbitrary graphs by Seo and Slater [12]. The problem is directly motivated by fault detection in processor networks: if each vertex in S is a diagnostic device monitoring its neighbors, the **total domination** condition ensures every processor is monitored, and the **location** condition ensures that any single faulty processor triggers a unique set of detectors.

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Variants such as error-correcting OLD sets have also been proposed to handle more complex fault scenarios [3].

The OLD problem is known to be NP-hard even for specific graph classes such as interval and permutation graphs, as shown by Foucaud et al. [4]. Jean and Rall [7] further explored the properties of locating-dominating sets, establishing connections to other domination parameters. Consequently, research has focused on finding bounds and exact values for specific families. Henning and Yeo [5] established tight bounds for cubic graphs, while Chellali et al. [2] provided a comprehensive analysis for trees.

More recently, attention has shifted to determining exact values for structured graph families constructed via operations. Argiroffo et al. [1] analyzed how the OLD number behaves under operations such as adding a universal vertex and taking the generalized corona. This line of inquiry has led to exact results for several regular structures: Savić et al. [10] determined the OLD number for classes of convex polytopes, Raza [9] solved the problem for rotationally-symmetric graphs including prisms, and Maksimović et al. [8] analyzed generalized Petersen graphs. Most recently, Savić [11] determined the exact value for Flower snarks. The methods in these works rely primarily on degree-based bounds or direct structural case analysis tailored to each specific family; such arguments do not extend directly to gear graphs, whose subdivision structure creates a mixed-degree setting in which standard bounds are not tight.

In this paper, we contribute to this body of work by determining exact values for three families: helm graphs, cycle coronae, and gear graphs. We focus on these families as they model fundamental network variations—hierarchical clusters (helms) and buffered rings with signal repeaters (gears)—providing a rigorous test case for how structural dilation using pendants or subdivisions affects the OLD number. A key contribution of this work is the analysis of gear graphs. Unlike families where standard degree-based bounds are tight, gear graphs require a more strictly local analysis. We employ a technique based on identifying forbidden local patterns in binary sequences to bridge the gap between standard bounds and the true value.

2. Preliminaries

We state a fundamental lemma on the distinguishability of pairs of vertices in a cycle.

LEMMA 2.1. *Let $n \geq 3$. For distinct indices $i, j \in \{0, \dots, n-1\}$, the unordered pairs $\{i-1, i+1\}$ and $\{j-1, j+1\}$ (indices modulo n) are distinct, except when $n = 4$ and $j \equiv i+2 \pmod{4}$.*

Proof. Let $P_i = \{i-1, i+1\}$ and $P_j = \{j-1, j+1\}$. If $P_i = P_j$, then either the elements match directly ($i-1 \equiv j-1 \implies i = j$) or they are swapped ($i-1 \equiv j+1$ and $i+1 \equiv j-1$). Summing the swapped congruences implies $2i \equiv 2j \pmod{n}$. Subtracting them implies $-2 \equiv 2 \pmod{n} \implies 4 \equiv 0 \pmod{n}$. For $n \geq 3$, this forces $n = 4$. In this case, indices i and $i+2$ yield identical pairs $\{i+1, i+3\}$ and $\{i+3, i+1\}$. $\square$

3. Helm graphs and cycle coronae

The *wheel graph* W_n consists of a cycle C_n (vertices $r_0, \dots, r_{n-1}$) and a hub h adjacent to all r_i .

- The **helm graph** H_n is obtained from W_n by attaching a pendant vertex u_i to each rim vertex r_i .
- The **cycle corona** $C_n \circ K_1$ is the subgraph of H_n induced by $V(H_n) \setminus \{h\}$.

See Figure 1 for an illustration of both graphs.

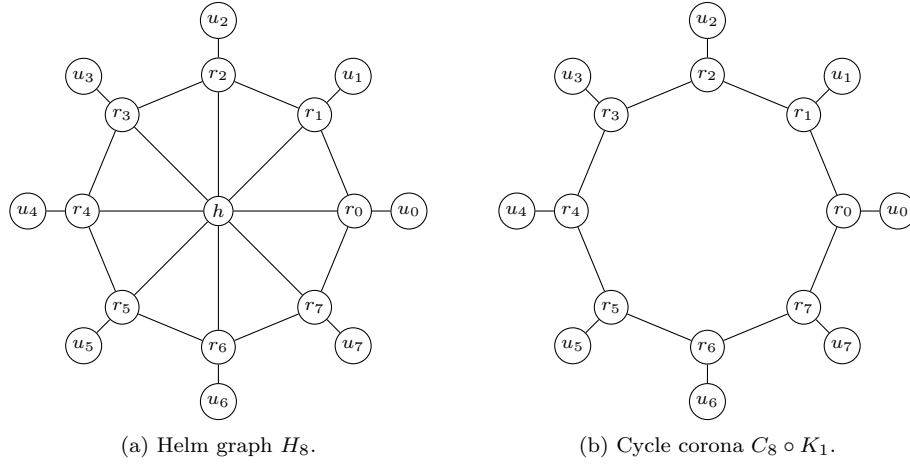


Figure 1: Helm and cycle corona graphs for $n = 8$.

THEOREM 3.1. *For every $n \geq 3$,*

$$\gamma_{\text{OLD}}(H_n) = \gamma_{\text{OLD}}(C_n \circ K_1) = \begin{cases} n & \text{if } n \neq 4, \\ 6 & \text{if } n = 4. \end{cases}$$

Proof. We present the proof for H_n ; the logic for $C_n \circ K_1$ is identical as the hub plays no role in the lower bound constraints derived from the pendants.

Lower bound. Let S be an OLD-set. Each pendant u_i has a single neighbor r_i . To dominate u_i , we must have $r_i \in S$. Since this holds for all $i \in \{0, \dots, n-1\}$, the set of rim vertices $R = \{r_0, \dots, r_{n-1}\}$ must be a subset of S . Thus $|S| \geq n$.

Upper bound ($n \neq 4$). Let $S = R$.

- **Domination:** u_i is dominated by $r_i \in S$; r_i is dominated by $r_{i-1} \in S$; h is dominated by all $r_i \in S$.
- **Location:** The intersections are $N(u_i) \cap S = \{r_i\}$, $N(r_i) \cap S = \{r_{i-1}, r_{i+1}\}$, and $N(h) \cap S = R$. Distinctness of intersections within the sets $\{u_i\}$ and $\{r_i\}$

follows from the distinctness of indices and Lemma 2.1. The sets corresponding to different types (pendants vs rim vertices vs hub) are distinguished by their cardinalities (1, 2, and n respectively), as $n \neq 2$.

The case $n = 4$. The set $S = R$ fails because $N(r_0) \cap S = \{r_1, r_3\} = N(r_2) \cap S$. *Minimality:* Since the lower bound argument shows $R \subseteq S$, we have $|S| \geq n = 4$. Suppose $|S| = 5$. Since $R \subseteq S$, any set of size 5 is of the form $S = R \cup \{x\}$. If $x = h$, then $N(r_0) \cap S = \{r_1, r_3, h\}$ and $N(r_2) \cap S = \{r_1, r_3, h\}$, so the collision persists. If $x = u_k$ (say u_0), then $N(r_1) \cap S = \{r_0, r_2\}$ and $N(r_3) \cap S = \{r_0, r_2\}$; the collision simply shifts to the other pair of rim vertices. By symmetry, adding any single pendant fails. Thus, at least 6 vertices are required. *Construction:* $S' = \{r_0, r_1, r_2, r_3, u_0, u_1\}$ is a valid OLD-set of size 6. The pendants u_0, u_1 modify the neighborhoods of r_0 and r_1 sufficiently to distinguish all pairs. $\square$

REMARK 3.2. The proof above relies entirely on the pendant structure: each u_i has only r_i as a neighbor, forcing $r_i \in S$ for all i and leaving no flexibility in the rim. The gear graph, treated in the next section, has no pendant vertices, and an analogous forcing argument yields only a loose lower bound. A more refined analysis via binary sequences is required.

4. Gear graphs

The gear graph G_n ($n \geq 3$) is obtained from W_n by subdividing every rim edge $r_i r_{i+1}$ with a vertex u_i . The vertex set is $V(G_n) = \{h\} \cup R \cup U$, where $U = \{u_0, \dots, u_{n-1}\}$. The neighborhoods are $N(u_i) = \{r_i, r_{i+1}\}$ and $N(r_i) = \{h, u_{i-1}, u_i\}$ (see Figure 2).

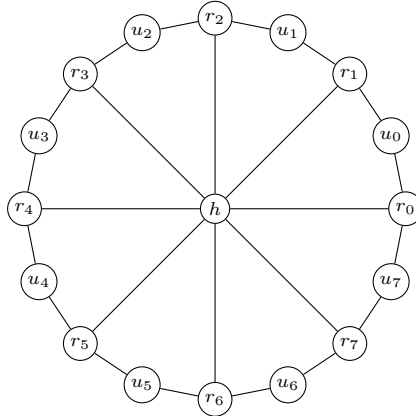


Figure 2: The gear graph G_8 .

4.1 Constraint analysis via binary sequences

To determine the exact value of $\gamma_{\text{OLD}}(G_n)$, we analyze the constraints on the rim and the subdivision ring separately. Let S be an OLD-set. We define two binary indicator sequences $\mathbf{x}, \mathbf{y} \in \{0, 1\}^n$:

$$x_i = \begin{cases} 1 & \text{if } r_i \in S \\ 0 & \text{if } r_i \notin S \end{cases} \quad \text{and} \quad y_i = \begin{cases} 1 & \text{if } u_i \in S \\ 0 & \text{if } u_i \notin S \end{cases}$$

Indices are taken modulo n . The OLD conditions impose local constraints on adjacent entries of these sequences: a violation of domination or location at some vertex corresponds to a specific short substring appearing in $\mathbf{x}$ or $\mathbf{y}$. We call such substrings *forbidden patterns*. The following lemma identifies them; their absence is then used in Lemma 4.2 to derive a density lower bound on the number of ones in each sequence.

LEMMA 4.1 (Forbidden Patterns). *For S to be a valid OLD-set:*

1. *The sequence $\mathbf{x}$ (rim) must not contain the substring ‘00’ nor ‘010’.*
2. *The sequence $\mathbf{y}$ (subdivision) must not contain the substring ‘010’.*
3. *If $h \notin S$, $\mathbf{y}$ contains no ‘00’. If $h \in S$, $\mathbf{y}$ contains at most one ‘00’.*

Proof. 1. Consider the vertices $u_i \in U$. $N(u_i) = \{r_i, r_{i+1}\}$. If $\mathbf{x}$ contains ‘00’ at indices $i, i+1$, then u_i has no neighbors in S (domination failure). If $\mathbf{x}$ contains ‘010’ at $i-1, i, i+1$, then $N(u_{i-1}) \cap S = \{r_i\}$ and $N(u_i) \cap S = \{r_i\}$, a location failure.

2. Consider the vertices $r_i \in R$. Their view of S is composed of the hub (if present) and neighbors in U . Specifically, $N(r_i) \cap U = \{u_{i-1}, u_i\}$. If $\mathbf{y}$ contains ‘010’ at $i-1, i, i+1$, then $N(r_i) \cap U = \{u_i\}$ and $N(r_{i+1}) \cap U = \{u_i\}$. Regardless of the hub’s status, this forces $N(r_i) \cap S = N(r_{i+1}) \cap S$, a location failure.

3. If $\mathbf{y}$ contains ‘00’ at positions $i-1, i$, then $u_{i-1} \notin S$ and $u_i \notin S$. Since $N(r_i) = \{h, u_{i-1}, u_i\}$, it follows that $N(r_i) \cap S \subseteq \{h\}$. If $h \notin S$, then $N(r_i) \cap S = \emptyset$, a domination failure. If $h \in S$, then $N(r_i) \cap S = \{h\}$. Since distinct vertices must have distinct intersections with S , at most one rim vertex r_i can satisfy $N(r_i) \cap S = \{h\}$. Therefore, the pattern ‘00’ can occur at most once in $\mathbf{y}$. $\square$

LEMMA 4.2 (Density bound). *Let $w \in \{0, 1\}^n$ be a circular binary sequence with n_1 ones and n_0 zeros.*

1. *If w contains no ‘00’ and no ‘010’, then $n_1 \geq n - \lfloor n/3 \rfloor$.*
2. *If w contains no ‘010’ and exactly one ‘00’, then $n_1 \geq n - \lfloor (n+2)/3 \rfloor$.*

Proof. The constraint “no ‘010’” implies that any zero (or block of zeros) must be separated by at least two ones. A single one between zeros creates ‘010’.

Case 1 (No ‘00’, no ‘010’): Zeros are isolated (blocks of length 1). Let B be the number of zero blocks. Then $n_0 = B$. Each zero block requires at least two ones to separate it from the next. Thus $n_1 \geq 2B = 2n_0$. Substituting into the total length $n = n_0 + n_1$:

$$n \geq n_0 + 2n_0 = 3n_0 \implies n_0 \leq \lfloor n/3 \rfloor.$$

Therefore, $n_1 = n - n_0 \geq n - \lfloor n/3 \rfloor$.

Case 2 (No ‘010’, exactly one ‘00’): Here, we have one zero-block of length 2, and $B - 1$ zero-blocks of length 1. The total count of zeros is $n_0 = 2(1) + 1(B - 1) = B + 1$. Thus, the number of blocks is $B = n_0 - 1$. The required number of ones to separate B blocks is still $2B$.

$$n_1 \geq 2B = 2(n_0 - 1).$$

Substituting into the total length:

$$n = n_0 + n_1 \geq n_0 + 2n_0 - 2 = 3n_0 - 2.$$

Rearranging gives $3n_0 \leq n + 2$, or $n_0 \leq \lfloor (n + 2)/3 \rfloor$. Therefore, $n_1 = n - n_0 \geq n - \lfloor (n + 2)/3 \rfloor$. $\square$

4.2 Exact values

THEOREM 4.3. *For every integer $n \geq 3$, the open-locating-dominating number of the gear graph is:*

$$\gamma_{\text{OLD}}(G_n) = \begin{cases} 5 & \text{if } n = 3, \\ 2 \lceil \frac{2n}{3} \rceil & \text{if } n \geq 4. \end{cases}$$

Proof. We first address the general case $n \geq 4$, and then treat the exceptional case $n = 3$.

Upper bound ($n \geq 4$). We utilize the density pattern from Lemma 4.2. We construct a set S by selecting indices i such that $i \not\equiv 2 \pmod{3}$.

$$S = \{r_i, u_i \mid i \not\equiv 2 \pmod{3}\}.$$

The indicator sequences for both R and U are periodic repetitions of the pattern ‘110’. The number of zeros in each sequence is $\lfloor n/3 \rfloor$, so the size of the set is $2(n - \lfloor n/3 \rfloor) = 2\lceil 2n/3 \rceil$.

We now demonstrate that S is a valid OLD-set. *Domination:* In both R and U , the maximum gap between selected vertices is 1 (the ‘0’ in ‘110’). Thus, no vertex in U (adjacent to r_i, r_{i+1}) or R (adjacent to u_{i-1}, u_i) is left undominated. *Location:* We compute the set intersections $N(v) \cap S$ modulo 3.

- For r_i : $N(r_i) \cap S = \{u_{i-1}, u_i\} \cap S$. If $i \equiv 0$, the intersection is $\{u_{i-1}, u_i\} \cap S = \{u_i\}$. If $i \equiv 1$, the intersection is $\{u_{i-1}, u_i\} \cap S = \{u_{i-1}, u_i\}$. If $i \equiv 2$, the intersection is $\{u_{i-1}, u_i\} \cap S = \{u_{i-1}\}$. These are distinct sets $\{u_0\}, \{u_0, u_1\}, \{u_1\}$.

- For u_i : $N(u_i) \cap S = \{r_i, r_{i+1}\} \cap S$. If $i \equiv 0$, the intersection is $\{r_i, r_{i+1}\}$. If $i \equiv 1$, the intersection is $\{r_i\}$. If $i \equiv 2$, the intersection is $\{r_{i+1}\}$.

All intersections are distinct and non-empty. The hub intersection $N(h) \cap S = R \cap S$ has size $\geq \lceil 2n/3 \rceil > 2$, distinguishing it from all other vertices.

Lower bound ($n \geq 4$). Let S be an OLD-set. We analyze two cases based on the hub h .

Case A: $h \notin S$. By Lemma 4.1, both $\mathbf{x}$ and $\mathbf{y}$ must be free of ‘00’ and ‘010’. By Lemma 4.2 (Case 1), both sequences must satisfy $n_1 \geq n - \lfloor n/3 \rfloor$. Thus, $|S| = |S \cap R| + |S \cap U| \geq 2(n - \lfloor n/3 \rfloor)$.

Case B: $h \in S$. The sequence $\mathbf{x}$ (rim) still cannot contain ‘00’ or ‘010’ (Lemma 4.1 part 1). Thus $|S \cap R| \geq n - \lfloor n/3 \rfloor$. However, the sequence $\mathbf{y}$ (subdivision) is allowed

to contain exactly one ‘00’ (Lemma 4.1 part 3). By Lemma 4.2 (Case 2), this relaxes the bound on $\mathbf{y}$ to $|S \cap U| \geq n - \lfloor (n+2)/3 \rfloor$. The total size is $|S| = 1 + |S \cap R| + |S \cap U|$.

$$|S| \geq 1 + \left(n - \left\lfloor \frac{n}{3} \right\rfloor \right) + \left(n - \left\lfloor \frac{n+2}{3} \right\rfloor \right).$$

We claim that this sum is never strictly less than the bound in Case A. We check the difference Δ between Case B and Case A:

$$\Delta = \left(1 + n - \left\lfloor \frac{n+2}{3} \right\rfloor \right) - \left(n - \left\lfloor \frac{n}{3} \right\rfloor \right) = 1 + \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor.$$

We verify for $n \pmod 3$:

- If $n = 3k$: $\Delta = 1 + k - k = 1 > 0$.
- If $n = 3k + 1$: $\Delta = 1 + k - \lfloor (3k+3)/3 \rfloor = 1 + k - (k+1) = 0$.
- If $n = 3k + 2$: $\Delta = 1 + k - \lfloor (3k+4)/3 \rfloor = 1 + k - (k+1) = 0$.

In all cases, $\Delta \geq 0$. Thus, including the hub in S does not reduce the cardinality below $2(n - \lfloor n/3 \rfloor)$. Intuitively, including h in S relaxes the constraint on $\mathbf{y}$ by permitting one ‘00’, but the hub itself must be counted in $|S|$, and the computation above shows this cost exactly cancels the gain.

Combining both cases, $\gamma_{\text{OLD}}(G_n) \geq 2(n - \lfloor n/3 \rfloor) = 2\lceil 2n/3 \rceil$.

The case $n = 3$. The general formula yields $2\lceil 2 \cdot 3/3 \rceil = 4$. We show that no OLD-set of size 4 exists for G_3 .

By Lemma 4.1 part 1, $|S \cap R| \geq 2$. By Lemma 4.1 parts 2–3, $|S \cap U| \geq 2$. Hence $|S| \geq 4$. Suppose $|S| = 4$. Then $h \notin S$ and both bounds are tight: $|S \cap R| = |S \cap U| = 2$, with exactly one vertex absent from each of R and U . Since exactly one rim vertex is absent, assume without loss of generality $r_2 \notin S$ (the three cases are related by the 3-fold rotational symmetry of G_3). Then $N(h) \cap S = \{r_0, r_1\}$. Since $N(u_0) = \{r_0, r_1\}$, we have $N(u_0) \cap S = \{r_0, r_1\} = N(h) \cap S$, a location failure between h and u_0 , regardless of which two vertices of U belong to S — a contradiction.

Hence $|S| \geq 5$, and therefore $\gamma_{\text{OLD}}(G_3) = 5$, with equality attained, for example, by $S = \{r_0, r_1, r_2, u_0, u_1\}$. $\square$

5. Conclusion

In this paper, we have determined the exact open-locating-dominating numbers for three structured graph families. We found that for $n \geq 3$, $\gamma_{\text{OLD}}(H_n) = \gamma_{\text{OLD}}(C_n \circ K_1) = n$ (with $\gamma_{\text{OLD}}(H_4) = 6$). Furthermore, we established that for gear graphs ($n \geq 4$), the value is $2\lceil 2n/3 \rceil$. While the results for helms and coronae rely on standard structural domination arguments, the gear graph presents a more complex scenario. The bipartite structure and the specific local arrangement of the subdivision vertices necessitate a density of $2/3$ on both the rim and the outer ring to satisfy the location property. This demonstrates that for certain graph families, local constraint analysis via indicator sequences provides a powerful tool for establishing tight bounds where

degree-based arguments fail. The binary sequence technique appears applicable more broadly to graph families with periodic subdivision structure. Natural directions for future work include determining the exact OLD number for subdivided wheel graphs and double-gear variants, as well as investigating whether the method extends to related parameters such as error-correcting OLD sets.

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