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HANDLE DECOMPOSITION FOR A CLASS OF COMPACT ORIENTABLE PL 4-MANIFOLDS

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Abstract. For a compact orientable PL 4-manifold M with boundary, let $\hat{M}$ be the singular manifold obtained by capping of ∂M. In this article, we explore the class of compact orientable PL 4-manifolds with empty or connected boundary, whose fundamental groups have rank 1, and their corresponding singular manifolds admit weak semi-simple crystallizations. First, we show that if M is a closed orientable PL 4-manifold belonging to this class, then there exist complementary submanifolds V and V' with a shared boundary such that $\mathcal{G}(V') \geq \mathcal{G}(M) \geq \mathcal{G}(V)$, where $\mathcal{G}(M)$, $\mathcal{G}(V)$, and $\mathcal{G}(V')$ denote the regular genera of M, V, and V', respectively.

Next, we provide a handle decomposition for a compact orientable PL 4-manifold M with connected non-spherical boundary from this class. If the rank of the fundamental group of $\hat{M}$, m', is 1, then M admits a handle decomposition that takes one of the following forms:

1) one 0-handle, one 1-handle, k 2-handles and one 3-handle, where $k = 2 + \beta_2(M) - \beta_1(M) - \beta_1(\hat{M})$, or

2) one 0-handle, two 1-handles, k 2-handles and one 3-handle, where $k = 3 + \beta_2(M) - \beta_1(M) - \beta_1(\hat{M})$.

Further, if $m' = 0$, then M admits a handle decomposition which consists of one 0-handle, one 1-handle and $\beta_2(M)$ 2-handles.

We further demonstrate that no manifold from this class with empty or connected spherical boundary can have a fundamental group isomorphic to a finite cyclic group. Finally, we provide a handle decomposition for such manifolds with empty or connected spherical boundary.

1. Introduction

A crystallization (Γ, γ) of a compact PL d-manifold is a specialized edge-colored graph that encodes the manifold's structure (see Subsection 2.1). Pezzana proved that every closed PL d-manifold has a crystallization (see [17]). This result was later broadened to include singular d-manifolds (cf. [7–12]). For a compact PL d-manifold M with

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boundary, a singular d-manifold $\hat{M}$ can be formed by capping off each boundary component of ∂M. There is a one-one correspondence between the class of compact PL 4-manifolds with empty or non-spherical boundary components and the class of singular manifolds (see Remark 2.2). If M is a compact PL 4-manifold with spherical boundary components, then $\hat{M}$ is a closed PL 4-manifold. If M is a closed PL 4-manifold, then $\hat{M} = M$.

The concept of "regular genus" for closed PL d-manifolds, introduced in [16], generalizes the classical genus in dimension 2 and the Heegaard genus in dimension 3 (see Subsection 2.2 for details). This idea was later extended to compact PL d-manifolds with non-spherical boundary components for $d \geq 2$ through the framework of singular d-manifolds (cf. [7–12]).

In [1], the concept of weak semi-simple crystallizations for closed PL 4-manifolds was introduced, while in [7], the notion of weak semi-simple gems for compact PL 4-manifolds with connected boundary was developed. This paper focuses on compact orientable PL 4-manifolds with empty or connected boundary, whose fundamental groups have rank 1 and whose associated singular manifolds admit weak semi-simple crystallizations.

In [10], the authors proved that the weak simple class (weak semi-simple for simply-connected 4-manifolds) of compact simply-connected 4-manifolds with empty or connected boundary admit a special handle decomposition that lacks 1-handles and 3-handles. The original problem for closed PL 4-manifolds was posed by Kirby and can be formulated as follows: "Does every simply-connected closed 4-manifold have a handlebody decomposition with no 1-handles and 3-handles?" A lot of work has been performed over the decades, including studies on manifolds with boundaries, such as Trace's work in [19, 20].

First, we prove that for a closed orientable 4-manifold M with the non-trivial cyclic fundamental group admitting a weak semi-simple crystallization, the combinatorial invariant regular genus of M lies between the regular genera of its complementary submanifolds V and V' (see Theorem 4.2).

In [3], it has been shown that for the class of closed orientable 4-manifolds with infinite cyclic fundamental group admitting semi-simple crystallizations, there exists a handle decomposition such that the number of 2-handles depends on the second Betti number of the manifold, and the number of other h-handles ($h \leq 4$) is at most 2.

In this article, we extend this work from the class of closed orientable PL 4-manifolds with rank 1 fundamental groups and admitting semi-simple crystallizations to a larger class of compact orientable PL 4-manifolds with empty or connected boundary that admit weak semi-simple crystallizations and have non-trivial cyclic fundamental groups. For a compact orientable PL 4-manifold M with connected non-spherical boundary within this class, we construct a handle decomposition. If the rank of the fundamental group of $\hat{M}$, m', is 1, then M admits a handle decomposition that takes one of the following forms:

1) one 0-handle, one 1-handle, k 2-handles and one 3-handle, where $k = 2 + \beta_2(M) - \beta_1(M) - \beta_1(\hat{M})$, or

2) one 0-handle, two 1-handles, k 2-handles and one 3-handle, where $k = 3 + \beta_2(M) -$

$\beta_1(M) - \beta_1(\hat{M})$.
Further, if $m' = 0$, then M admits a handle decomposition which consists of one 0-handle, one 1-handle and $\beta_2(M)$ 2-handles (see Theorem 4.4). In proving this theorem, we shall extensively use the notion of regular neighborhoods [18]. The proof of this theorem implies that any manifold with empty or connected spherical boundary of this class can never have a finite cyclic fundamental group, and we also provide a handle decomposition for such manifolds (see Corollary 4.5).

2. Preliminaries

The crystallization theory provides a tool for representing piecewise-linear (PL) manifolds of any dimension combinatorially using edge-colored graphs.

2.1 Crystallization

In this article, we study multigraphs $\Gamma = (V(\Gamma), E(\Gamma))$ without loops, where the edges are assigned colors (or labels) from the set $\Delta_d := \{0, 1, \dots, d\}$. The elements of Δ_d are called the *colors* of Γ. A $(d+1)$-*regular colored graph* is defined as a pair (Γ, γ), where each vertex of Γ has degree $d+1$, and there exists a surjective map $\gamma : E(\Gamma) \to \Delta_d$ such that any two adjacent edges e_1 and e_2 satisfy $\gamma(e_1) \neq \gamma(e_2)$. When the edge coloring is clear from the context, we may simply refer to the $(d+1)$-regular colored graph as Γ. For $\mathcal{C} \subseteq \Delta_d$ with cardinality k, the graph $\Gamma_{\mathcal{C}} = (V(\Gamma), \gamma^{-1}(\mathcal{C}))$ is a k-regular colored graph with edge-coloring $\gamma|_{\gamma^{-1}(\mathcal{C})}$. For a color set $\{j_1, j_2, \dots, j_k\} \subset \Delta_d$, $g(\Gamma_{\{j_1,j_2,\dots,j_k\}})$ or $g_{\{j_1,j_2,\dots,j_k\}}$ denotes the number of connected components of the graph $\Gamma_{\{j_1,j_2,\dots,j_k\}}$. A graph (Γ, γ) is called *contracted* if the subgraph $\Gamma_{\hat{j}} = \Gamma_{\Delta_d \setminus \{j\}}$ is connected for all $j \in \Delta_d$. We refer to [6] for standard terminologies on graphs. All spaces and maps will be considered in PL-category [18].

For a $(d+1)$-regular colored graph (Γ, γ), a corresponding colored d dimensional simplicial cell complex $\mathcal{K}(\Gamma)$ is constructed as follows:

- for each vertex $v \in V(\Gamma)$, take a d-simplex $\sigma(v)$ with vertices labeled by Δ_d,
- corresponding to each edge of color j between $v_1, v_2 \in V(\Gamma)$, identify the $(d-1)$-faces of $\sigma(v_1)$ and $\sigma(v_2)$ opposite to j-labeled vertices such that the same labeled vertices coincide.

The geometric carrier $|\mathcal{K}(\Gamma)|$ is a d-pseudomanifold, and (Γ, γ) is referred to as a *gem* (graph encoded manifold) of any d-pseudomanifold homeomorphic to $|\mathcal{K}(\Gamma)|$, or equivalently, it is said to represent the d-pseudomanifold. Since the vertex set of $\mathcal{K}(\Gamma)$ can be $(d+1)$-colored, we refer to $\mathcal{K}(\Gamma)$ as a *colored triangulation* of the d-pseudomanifold $|\mathcal{K}(\Gamma)|$. For further details on CW complexes and related notions, see [5].

From this construction, it is straightforward to observe that for any subset $\mathcal{C} \subset \Delta_d$ of cardinality $k+1$, $\mathcal{K}(\Gamma)$ has as many k-simplices with vertices labeled by $\mathcal{C}$ as there are connected components in $\Gamma_{\Delta_d \setminus \mathcal{C}}$ [14]. The *disjoint star* of a simplex $\sigma \in \mathcal{K}(\Gamma)$ is a simplicial cell complex comprising all d-dimensional simplices of $\mathcal{K}(\Gamma)$ that contain σ as a face, with the identification of their $(d-1)$-faces containing σ as in $\mathcal{K}(\Gamma)$. The *disjoint link* of $\sigma \in \mathcal{K}(\Gamma)$ is the subcomplex of its disjoint star formed by all simplices that do not intersect σ.

Definition 2.1. A *singular PL d-manifold* is a closed d dimensional polyhedron with a colored triangulation such that the disjoint links of its vertices are closed $(d-1)$-manifolds. A vertex whose link is not a sphere is called a singular vertex. It follows that a closed (PL) d-manifold is simply a singular (PL) d-manifold with no singular vertices.

From the correspondence between $(d+1)$-regular colored graphs (Γ, γ) and d dimensional pseudomanifolds, it can be easily visualised that:
1) $|\mathcal{K}(\Gamma)|$ is a closed PL d-manifold if and only if for each $c \in \Delta_d$, $\Gamma_{\hat{c}}$ represents $\mathbb{S}^{d-1}$.
2) $|\mathcal{K}(\Gamma)|$ is a singular PL d-manifold if and only if for each $c \in \Delta_d$, $\Gamma_{\hat{c}}$ represents a closed PL $(d-1)$-manifold.

If $\Gamma_{\hat{j}}$ does not represent the $(d-1)$-sphere, then the color j is called a singular color.

Remark 2.2 ([10, 12]). There is a one-to-one correspondence between the class of singular (PL) d-manifolds and the class of all compact (PL) d-manifolds with empty or non-spherical boundary components. If M is a singular d-manifold, then removing a small open neighborhood around each of its singular vertices yields a compact d-manifold $\check{M}$ with non-spherical boundary components. Moreover, $M = \check{M}$ if and only if M is a closed d-manifold. Conversely, if M is a compact d-manifold with non-spherical boundary components, a corresponding singular d-manifold $\hat{M}$ can be obtained by capping off each boundary component of ∂M. If M is closed, then $M = \hat{M}$.

From this point forward, when we refer to a graph representing a compact PL 4-manifold M with non-spherical boundary components, we mean the graph that represents the corresponding singular manifold $\hat{M}$, obtained by capping off each boundary component of M with a cone. In this article, we focus on compact PL 4-manifolds M with empty or connected non-spherical boundary. As a result, we will consider only 5-regular colored graphs. Without loss of generality, we assume that color 4 is the only possible singular color of Γ, the gem of $\hat{M}$.

Definition 2.3. A $(d+1)$-regular colored gem of a singular d-manifold with at most one singular vertex is called a *crystallization* of M if it is contracted.

In the colored triangulation of M corresponding to a crystallization, there are precisely $d+1$ vertices. The foundation of crystallization theory lies in Pezzana's existence theorem [17], which proves that every closed PL d-manifold admits a crystallization. Additionally, the existence of crystallizations was established for singular (PL) d-manifolds with a single singularity in [12]. This concept has since been further developed in various articles [7–11].

2.2 Regular Genus

It is well known that a $(d+1)$-regular colored bipartite (resp. non-bipartite) graph (Γ,γ) admits regular embedding into the orientable (resp. non-orientable) surface F_ε (corresponding to cyclic permutation ε of Δ_d) of Euler characteristic

$$\chi_\varepsilon(\Gamma) = \sum_{i\in\mathbb{Z}_{d+1}} g_{\{\varepsilon_i,\varepsilon_{i+1}\}} + (1-d)\frac{V(\Gamma)}{2},$$

where subscripts are taken in modulo d. Additionally, (Γ,γ) does not admit any regular embedding into a non-orientable (resp. orientable) surface [15]. Using this, the concept of *regular genus* of closed and singular manifolds was introduced in [16] and [12], respectively, extending the notions of *genus of surfaces* and *Heegard genus of 3-manifolds*. The regular genus is a PL-invariant for PL manifolds of any dimension. It is known that a gem (Γ,γ) of a manifold M is a bipartite graph if and only if M is orientable. For $d\geq 3$, the regular genus $\rho(\Gamma)$ of (Γ,γ) is defined as

$$\rho(\Gamma) = \min\{\rho_\varepsilon(\Gamma) \mid \varepsilon \text{ is a cyclic permutation of } \Delta_d\},$$

where the genus (resp. half of genus) $\rho_\varepsilon(\Gamma)$ of F_ε satisfies

$$\rho_\varepsilon(\Gamma) = 1 - \frac{\chi_\varepsilon(\Gamma)}{2}.$$

The regular genus of M (closed or singular PL manifold) is defined as

$$\mathcal{G}(M) = \min\{\rho(\Gamma) \mid (\Gamma,\gamma) \text{ represents } M\}.$$

For a compact PL 4-manifold M with connected non-spherical boundary, the (generalized) regular genus $\mathcal{G}(M)$ of M is defined as the regular genus $\mathcal{G}(\hat{M})$ of $\hat{M}$.

Proposition 2.4 ([11]). *Let M be a compact PL 4-manifold with connected non-spherical boundary. Then $\mathcal{G}(M) \geq 2\chi(M) + 3m + 2m' - 2$, where m and m' are the ranks of the fundamental groups of M and $\hat{M}$, respectively.*

In [2], a lower bound is provided for the regular genus of compact PL 4-manifolds with boundary. However, the definition of regular genus of compact PL 4-manifolds with boundary used there differs from the definition of regular genus presented in this article.

3. Weak semi-simple crystallizations of compact 4-manifolds

In [1] and [7], the concept of weak semi-simple crystallizations of compact PL 4-manifolds with empty or connected non-spherical boundary was introduced. These crystallizations are minimal with respect to regular genus among the graphs representing the same 4-manifold.

Definition 3.1. Let M be a compact PL 4-manifold with empty or connected non-spherical boundary. A 5-regular colored graph Γ representing M is called *weak semi-simple* with respect to a permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_4 = 4)$ if $g_{\{\varepsilon_j,\varepsilon_{j+2},\varepsilon_{j+4}\}} = m+1$

$\forall\ j \in \{0,2,4\}$ and $g_{\{\varepsilon_j,\varepsilon_{j+2},\varepsilon_{j+4}\}} = m' + 1\ \forall\ j \in \{1,3\}$ (additions are modulo 5 in subscripts), where m and m' are the ranks of the fundamental groups of M and $\hat{M}$, respectively.

Note that $m' \leq m$ and if Γ is weak semi-simple, then it is contracted as well, and hence Γ is a crystallization.

Let (Γ, γ) be a crystallization of a singular PL 4-manifold M and $\mathcal{K}(\Gamma)$ be the corresponding colored triangulation with the vertex set Δ_4. If $B \subset \Delta_4$, then $\mathcal{K}(B)$ denotes the subcomplex of $\mathcal{K}(\Gamma)$ generated by the vertices $i \in B$. If Sd $\mathcal{K}(\Gamma)$ is the first barycentric subdivision of $\mathcal{K}(\Gamma)$, then $F(i,j)$ (resp. $F(i,j,k)$) is the largest subcomplex of Sd $\mathcal{K}(\Gamma)$, disjoint from Sd $\mathcal{K}(i,j) \cup$ Sd $\mathcal{K}(\Delta_4 \setminus \{i,j\})$ (resp. Sd $\mathcal{K}(i,j,k) \cup$ Sd $\mathcal{K}(\Delta_4 \setminus \{i,j,k\})$). Then the polyhedron $|F(i,j)|$ (resp. $|F(i,j,k)|$) is a closed PL 3-manifold which partitions M into two 4-manifolds $N(i,j)$and$N(\Delta_4 \setminus \{i,j\})$ (resp. $N(i,j,k)$and$N(\Delta_4 \setminus \{i,j,k\})$) with $|F(i,j)|$ (resp. $|F(i,j,k)|$) as common boundary. Further, $N(i,j)$ (resp. $N(i,j,k)$) is a regular neighbourhood of $|\mathcal{K}(i,j)|$ (resp. $|\mathcal{K}(i,j,k)|$) in $|\mathcal{K}(\Gamma)|$ (see [3, 10, 13, 16] for more details). Thus, M has a decomposition as $M = N(i,j) \cup_\phi N(\Delta_4 \setminus \{i,j\})$, where ϕ is a boundary identification.

Definition 3.2 (Complementary Submanifolds). Let M be a compact orientable PL 4-manifold with empty or connected boundary, and $\hat{M}$ be its corresponding singular manifold. Let $\hat{M}$ admit weak semi-simple crystallization (Γ, γ). Without loss of generality, we write $\hat{M} = N(1,4) \cup N(0,2,3)$. We denote $N(1,4)$ and $N(0,2,3)$ by V and V' respectively. We call V and V' the *complementary submanifolds* of $\hat{M}$.

Lemma 3.3. *Let M be a compact orientable PL 4-manifold with empty or connected non-spherical boundary, and $\hat{M}$ be its corresponding singular manifold. Let M admit weak semi-simple crystallization (Γ, γ) and V, V' be the complementary submanifolds as in Definition 3.2. Then $V \cap V'$ is orientable and V is PL homeomorphic to the boundary connected sum of $\#_{m'}(\mathbb{S}^1 \times \mathbb{B}^3)$ and $C(\partial M)$, the cone over ∂M, where m' is the rank of $\pi_1(\hat{M})$.*

Proof. Using the Mayer-Vietoris exact sequence for the triple $(\hat{M}, V, V')$, we obtain $0 \to H_4(\hat{M}) \to H_3(\partial V) \to 0$. This shows that $\hat{M}$ is orientable if and only if ∂V is orientable. Since M is orientable, it follows that $\hat{M}$ is orientable, and thus ∂V is also orientable. Consequently, as $\partial V = V \cap V'$, $V \cap V'$ is orientable.

Given that $g_{\{0,2,3\}} = m' + 1$, there are exactly $m' + 1$ edges of color $\{1,4\}$. Therefore, V is PL-homeomorphic to the boundary connected sum $\#_{m'}(\mathbb{S}^1 \times \mathbb{B}^3) \# C(\partial M)$, where $C(\partial M)$ is the cone over ∂M. This further implies that $\partial V = \#_{m'}(\mathbb{S}^1 \times \mathbb{S}^2) \# \partial M$. $\square$

The following result on isomorphism between cohomology and homology groups of a 4-manifold with a connected boundary and its associated singular manifold, respectively, was given in [10, Proposition 5].

Proposition 3.4. *Let M be a compact orientable PL 4-manifold with connected boundary and $\hat{M}$ be the singular manifold by capping off its boundary. Then*

$$H_k(\hat{M}) \cong H^{4-k}(M) \text{ for } k \in \{1,2,3,4\}.$$

Lemma 3.5. *Let M be a compact orientable PL 4-manifold with empty or connected non-spherical boundary admitting a weak semi-simple crystallization. Let V and V' be the spaces, as in Definition 3.2. Then*

$$\beta_1(M) + \beta_1(\hat{M}) - \beta_2(M) + \beta_2(V') - \beta_1(V') = m', \tag{1}$$

where m' is the rank of the fundamental group of $\hat{M}$.

Proof. Since V' is regular neighbourhood of 2-dimensional polyhedron $|\mathcal{K}(0,2,3)|$, it deformation retracts onto $|\mathcal{K}(0,2,3)|$ and thus $H_i(V') = 0$ for $i > 2$. The Mayer-Vietoris sequence of the triple $(\hat{M}, V, V')$ gives the following long exact sequence.

$$\begin{array}{l} 0 \longrightarrow H_3(\hat{M}) \longrightarrow H_2(\partial V) \longrightarrow H_2(V) \oplus H_2(V') \longrightarrow H_2(\hat{M}) \longrightarrow H_1(\partial V) \\ \hspace{30em}\downarrow \\ \hspace{17em} 0 \longleftarrow H_1(\hat{M}) \longleftarrow H_1(V) \oplus H_1(V') \end{array}$$

By Proposition 3.4 and Universal Coefficient Theorem, we have $H_3(\hat{M}) \cong H^1(M) \cong FH_1(M)$ and $H_2(\hat{M}) \cong H^2(M) \cong FH_2(M) \oplus TH_1(M)$. Now,

$$\begin{aligned} H_2(\partial V) &= H_2\big(\#_{m'}(\mathbb{S}^1 \times \mathbb{S}^2)\#\partial M\big) = H_2\big(\#_{m'}(\mathbb{S}^1 \times \mathbb{S}^2)\big) \oplus H_2(\partial M) \\ &\cong \oplus_{m'}\mathbb{Z} \oplus H^1(\partial M) \cong \oplus_{m'}\mathbb{Z} \oplus FH_1(\partial M), \end{aligned}$$

where the third isomorphism is due to Poincaré duality. Similarly, we have $H_1(\partial V) = \oplus_{m'}\mathbb{Z} \oplus H_1(\partial M)$. Also, for $i = 1,2$; $H_i(V) = H_i\big(\#_{m'}(\mathbb{S}^1 \times \mathbb{B}^3)\big) \oplus H_i(C)$, where C is cone over ∂M. This implies, $H_2(V) = 0$ and $H_1(V) = \oplus_{m'}\mathbb{Z}$. Therefore, the above exact sequence simplifies to the following.

$$\begin{array}{l} 0 \longrightarrow FH_1(M) \longrightarrow \oplus_{m'}\mathbb{Z} \oplus FH_1(\partial M) \longrightarrow H_2(V') \longrightarrow FH_2(M) \oplus TH_1(M) \\ \hspace{33em}\downarrow \\ \hspace{12em} 0 \longleftarrow H_1(\hat{M}) \longleftarrow \oplus_{m'}\mathbb{Z} \oplus H_1(V') \longleftarrow \oplus_{m'}\mathbb{Z} \oplus H_1(\partial M) \end{array}$$

Since the alternate sum of the number of free generators of finitely generated abelian groups in an exact sequence is zero, the result follows. □

It is known that every compact PL 4-manifold M with empty or connected boundary admits a handle decomposition, i.e.,

$$\begin{aligned} M = H^{(0)} &\cup (H_1^{(1)} \cup \cdots \cup H_{d_1}^{(1)}) \cup (H_1^{(2)} \cup \cdots \cup H_{d_2}^{(2)}) \\ &\cup (H_1^{(3)} \cup \cdots \cup H_{d_3}^{(3)}) \text{ (if } M \text{ is closed) } \cup H^{(4)}, \end{aligned}$$

where $H^{(0)} = \mathbb{D}^4$ and each k-handle $H_i^{(k)} = \mathbb{D}^k \times \mathbb{D}^{4-k}$ (for $1 \leq k \leq 4$, $1 \leq i \leq d_k$) is attached via a map (embedding) $f_i^{(k)} \colon \partial\mathbb{D}^k \times \mathbb{D}^{4-k} \to \partial(H^{(0)} \cup \ldots \cup (H_1^{(k-1)} \cup \ldots \cup H_{d_{k-1}}^{(k-1)}))$.

Lemma 3.6. *Let M be a compact orientable PL 4-manifold with empty or connected non-spherical boundary admitting a weak semi-simple crystallization (Γ, γ) and V, V' be as in Definition 3.2. Then, $rank(\pi_1(\hat{M})) \leq rank(\pi_1(V'))$.*

Proof. Let m' be the rank of the fundamental group of $\hat{M}$. We have $\hat{M} = V \cup V' = \big(\#_{m'}(\mathbb{S}^1 \times \mathbb{B}^3)\#C(\partial M)\big) \cup V'$ (up to homeomorphism), where $C(\partial M)$ is the cone over ∂M. Also, $V \cap V' = \partial V = \partial V' = (\#_{m'}\mathbb{S}^1 \times \mathbb{S}^2)\#\partial M$ from Lemma 3.3. We will use Seifert-Van Kampen Theorem. Let $i_1 : \pi_1(V \cap V') \to \pi_1(V)$ and $i_2 : \pi_1(V \cap V') \to \pi_1(V')$ be the maps induced from inclusion maps $j_1 : V \cap V' \to V$ and $j_2 : V \cap V' \to V'$, respectively.

Since $V \cap V' = \#_{m'}(\mathbb{S}^1 \times \mathbb{S}^2)\#\partial M$, its fundamental group is generated by atleast m' elements, say $\pi_1(V \cap V') = \langle \alpha_1, \ldots, \alpha_{m'}, \gamma_1, \ldots, \gamma_{n_1} | d_1, d_2, \ldots, d_{n_2}\rangle$ where generator α_i comes from $\#_{m'}(\mathbb{S}^1 \times \mathbb{S}^2)$ for all $1 \leq i \leq m'$, and the remaining generators come from ∂M. Also, we have $\pi_1(V) = \langle \alpha_i \mid 1 \leq i \leq m'\rangle$ and hence $i_1(\alpha_i) = \alpha_i,\ i_1(\gamma_j) = \langle e\rangle$. Now, let $\pi_1(V') = \langle \beta_1, \ldots, \beta_{n_3} | z_1, \ldots, z_{n_4}\rangle$. Applying the theorem, we get

$$\pi_1(\hat{M}) = \langle \alpha_1, \ldots, \alpha_{m'}, \beta_1, \ldots, \beta_{n_3} | z_1, \ldots, z_{n_4}, \alpha_i = i_2(\alpha_i), e = i_2(\gamma_j), 1 \leq i \leq m', 1 \leq j \leq n_1)\rangle,$$

which proves the result. □

The following is a result that gives a characterization of having a weak semi-simple crystallization for a closed PL 4-manifold.

Proposition 3.7 ([1,4]). *Let M be a closed PL 4-manifold. Then, M admits a weak semi-simple crystallization if and only if $\mathcal{G}(M) = 2\chi(M) + 5m - 4$, where m is the rank of the fundamental group of M.*

4. Proof of main results

In this section, we consider the class of compact orientable PL 4-manifolds with empty or connected boundary that have fundamental groups of rank 1 (i.e., $m = 1$) such that their corresponding singular manifolds admit weak semi-simple crystallizations. This implies that the fundamental group of a manifold from this class is either $\mathbb{Z}$ or $\mathbb{Z}_k$ for some $k,\ k \geq 2$. We will significantly rely on the following known fact to prove our main results.

Proposition 4.1 ([18]). *Let M be a manifold and $X \subset int\ M$ be a polyhedron. If X collapses onto Y, then a regular neighborhood of X in M is PL-homeomorphic to a regular neighborhood of Y in M.*

Theorem 4.2. *Let M be a closed orientable PL 4-manifold with $rank(\pi_1(M)) = 1$. Let (Γ, γ) be a weak semi-simple crystallization of M and V, V' be as in Definition 3.2. Then $\mathcal{G}(V') \geq \mathcal{G}(M) \geq \mathcal{G}(V)$.*

Proof. Since $M = V \cup V'$ admits weak semi-simple crystallization, $\mathcal{G}(M) = 2\chi(M) + 1$ by Proposition 3.7. Since V' is a 4-manifold with connected boundary, Proposition 2.4 and Lemma 3.6 imply that $\mathcal{G}(V') \geq 2\chi(V') + 3\ rank(\pi_1(V')) - 2 = 2\chi(V') + 1$. Also, $\chi(V) = \chi(\mathbb{S}^1 \times \mathbb{B}^3) = 0$ and $\chi(V \cap V') = \chi(\mathbb{S}^1 \times \mathbb{S}^2) = 0$, $\chi(M) = \chi(V) + \chi(V') - \chi(V \cap V') = \chi(V')$. Thus, $\mathcal{G}(V') \geq 2\chi(V') + 1 = 2\chi(M) + 1 = \mathcal{G}(M)$ and one inequality follows. On the other hand, $\mathcal{G}(\mathbb{S}^1 \times \mathbb{B}^3) = 1$ (cf. [11]) implies $\mathcal{G}(M) \geq \mathcal{G}(V)$ and the result follows. □

Lemma 4.3. *Let M be a compact orientable PL 4-manifold with empty or connected non-spherical boundary, whose fundamental group has rank 1. Let $\hat{M}$ admit weak semi-simple crystallization (Γ, γ) and V, V' be as in Definition 3.2. Then V', a regular neighborhood of $\mathcal{K}(0,2,3) = \mathcal{K}$, is PL homeomorphic to a regular neighborhood of one of the following: $\mathbb{S}^1 \vee \mathbb{S}^1 \vee (\mathbb{S}^2)^\alpha$, $\mathbb{S}^1 \vee (\mathbb{S}^2)^\alpha$ or a space with the trivial fundamental group, where $\alpha = \beta_2(\mathcal{K})$ and $(\mathbb{S}^2)^\alpha$ denotes the wedge product of α copies of the 2-sphere $\mathbb{S}^2$.*

Proof. We shall analyze $\mathcal{K}(0,2,3)$ to prove this. Let k be the number of $\{2,3\}$-labeled edges, and let $d_1, d_2, \dots, d_k$ denote the $\{2,3\}$-labeled edges. Also, by the definition of weak semi-simple and the correspondence between the number of components of Γ and the number of simplices in the corresponding simplicial cell complex, there are exactly two $\{0,2\}$- and $\{0,3\}$-labeled edges each. We denote the two edges with endpoints labeled by color 0 and 2 (resp. 0 and 3) by l_1 and l_2 (resp. by r_1 and r_2). Thus, there can be at most four distinct triangles with vertices $0, 2, 3$ passing through the edge d_i for each $i \in \{1, 2, \dots k\}$, as shown in Figure 1. Let $A^i_{11}, A^i_{12}, A^i_{21}$ and A^i_{22} be these four triangles.

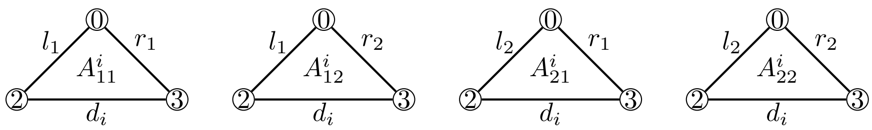

Figure 1: The four possible 2-simplices passing through one $\{2,3\}$-labeled edge.

Since the two triangles with all three edges coinciding, make one 2-sphere $\mathbb{S}^2$, the triangles with the same boundary give the copies of 2-sphere. So, from here on in this proof, we assume that no two triangles have the same boundary.

Let p_i be the cardinality of 2-simplices with vertices labeled by $\{0,2,3\}$ that pass through the d_i edge.

Case 1 $(p_i + p_j \geq 5 \ \forall \ i, j)$: We consider the triangles in $\mathcal{K}(0,2,3)$ that pass through d_i and d_j, $i \neq j$. The non-trivial loops (if any) should be along the edges $l_1 l_2$, $r_1 r_2$ and $d_i d_j$. It is not difficult to observe that if $p_i + p_j \geq 5$, then all these three possible non-trivial loops $l_1 l_2$, $r_1 r_2$ and $d_i d_j$ are homotopic to identity, i.e., the fundamental group of $\mathcal{K}(0,2,3)$ is trivial. Since i and j are arbitrary, we make the general statement on similar lines that for k number of $\{2,3\}$-labeled edges, say $d_1, d_2, \dots, d_k$, if $p_i + p_j \geq 5$ $\forall \ i, j \in \{1, 2, \dots, k\}$ then the fundamental group of $\mathcal{K}(0,2,3)$ is trivial.

Case 2 $(p_i + p_j \geq 4 \ \forall \ i, j$ and $p_i + p_j = 4$ for some i, j but $p_i + p_j \neq 4 \ \forall \ i, j)$: If $p_i = 2 = p_j$, $i \neq j$ and $p_k = 3$ or 4 then the loops $l_1 l_2$, $r_1 r_2$, $d_i d_k$ and $d_j d_k$ are homotopic to identity by using the proof of Case 1 because $p_i + p_k$, $p_j + p_k \geq 5$. Now, we have to check whether the loop $d_i d_j$ is trivial. Without loss of generality, we assume that one triangle passing through d_i is of the form A^i_{11} in Figure 1. If any of the two triangles passing through d_j is of the form A^j_{11}, then it forms a cone-like shape, and obviously $d_i d_j$ is homotopic to a constant loop. If one triangle is of the form A^j_{12} (resp., A^j_{21}) in Figure 1 then d_id_j is homotopic to r_1r_2 (resp., l_1l_2) which is further homotopic to a constant loop. If one triangle is of the form A^j_{22} then another triangle passing through d_j is among A^j_{11}, A^j_{12} or A^j_{21} because $p_j = 2$. Then, A^i_{11} with any of these three makes d_id_j contractible. Thus, we summarise that $\mathcal{K}(0,2,3)$ has the trivial fundamental group.

Case 3 ($p_i = 1$ for some i and $p_j = 3$ or 4 for some j): Let $p_k = 1$ for some k. As the other two edges of the triangle passing through d_k coincide with the edges of other triangles in $\mathcal{K}(0,2,3)$ passing through the $\{2,3\}$-labeled edge d_j, we can deform thus the triangle passing through d_k from the free edge d_k. On collapsing, we get the space consisting of all the triangles except the one passing through d_k, and this case reduces to **Case 1** or **Case 2**. Therefore, $\mathcal{K}(0,2,3)$ has the trivial fundamental group.

Case 4 ($p_i = 2 \ \forall \ i$): This case consists of the two triangles passing through each d_i. Let $A^i_{pq} + A^i_{rs}$ be the set of two triangles as in Figure 1 passing through d_i, where $pq, rs \in \{11, 12, 21, 22\}$. We denote the possible combinations of triangles as follows:

Combination	$A^i_{11}+A^i_{12}$	$A^i_{11}+A^i_{21}$	$A^i_{11}+A^i_{22}$	$A^i_{12}+A^i_{21}$	$A^i_{12}+A^i_{22}$	$A^i_{21}+A^i_{22}$
Denoted by	I_i	II_i	III_i	IV_i	V_i	VI_i

Let us first consider that $k = 2$, that is, the number of 23-colored edges are two, say d_i, d_j. We observe the following points:

1) The combination of the type $III_i + IV_j$ is not possible; otherwise, $\hat{M}$ will turn out to be non-orientable.

2) The combinations $I_i + VI_j$ and $II_i + V_j$ are the same up to the renaming of edges.

Now, as we did in **Case 1**, it is easy to check that all possible combinations are contractible except those shown in Figure 2 up to the renaming of edges. We tabulate all the non-contractible combinations up to the renaming of edges and their corresponding spaces such that they have homeomorphic regular neighborhoods:

$I_i + VI_j$	$III_i + III_j$	$III_i + IV_j$
$\mathbb{S}^2 \vee \mathbb{S}^1$	$\mathbb{S}^2 \vee \mathbb{S}^1$	$\mathbb{RP}^2$

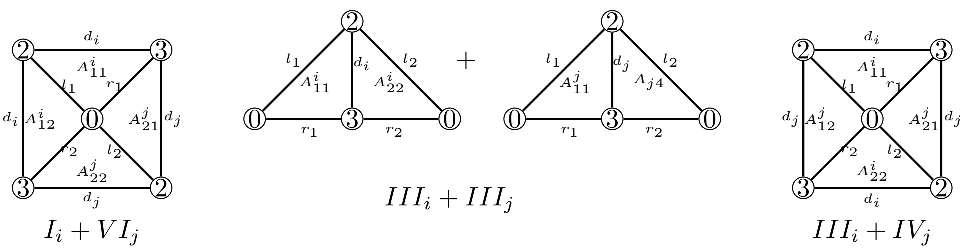

Figure 2: Possible non-contractible combinations up to the renaming of edges for $k = 2$.

Let us consider $k = 3$. We observe that we get one more $\mathbb{S}^2$ in the corresponding space if

1) we add I_k or VI_k to $I_i + VI_j$;

2) if we add III_k to $III_i + III_j$;

else the space has the trivial fundamental group. On a similar basis, the above statement is generalized on k as below. Let I^r denote the triangles of type I_{i_1}, $I_{i_2}, \ldots, I_{i_r}$ passing through r different edges with points labeled by $\{2,3\}$.

$$\begin{array}{cc} I^r + VI^s & III^r \\ (\mathbb{S}^2)^{r+s-1} \vee \mathbb{S}^1 & (\mathbb{S}^2)^{r-1} \vee \mathbb{S}^1, \end{array}$$

where $(\mathbb{S}^2)^\alpha$ denoted the wedge product of α copies of $\mathbb{S}^2$.

Besides these combinations for the k number of $\{2,3\}$-labeled edges, we get the simply-connected space.

Case 5 ($p_i \in \{1,2\}\ \forall\, i$ but $2 \neq p_i \neq 1\ \forall\, i$): Let $p_j = 1$. From the same observation as in **Case 3**, either we have the space where we can deform the triangle through the free edge d_j and the result is unaffected, or we have the combinations as in Figure 3. The space to which this combination corresponds with respect to the regular neighborhood is $S^1 \vee (S^2)^{i_r - 2}$.

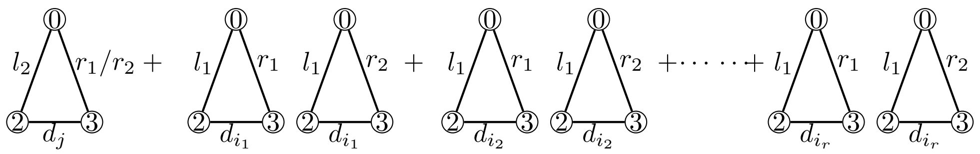

Figure 3

Case 6 ($p_i = 1\ \forall\, i$): Keeping the observation of **Case 3** in consideration, all the combinations in this case can deform as in Figure 4 and its corresponding space is the wedge product of two 1-spheres i.e., $\mathbb{S}^1 \vee \mathbb{S}^1$.

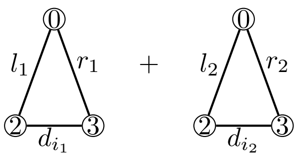

Figure 4

With all these cases, the lemma follows. ☐

Theorem 4.4. *Let M be a compact orientable PL 4-manifold with connected non-spherical boundary admitting a weak semi-simple crystallization, such that the rank of its fundamental group is 1. If the rank of the fundamental group of $\hat{M}$, m', is 1, then M admits a handle decomposition that takes one of the following forms:*
1) one 0-handle, one 1-handle, k 2-handles and one 3-handle, where $k = 2 + \beta_2(M) - \beta_1(M) - \beta_1(\hat{M})$, or

2) one 0*-handle, two* 1*-handles,* k 2*-handles and one* 3*-handle, where* $k = 3+\beta_2(M)-\beta_1(M)-\beta_1(\hat{M})$*.*
Further, if $m' = 0$*, then* M *admits a handle decomposition which consists of one* 0*-handle, one* 1*-handle and* $\beta_2(M)$ 2*-handles.*

Proof. Let M admit a weak semi-simple crystallization Γ with respect to the permutation $\epsilon = (0,1,2,3,4)$. We write $\hat{M} = N(1,4) \cup N(0,2,3) = V \cup V'$, where $V = \#_{m'}(\mathbb{S}^1 \times \mathbb{B}^3)\#C(\partial M)$ by Lemma 3.3. Since $m = 1$, $m' = 0, 1$ and $\pi_1(M) = \mathbb{Z}_k$ (or $\mathbb{Z}$). Lemma 4.3 implies that V', a regular neighborhood of $\mathcal{K} = \mathcal{K}(0,2,3)$, is PL homeomorphic to a regular neighborhood of a CW complex T, which is either $\mathbb{S}^1 \vee \mathbb{S}^1 \vee (\mathbb{S}^2)^{\beta_2(\mathcal{K})}$, $\mathbb{S}^1 \vee (\mathbb{S}^2)^{\beta_2(\mathcal{K})}$ or a simply-connected space, where $(\mathbb{S}^2)^{\gamma}$ denote the wedge product of γ copies of the 2-sphere $\mathbb{S}^2$.

If $m' = 1$, then due to Lemma 3.6, T cannot be simply-connected, and we are left with two possible choices for T.
Case 1 $\big(T = \mathbb{S}^1 \vee (\mathbb{S}^2)^{\beta_2(\mathcal{K})}\big)$: In this case, CW complex T consists of one 0-cell, one 1-cell and $\beta_2(\mathcal{K})$ number of 2-cells. For $0 \le j \le 2$, small regular neighbourhood of j-cell of T is a j-handle $H^{(j)}$. Since V' deformation retracts onto $\mathcal{K}$, $\beta_2(V') = \beta_2(\mathcal{K})$, we can represent V' as $V' = H^{(0)} \cup H^{(1)} \cup \Big(H^{(2)}_1 \cup \cdots \cup H^{(2)}_{\beta_2(V')}\Big)$.

Now, observe that boundary identification between V and V' is an attachment of one 3-handle. Also, $\beta_2(V')$ equals $2+\beta_2(M)-\beta_1(M)-\beta_1(\hat{M})$ from (1). Therefore, $M = H^{(0)} \cup H^{(1)} \cup \Big(H^{(2)}_1 \cup \cdots \cup H^{(2)}_{\beta_2(V')}\Big) \cup H^{(3)}$, where $\beta_2(V')$ is equal to $\beta_2(M)$, $\beta_2(M)+1$ or $\beta_2(M)+2$.

Case 2 $\big(T = \mathbb{S}^1 \vee \mathbb{S}^1 \vee (\mathbb{S}^2)^{\beta_2(\mathcal{K})}\big)$: In this case, the CW complex T consists of one 0-cell, two 1-cell and $\beta_2(\mathcal{K})$ number of 2-cells. Proceeding in the same manner as **Case 1**, we can represent V' as $V' = H^{(0)} \cup \big(H^{(1)}_1 \cup H^{(1)}_2\big) \cup \Big(H^{(2)}_1 \cup \cdots \cup H^{(2)}_{\beta_2(V')}\Big)$.

Now, observe that boundary identification between V and V' is an attachment of one 3-handle. Thus, $M = H^{(0)} \cup \big(H^{(1)}_1 \cup H^{(1)}_2\big) \cup \Big(H^{(2)}_1 \cup \cdots \cup H^{(2)}_{\beta_2(V')}\Big) \cup H^{(3)}$, where $\beta_2(V')$ is equal to $\beta_2(M)+1$, $\beta_2(M)+2$ or $\beta_2(M)+3$ from (1).

Now, if $m' = 0$ then we get $M = V'$ using Lemma 3.3. Since $m = 1$, we get $T = \mathbb{S}^1 \vee (\mathbb{S}^2)^{\beta_2(\mathcal{K})}$ and thus M has a handle decomposition as $M = H^{(0)} \cup H^{(1)} \cup \Big(H^{(2)}_1 \cup \cdots \cup H^{(2)}_{\beta_2(M)}\Big)$. This proves the theorem. □

When M is a compact orientable PL 4-manifold with connected spherical boundary, $\hat{M}$ is a closed orientable manifold, and considering (Γ, γ) to be a weak semi-simple crystallization of $\hat{M}$, Lemmas 3.3, 3.5, 3.6 and 4.3 hold true for this case as well.

Corollary 4.5. *Let* M *be a compact orientable PL* 4*-manifold with empty or connected spherical boundary having the fundamental group of rank* 1*. Let* (Γ, γ) *be a weak semi-simple crystallization representing* $\hat{M}$ *and* V, V' *be as in Definition 3.2. Then* $\pi_1(M) = \mathbb{Z}$ *and* M *admits a handle decomposition as follows:*

$$M = H^{(0)} \cup H^{(1)} \cup \Big(H^{(2)}_1 \cup \cdots \cup H^{(2)}_{\beta_2(M)}\Big) \cup H^{(3)} \text{ (if } M \text{ is closed) } \cup H^{(4)}.$$

Proof. By the Seifert-Van Kampen Theorem, we deduce that $\pi_1(\hat{M}) = \pi_1(M)$ if M has a spherical connected boundary. Also, from the proof of Lemma 3.6, we have $\pi_1(\hat{M}) = \pi_1(V')$ if M has empty or connected spherical boundary. Since $m = 1$, $T = \mathbb{S}^1 \vee (\mathbb{S}^2)^{\beta_2(\mathcal{K})}$ by Lemma 4.3. Thus, M with empty or connected spherical boundary cannot have a finite cyclic group as its fundamental group. Due to (1), we get $\beta_2(V') = \beta_2(M)$. Thus,

$$M = H^{(0)} \cup H^{(1)} \cup \left(H_1^{(2)} \cup \cdots \cup H_{\beta_2(M)}^{(2)}\right) \cup H^{(3)} \text{ (if } M \text{ is closed) } \cup H^{(4)}.$$

This proves the result. □

Remark 4.6. In [3], the authors proved that if M is a closed orientable PL 4-manifold having a semi-simple crystallization with the fundamental group isomorphic to $\mathbb{Z}$, then it admits a handle decomposition as one of the following types:

1) one 0-handle, two 1-handles, $1+\beta_2(M)$ 2-handles, one 3-handle and one 4-handle,

2) one 0-handle, one 1-handle, $\beta_2(M)$ 2-handles, one 3-handle and one 4-handle,

where $\beta_2(M)$ denotes the second Betti number of the manifold M with $\mathbb{Z}$ coefficients. Clearly, Corollary 4.5 refines this result by providing the precise handle decomposition of M.

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